Basic Real Analysis

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Pointwise and Uniform Convergence –Part IV

Lecture No 65

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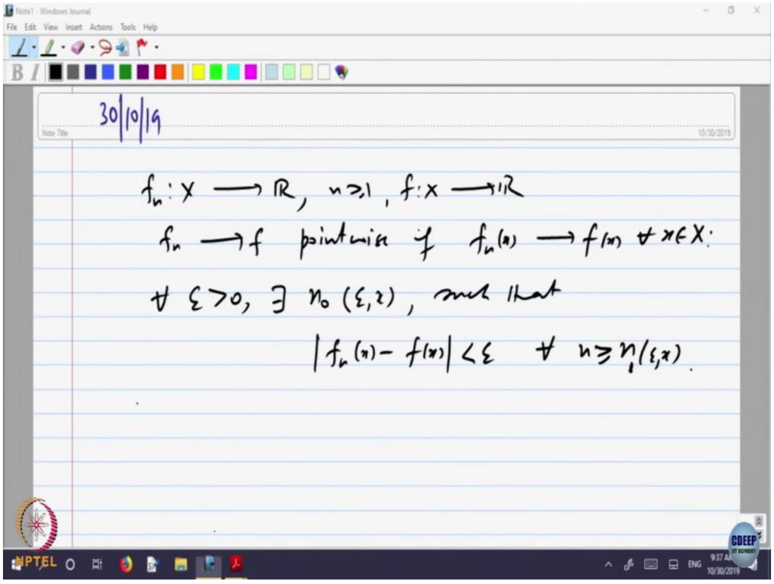

So, let us just recall we were looking at Pointwise and Uniform Convergence of sequences of functions. So, let me just recall that we said that fn is a sequence of functions defined on n. We say, fn converges to f pointwise if fn of x, also f is also a function from x to R, converges to f of x for every x belonging to the domain.

And that means, we should not forget what that means; means for every epsilon bigger than 0, there is a stage n naught which may depend upon, in general it will depend upon epsilon and the point x, such that mod of fn x minus f of x is less than epsilon for every n bigger than or equal to that natural number n naught, which may depend upon n and x. So, that stage may depend upon epsilon of course and also on x.

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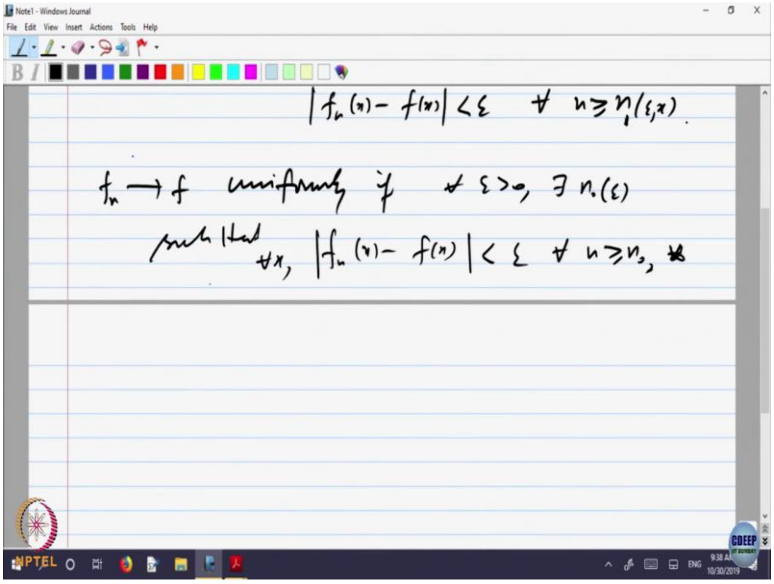

So, then we define what is called fn converges to f uniformly. If, same thing if for every epsilon bigger than 0 there is a stage n naught which now will not depend on the point, but it will depend only upon epsilon. There is a stage, there is a natural number n naught, such that fn x minus f of x is less than epsilon for every n bigger than n naught and for every, so probably we should mention that, for every x this happens.

So, we gave a lot of examples of sequences which converge pointwise, which converge uniformly and we had started looking at properties of uniform convergence. We said that, pointwise convergence need not preserve various properties, namely if each fn is continuous f may not be continuous, each fn is differentiable then f may not be differentiable, and if each fn is integrable, f may not be integrable.

