Basic Real Analysis
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Series of Numbers –Part I
Lecture No 66

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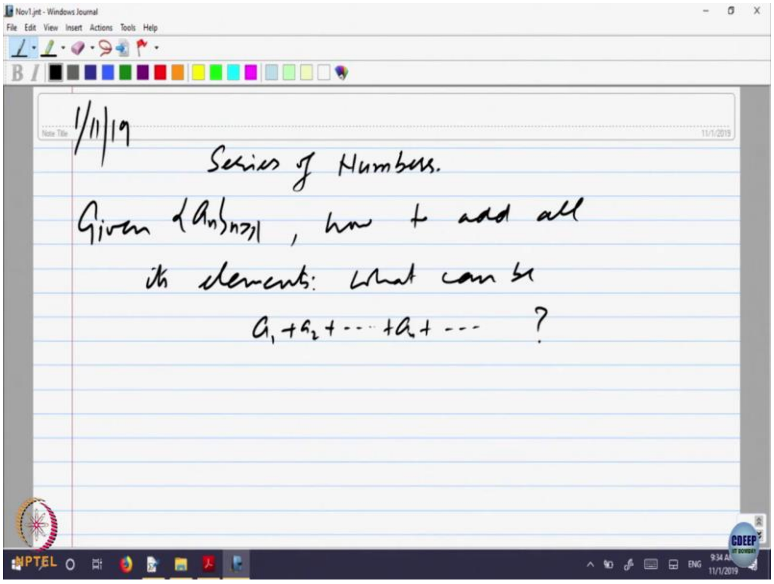

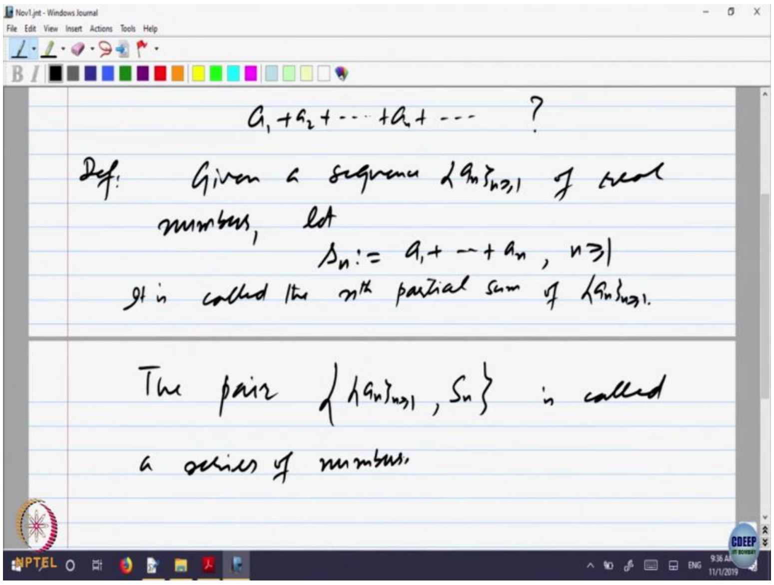

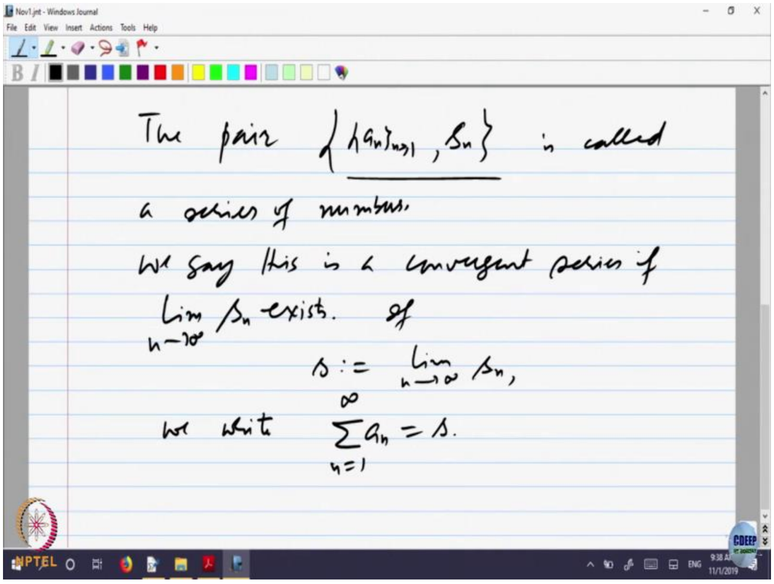

The basic idea is, given a sequence an, so given a sequence an of numbers how to add all its elements. So that is saying that, same as saying what can be a1 plus a2 plus, what this quantity can be?