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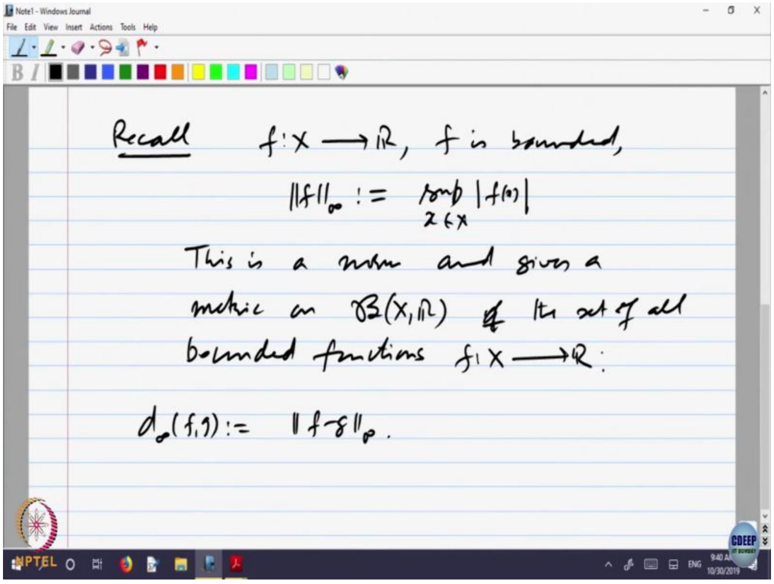

So, to analyze these properties for uniform convergence functions let us recall, if f is a function defined on X to R and f is bounded. We define what is called the infinity norm, that is supremum x belonging to X of mod f x. So for, and we observed that this is a norm and gives a metric or so I think we called it B X, R on the set of bounded functions with domain X and taking values in R. So what is that metric, that metric is d infinity f, g is equal to norm of f minus g.

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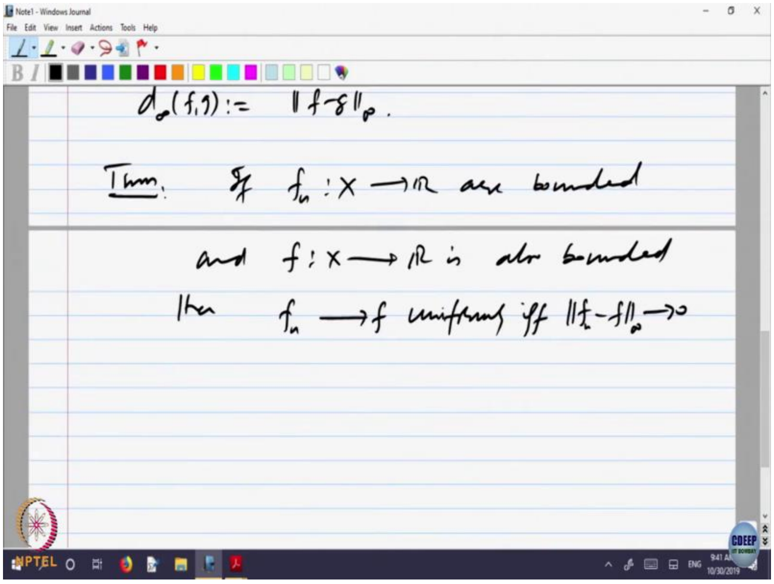

So, the relation of uniform convergence with this norm is, let us put that as a theorem. We proved it last time, that if fn, if fn X to R are bounded and f X to R is also bounded then fn converges to f uniformly, if and only if norm of fn minus f goes to 0. So, we had proved this theorem.