So, to define this properly, let us make a definition. So, given a sequence an of real numbers, let us define Sn to be equal to a1 plus up to an, n bigger than 1. So, that is the sum of first n terms of the sequence. So, this Sn, it is called the nth partial sum of the sequence an.

So, this is called the nth partial sum and let us give it a name, so the pair, so here is a sequence and here is the partial sum Sn, is called, we will call it a series.

So, a sequence together with its partial sums is called a series of numbers and we say this series, this is a convergent series if, I denote it with small Sn, if Sn limit n going to infinity Sn exists. If we look at the partial sums and take the limit of that, if that limit exists then we say that the series is convergent, and if this limit is equal to S, so Sn we write, sigma an, n equal to 1 to infinity equal to S.

Now let us observe, something so that we do not have to write this cumbersome notation of series being this way. That given a sequence an, Sn's are defined, partial sums are defined, so we know Sn's.

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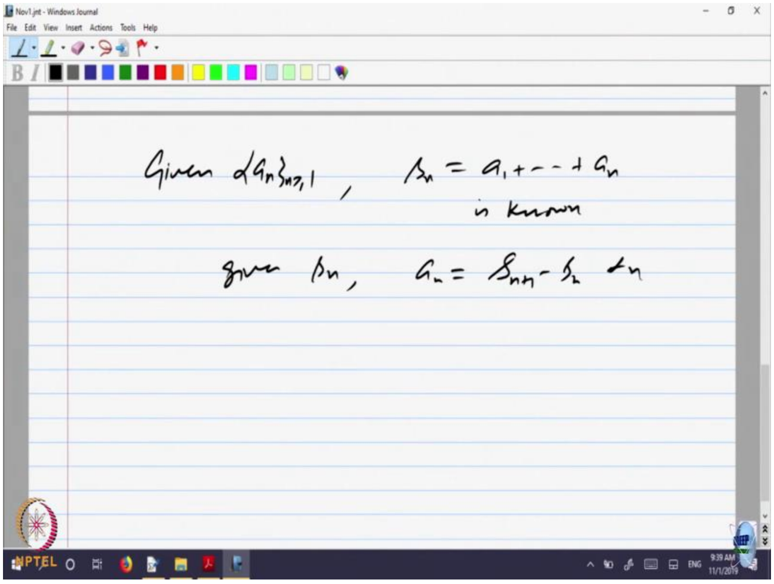

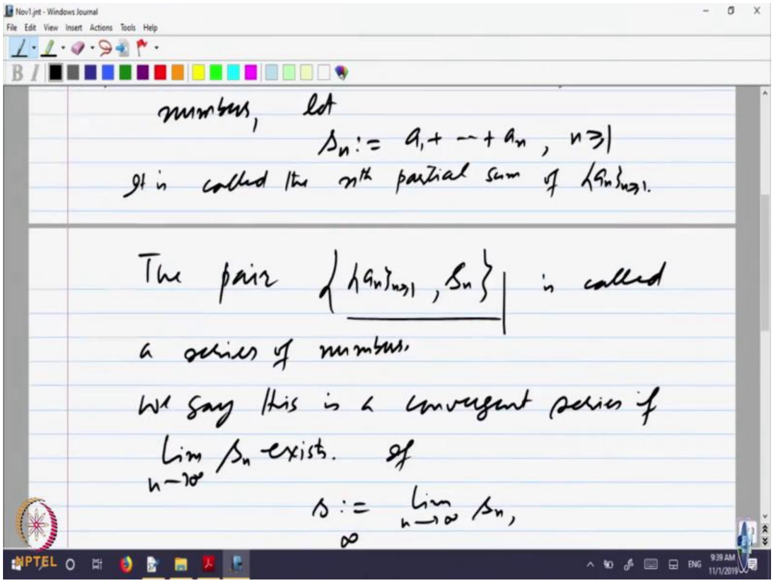

And supposing we say that, we give you the sequence Sn of numbers which is the partial sums of some sequence, then the sequence also is known. So how is that? So, given an, Sn which was defined as a1 plus an is known and conversely given Sn’s. What is an, that is Sn plus 1 minus Sn for every n. So, giving the sequence or its partial sums that data both are equivalent to each other, you give one data, you get the other data. So, for that reason we do not write all the time a series to be like this. We just, given a sequence an.