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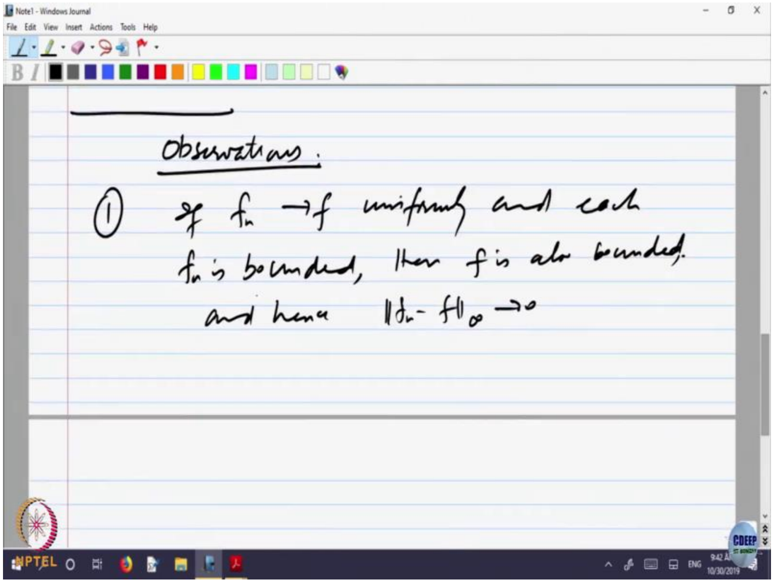

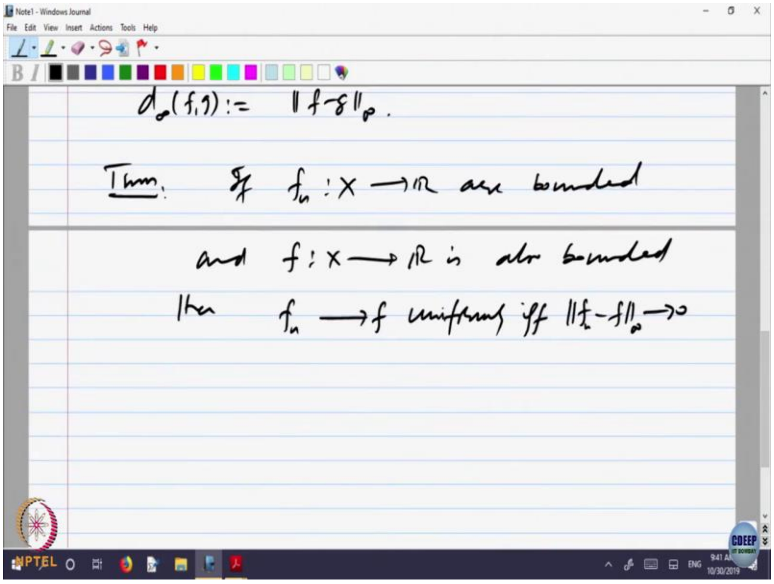

Let me also point out that here, so let us observe a few things, some observations. One, if fn converges to f uniformly and each fn is bounded then f is also bounded. So, if a sequence of functions converges uniformly to a some function f, and each fn is bounded, then the function f is also bounded. And hence, you can apply earlier criteria because there we assumed f is also bounded.

So, one way, if f is uniformly convergent, each all fn's are bounded then f is also bounded. And hence, you can write and hence norm of fn minus f converges to 0. That is a consequence of the earlier theorem.

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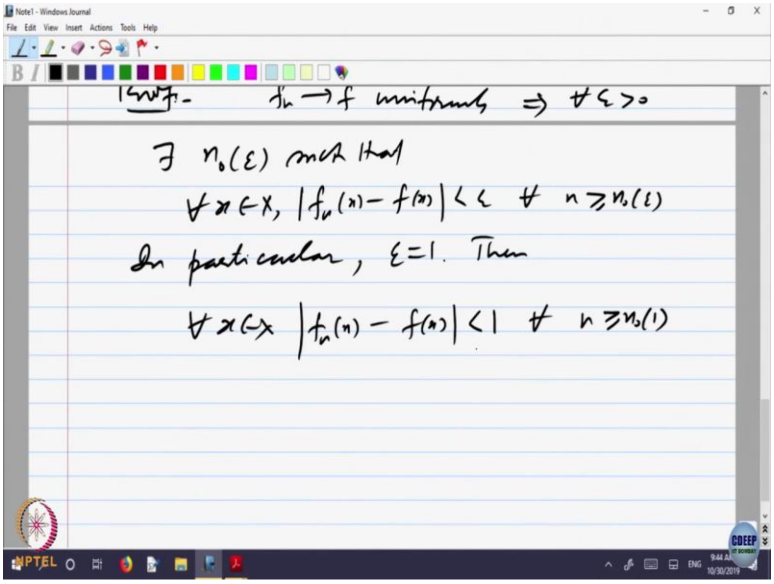

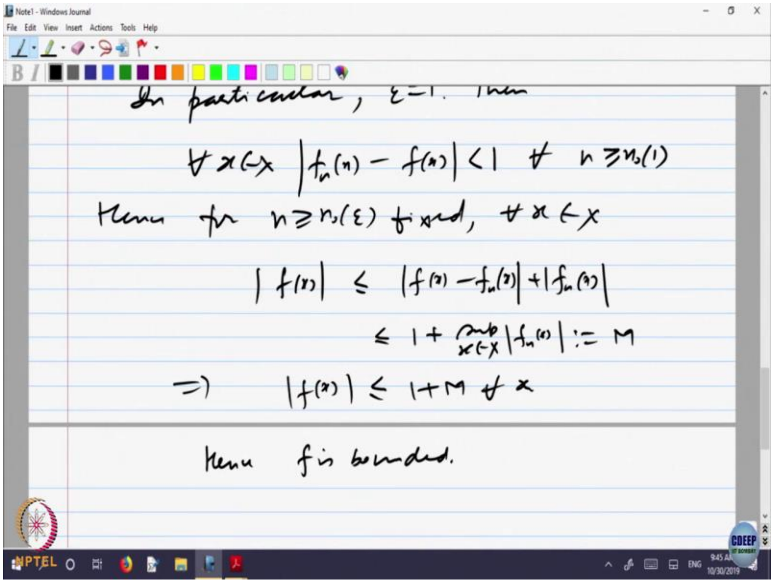

So, let us prove this. So, fn converges to f uniformly. So that means, what does that mean, that means for every epsilon bigger than 0, there is a stage n naught which depends only on epsilon, such that norm of, absolute value of fn x minus f of x such that for every x belonging to X is less than epsilon for every n bigger than n naught. That is the definition of uniform convergence.

So, let us specialize it. So, in particular, it is not necessary but anyway, let us take epsilon equal to 1, then for every x belonging to X we have fn x minus f of x will be less than 1 for every n bigger than that stage n naught 1.

That means fn is close to f by distance 1 and fn's are bounded anyway, so I can just apply triangle inequality. Hence, for n greater than n naught epsilon fixed, so fix one, any one of the numbers, mod of fx I want to show it is bounded, it is bounded by some scalar for every x, then for every x belonging to X. I know f of x is close to fn x, and fn's are bounded anyway. So, I can use the triangle inequality.

Then for every x belonging to X this is true, this is less than or equal to 1 plus supremum over x belonging to X of fn x, that exists. So, you can call this number as M if you like, so implies mod fx is less than or equal to 1 plus M, for every x. So hence, f is bounded. So, if a bounded sequence of real valued functions converges uniformly then the limit also is a bounded function. So, that is 1.

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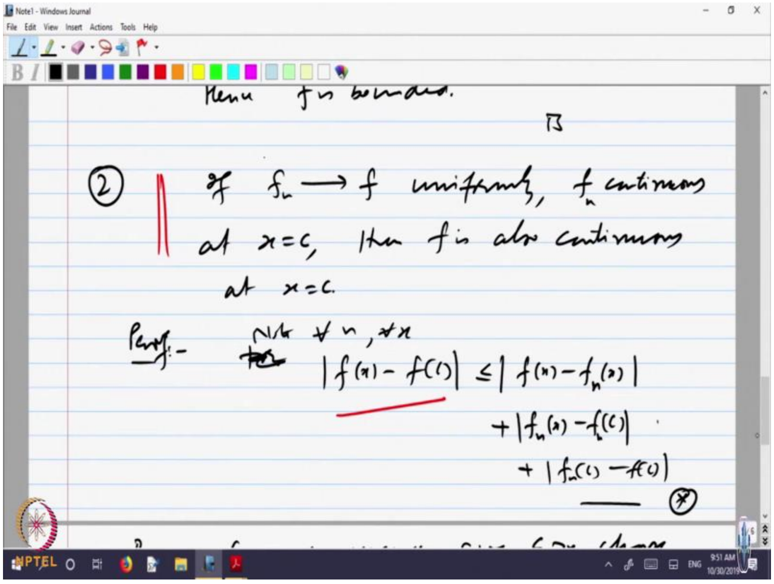