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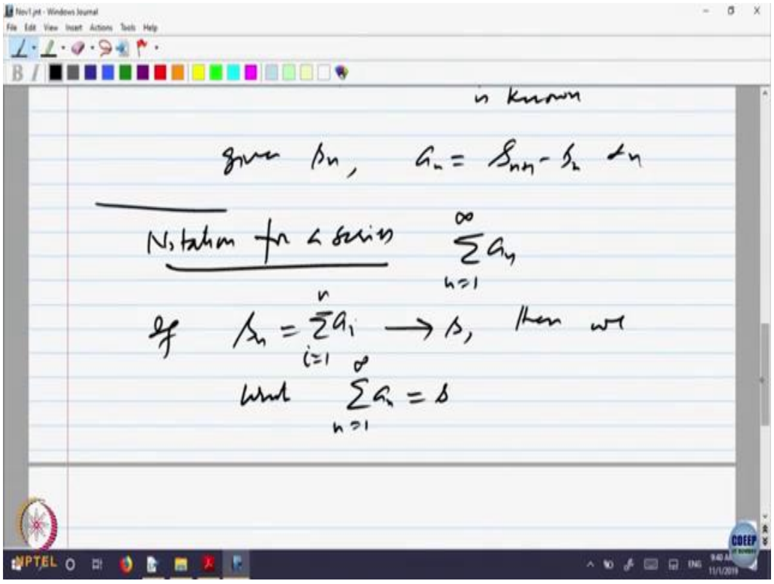

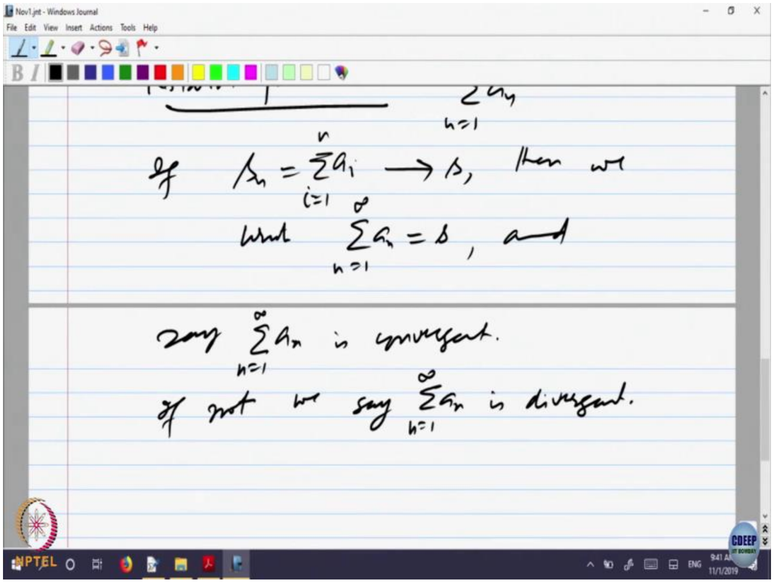

So, notation for a series is sigma an. Given a sequence an, we write the corresponding series as this. This does not mean, you should not take as if it says the sum exists. It is just the notation, for that series.

If convergent, so if it is convergent, if the partial sums Sn's which we defined as sigma i equal to 1 to n ai converges to S then we write sigma an equal to 1 to infinity is equal to S. And say, an is convergent, if not, if it is not convergent we say it is divergent, then we say it is divergent.

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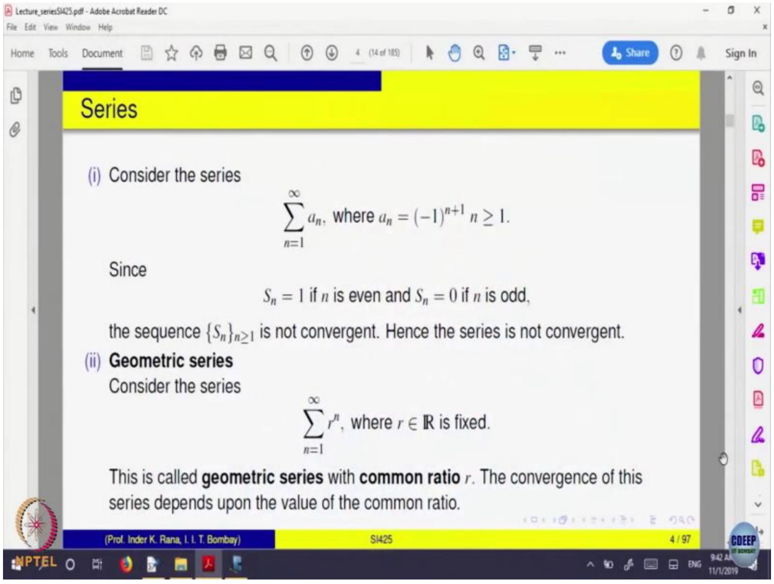