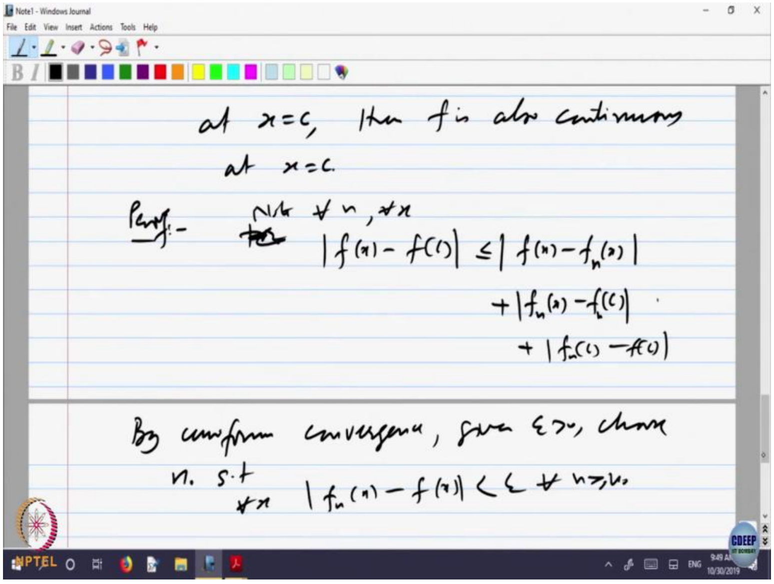

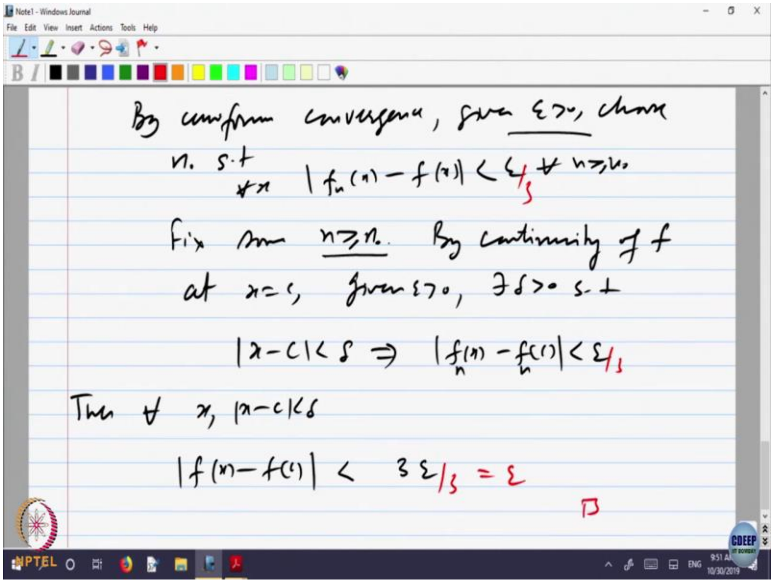

Let us look at, second property that we looked at last time, so let us do it again anyway. That if, fn converges to f uniformly, f continuous, each fn is continuous at x equal to c, then f is also, then f is also continuous at x is equal to c. So, continuity is also preserved under uniform convergence. So, let us prove that.

So to prove this, what we have to do, so we want to make for, we want to analyze the distance between f of x and f of c. We want to say that this distance can be made small whenever x is close to c, for continuity. And what we know is, fn is converging uniformly to f. So close to f, there is a fn, at every point and fn is also continuous.

So both these facts, so let us note this I can add, so let me write less than or equal to, f of x minus fn of x for any n, plus, now fn x minus fn at c, that will be small because of continuity of fn, so plus fn of c, and the last is f of c.

Why I am doing this, I want the left hand side to be small. But close to f there is a fn, because fn is converging uniformly, so I can estimate change f to fn. And fn's are given to be continuous, so I can use continuity for the second term of fn. And finally, once again, close to fn there is a f, fn is close. So, all these 3 terms can be made small. So how do I write it, so note for every n, for every x this is true.

So, by uniform continuity, by uniform convergence, by uniform convergence. So, this is how you think and now how you write for uniform convergence, given epsilon bigger than 0 choose the stage n naught such that mod fn x minus f of x is less than epsilon for every n bigger than n naught and for every x.

So, that is by uniform convergence. So now, let us fix some n bigger than n naught, bigger than this n naught fix. By continuity of f at, by continuity of f at the point x is equal to c, given epsilon as before, that is already fix, there is a delta such that mod of x minus c less than delta implies f of x minus f of c, fn so fn of c is less than epsilon.

So, there is continuity of fn, we are given fn is continuous. So, we have fixed n bigger than n naught, already epsilon is fixed, so find the delta size that this happens. Then for every x with x minus c less than delta f of x minus f of c will be less than or equal to, so go back, our starting point. Let us call that as star, when we said that f is close to fn, fn is continuous so that triangle inequality, the first inequality. So, that is what motivated us, so in the star use these things, is less than, at least strictly less than 3 epsilon.

And if you wanted nice thing, then you could have gone back and modified everything, this by epsilon by 3, this by epsilon by 3 and this by epsilon by 3 and that could have been epsilon, because epsilon is arbitrary. So, you can always make it as small as you want. So, that will prove.

So that proves, basically keep in mind what we want to show, we want to make this thing small, and we are given that fn's are converging to f uniformly and each fn is continuous. So, bring in fn's. So that is, continuity is preserved under uniform convergence.

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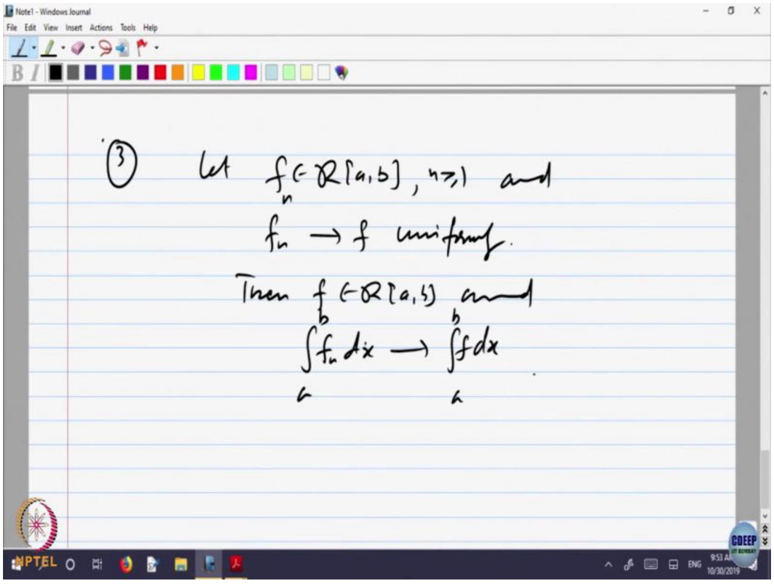

So, let us look at next, something which is also preserved. So, third, fourth, I don't know what is it, property number 3. Let us look at integrability. Let, f belong to R a, b, fn's belong to R a, b and greater than equal to 1, or Riemann integrable functions on the interval a, b and fn converges to f uniformly. Then f is also Riemann integrable, so then f belongs to R a, b, and integral fn d mu, dx converges to integral f dx, a to b.

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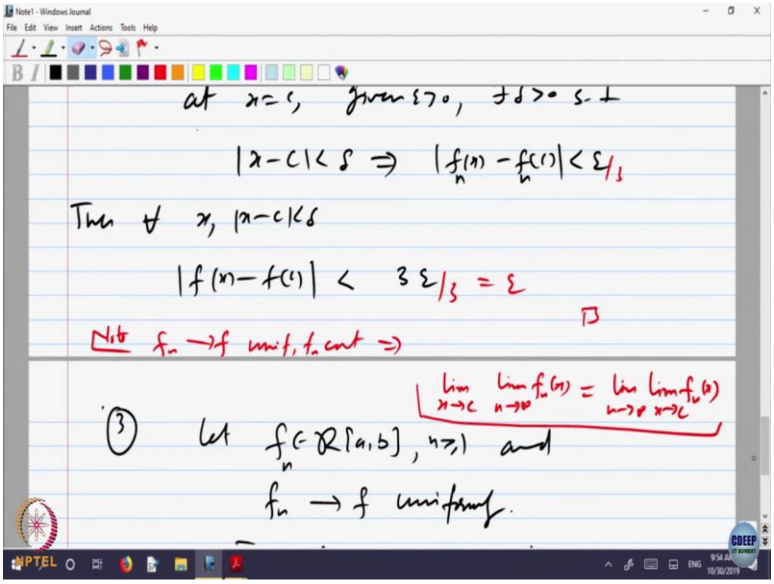