Let us look at some examples, they are relatively simple examples. So, let us look at some examples. So, let us look at the series sigma an where the nth term is minus 1 to the power n plus 1 for n bigger than or equal to 1. So, this is a sequence an, what are the terms of the sequence, 1 plus n equal to 1, or so minus 1, minus 1, plus 1, minus 1, plus 1 and so on.

We know that as a sequence this sequence is not convergent, it fluctuates. Let us try to form the partial sum, Sn. So, what will be the partial sum, depends on whether the n is even or odd. So partial sum will be equal to 0 if n is even, the terms will cancel out otherwise it will be minus 1 or plus 1. So, partial sums do not converge. So, we can say that this series minus 1 to the power n plus 1, is not a convergent series by the definition itself. So, this is not convergent series.

The simplest example of a convergent series is the one which we start looking at in our schools, normally called the geometric series. So, what is a geometric series, so it is the series where the nth term an is r to the power n, where r is a fixed real number.

So, take a real number. First term is r, second term is r square and so on. And this number r is called the common ratio because it is a ratio of an plus 1 and an. So, when is it convergent, we all know that is convergent when mod of r is strictly less than 1.

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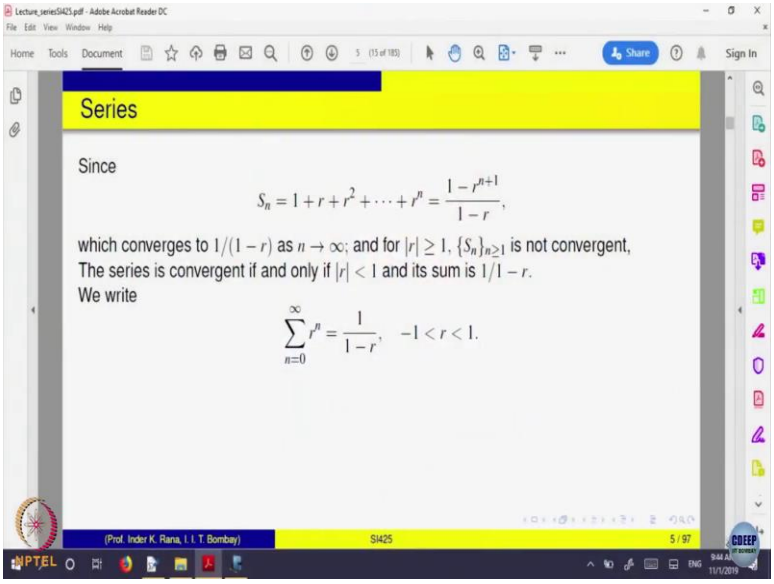

And why, how is that, what is the proof of that? One can find what is Sn, one can write 1 plus r plus this Sn is equal to 1 minus r to the power n plus 1 over 1 minus r. That is easy to find if you write Sn something multiplied by a, it shifts the powers and subtract and you could easily compute what is Sn.

So, this formula that Sn is equal to 1 minus r to the power n plus 1 over 1 minus r, we do it in our schools but, and it is not difficult to find. And if r, so the question is whether r to the power n plus 1 converges to something or not, as n goes to infinity. And we know, we have done it in sequences that x to the power n converges to 0, if and only if mod x is strictly less than 1.

So, using that fact this is convergent, if and only if mod x is less than 1 and in that case the sum is equal to r to the power n minus 1 will go to 0. So, it is 1 over 1 minus r. So, the simplest example of a series which is convergent and this will be sort of used again and again, a geometric series common ratio is less than 1 is convergent. You will see how, this is one of the building blocks for analyzing series.

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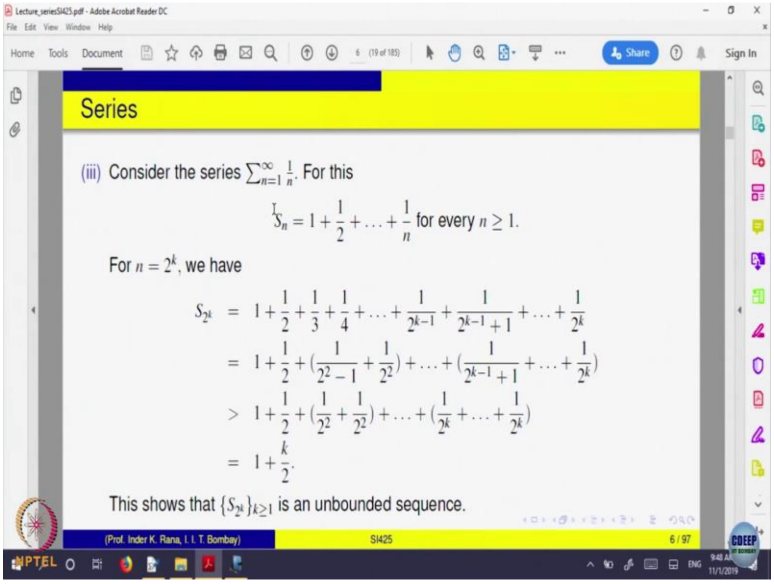

Let us look at one more example, let us look at Sn equal to 1 over n. The first one was, minus 1 plus 1 and now it is 1 over n. So, what is Sn, Sn is 1 plus 1 by 2 plus up to 1 by n, and the terms are non negative. So, it seems Sn is going to increase, you are adding more and more non negative numbers. So, S1 is 1, and S2 is equal to 1 plus half so on, so something is increasing.

But the question is how much does it increase? Because if the partial sums Sn's are increasing, we know it is increasing, but if it is bounded then they will converge, by the property of real numbers. So, is it bounded or not bounded, if it is not bounded above then it will not converge. So, to analyze that one has to make some estimates.

So, let us look at, for n equal to 2 to the power k, let us compute this quantity. So n to the, 2 to the power k. So, this is the 1, plus 1 by 2 and so on over 1 to 2 to the power k and now you pair up. See, 1 over 4 is 1 over 2 square 1 over 3 is 1 over 2 square minus 1. So, make this pairing and 2 to the power k is even. So, you can pair up.

Once you pair up, now this quantity 1 over 2 square minus 1, it is bigger than, if I increase the denominator it is bigger than 1 over 2 square. So, I get bigger than, I do it everywhere, and this is 2 by 2 square and k by 2 to the power k. So, this is bigger than 1 plus k by 2.

So, these kinds of estimates one has to do, to analyze a series. So, what we are saying is for n equal to 2 to the power k, the sum S, 2 to the power k is 1 plus k by 2. So, what happens to these partial sums for n equal to 2 to the power k, as k goes to infinity. It is bigger than k by 2, so it goes to infinity.

So, at least for the partial sums we have got a sub sequence, when n is equal to 2 to the power k. The partial sums has a sub sequence which goes to infinity, is non negative, so it is not bounded above. So, there is a sub sequence which is not bounded above of partial sums. So, the sequence itself cannot be bounded above.

So, sequence of partial sums is not convergent because it is non negative, it is not bounded above. For given any n, you can always find 2 to the power k such that S to the power, such that 2 to the power k is bigger than n.

Given any natural number n, you can always find k, such that 2 to the power k is bigger than n, that increases faster than n, you can easily prove that. So, S to the power 2, S partial sum up to 2 to the power k will be bigger than the partial sum up to n. So, that also will go to infinity. So, that shows it is not bounded, so it is not convergent. So, what does it imply, it implies that the series, 1 over n is not convergent. This is how by definition itself alone we are trying to analyze. Because it is non negative we can make estimates.

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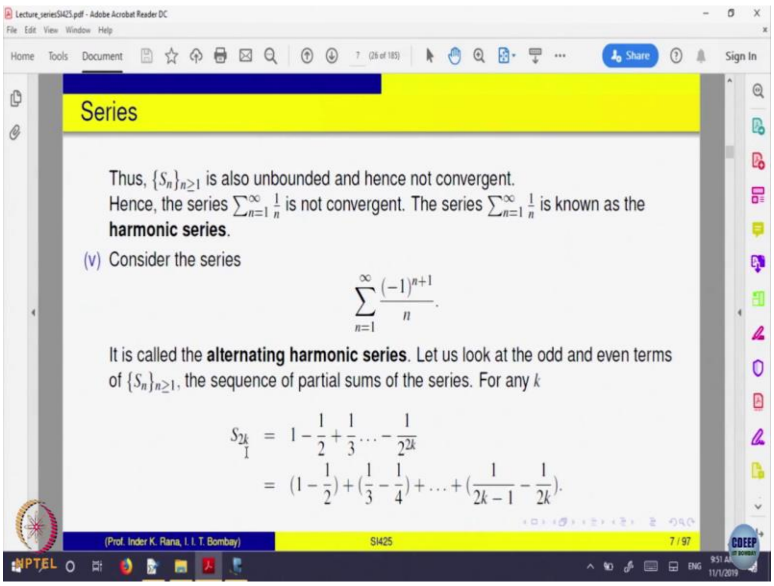

Let us do one more estimation like this, this is interesting. We had 1 over n, let us look at, so the series 1 over n is called harmonic series because of a different reason.

Let us consider the harmonic series, but now the terms are coming plus and minus; so minus 1 to the power n plus 1 divided by n, so it starts with n equal to 1. It is 1 minus 1 by 2 plus 1 by 3 and so on, alternative. Let us try to find out, whether the partial sums for this converge or not, somewhere. So, let us make some estimates.

Once again as before let us try to look at n equal to 2k. So, n equal to 2k, look at the partial sums up to the terms 2k, I think there is something wrong here. This is not 2 to the power 2k, it is just 2k, so there is a typo here. Now how many terms are there, even number of terms, 2k, so I am taking the sums of first 2k terms first. So, I can pair them.

So, the first one, 1, so what I am doing is I am pairing up, so that it is sum of non negative terms, 1 minus 1 by 2 plus 1 by 3 minus 1 by 4 and so on. So, I have grouped them in 2, 2 a pairing. And 1 by 3 minus 1 by 4, that is non negative. So, each bracket is a non negative number.

So, S to the, what does it tell you about these partial sums, that for n equal to 2k, the partial sums are increasing. Because each bracket is non negative, and k plus 1, one more bracket will come, some non negative thing will be added up. So, S2k is a sequence of non negative real numbers. Let us see, what happens when it is odd.

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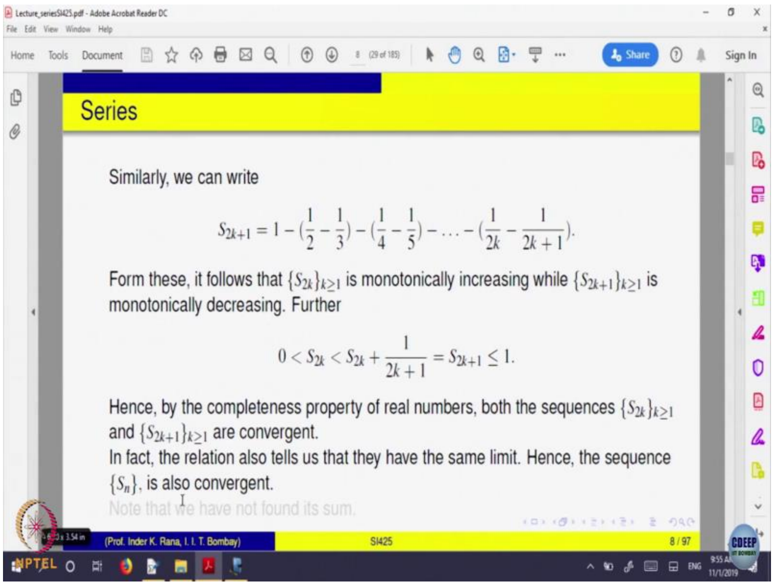

So, let us compute the same thing, for 2k plus 1. So, S2k plus 1, one more term will be added there. So, what is relation between them, S2k plus 1 over 2k plus 1, that is equal to S2k plus 1. One more term is added there and it is plus here, because 2k plus 1 is odd, so minus 1 to the power n plus 1 it was. So, this is a relation between 2k and 2k plus 1.

So, if I look at this sequence of 2k and 2k plus 1, what is a difference between these 2, this one is increasing S2k, S2k plus 1 when I pair them up, what can you say about the sequence S 2k plus 1? 1 minus something, minus again something, I am subtracting and each bracket is non negative.

So, more and more things are being subtracted. So S2k plus 1 is decreasing as k increases but S2k is increasing and this is relation between them, S2k is less than 2k plus 1 and all are bounded between 0 and 1.

So, what does it imply, S2k is increasing and bounded, so that will converge. S2k minus 1 that also is decreasing is a monotonous sequence, bounded below so that also will converge, so both of them converge. And what is the difference between the 2, between this S2k and S2k plus 1, the difference is 1 over k plus 1, so the difference can be made as small as you want. So, the sequence of partial sums, the even partial sums and the odd partial sums both converge to the same value. We have got a sequence, where the odd and the even both converge, the sub sequence of odd terms and the sub sequence of even terms both converge to the same value.

So, here is an exercise show that a sequence itself is convergent. So, take it as an exercise in sequences. You have got a sequence an of numbers, such that if I take the sub sequence of even, so a2, a4, a6 that sub sequence and look at the sub sequence a1, a3 and so on, both converge to the same value. Then claim, that the sequence itself should converge to that value, the sequence itself is convergent. It is a very small exercise, it is a good exercise to go back and revise your notion of sequences, just definition.

So that will mean what, what will that mean, odd and even both converge so the sequences themselves converge. So that means, Sn is convergent, Sn itself is convergent that was the exercise we were seeing, and as a result this series is convergent.

So, the interesting thing is the series 1 over n is not convergent, but minus 1 to the power n plus 1, divided by n, that is a convergent series, alternate plus and minus terms if you make it, that is called the alternating harmonic series.

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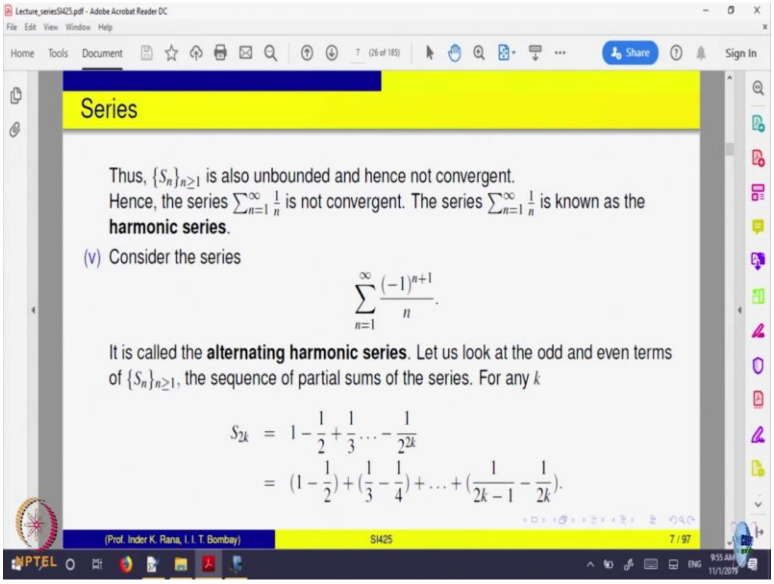

So that is called, so this series, this series is called alternating harmonic series. That is convergent. So, I am just giving you some examples to illustrate that how definition can be used to prove something is convergent or not. And it becomes slightly cumbersome every time estimating the partial sums and trying to see whether it is convergent or not. So, these are the examples which illustrate that.

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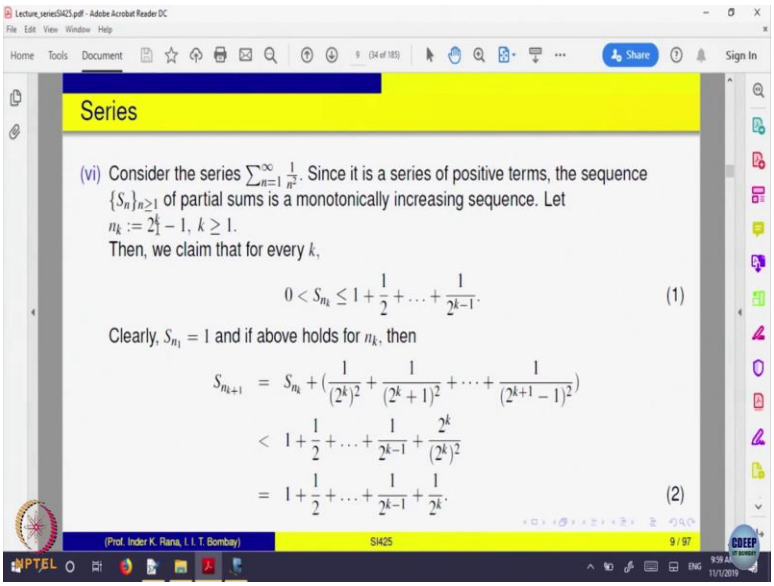

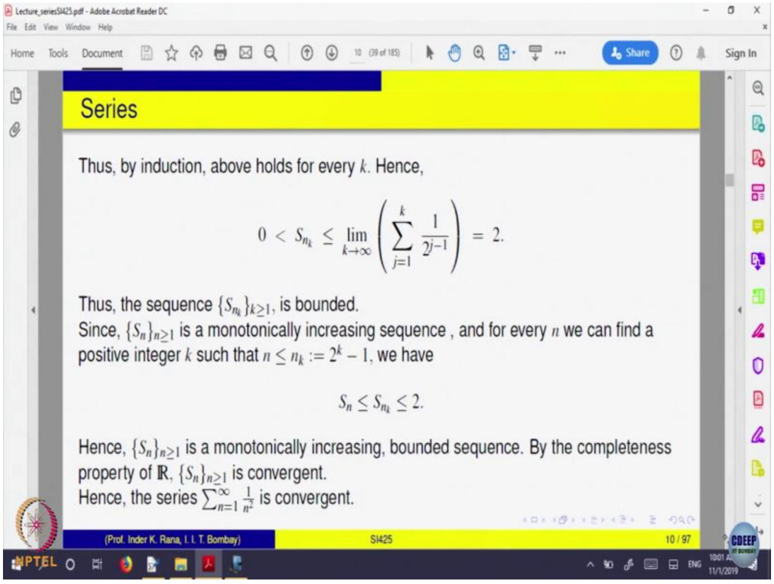

So naturally, let us look at one more, probably, 1 over n square. So, the series is nth term is 1 over n square. This is a series of non-negative terms. 1 over n square is nonnegative. So, partial sums will be a monotonically increasing sequence of numbers. The question is whether it is bounded above or not. If it is bounded above, the partial sums then the series will converge, if not then it will diverge.

Again, let us try to estimate. The claim is that, if I look at nk which is 2 to the power k minus 1, then Snk is less than equal to sum of the geometric series, 1 by 2 plus 1, 1 by 2, 1 by 2 cube and so on. So, we are trying to bring in somewhere again, something known kind of a thing, and this proof for every k we want to prove something.

So, what is the technique of proving something for every natural number, the only thing we know is by induction. So apply induction, Sn1 is equal to 1, so it is true, nk plus 1. So, what will be it, that is Snk plus something, plus the remaining terms which are being added, and that squares, 2k squares.

So, you can make it less than, 1 over 2 to the power k is less than 1 over 2 and so on. So, this becomes less than the geometric series. So, estimates basically. So, once you do that, once you know this is true, so what happens to the series Snk, it is less than this. And this is a convergent series, we know that.

So, what happens to the limit, as k goes to infinity, the geometric series. It was bounded by the geometric series, so we know the sum of n terms, and goes to infinity. So, what we are saying is there is a sub sequence nk, there is a subsequence nk of partial sums Sn's which are bounded between 0 and 2.

Can you say that, that implies Sn itself is bounded? What is nk, what was nk, nk was 2 to the power k minus 1. We were saying that if I take n to be, nk to be this then Snk is bounded. Can we claim that Sn itself is bounded? Keep in mind they are non negative.

Once again, given any n, given any n you can find a k, such that n is less than nk. Given any natural number n, you can find a power of 2 to the power k, such that n is less than 2 to the power k minus 1.

Yes or no? Yes? Natural numbers, 2 to the power, they are going to increase faster than n anyway, much faster. So that means what, and they are non negative terms. So given any n, there is a k such that n is less than nk.

Can I say Sn is less than S of nk, yeah because they are non negative terms. Sn is increasing, and that is bounded by 2, so each Sn is bounded by 2. Each partial sum is bounded by 2, because the sequence of partial sums is monotonically increasing and for n equal to 2 to the power k minus 1, it is bounded by 2. And coupled this with the fact that, given any n, you can find a natural number k, such that n is less than 2 to the power k minus 1.

So, Sn will be less than the partial sum up to 2 to the power k minus 1 which is less than 2. So, each Sn is bounded by 2, it is monotonously increasing so they will be convergent. Because they are non negative, so 1 over n square is a convergent series.

So, this is convergent, monotonically increasing and bounded so it is convergent. Here what was helping us is because it is series of non negative terms, partial sums are monotonously increasing we have to only analyze whether they are bounded above or not.