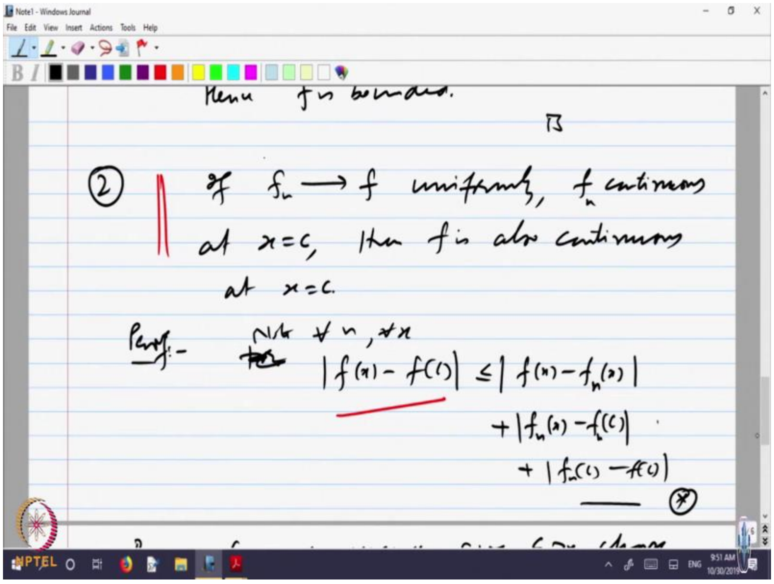

By the way, I think, let me just make a point here that this theorem when we said uniform limit of continuous functions is continuous, another way of writing that. So let me put a note; fn converges to f uniformly fn continuous implies, see we look at limit fn of x, n going to infinity, that is f of x, and as x goes to c that is continuity.

So, limit x going to c is same as, you can write it as, you can take the limit inside. So, it is limit x going to, n going to infinity, let me write n going to, n going to infinity limit x going to c of fn of x. So, limit fn will be f of c and f is continuous.

So this, conclusion of this you can write it as like this. It is like interchanging 2 orders of taking limiting operations, limit x going to c, limit n going to infinity is same as limit n going to infinity and limit x going to c, interchange of limits operations are possible whenever the convergence is uniform. So, that is another way of writing this theorem, so which is useful way of observing. There are 2 limit operations, so you can interchange whenever there is uniform convergence, that is what it says

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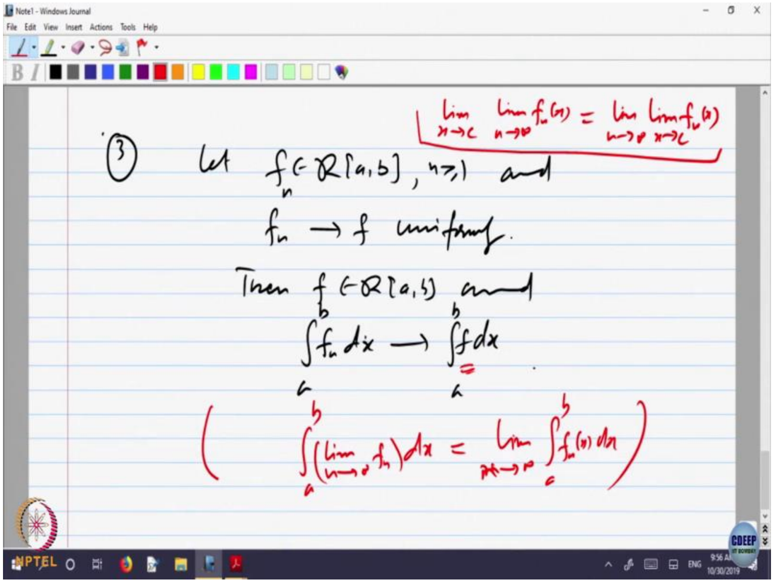

So, that is something, something coming here also, so I can write this as, so it says a to b limit n going to infinity fn dx. What is the right hand side, f is the limit, and the right hand side f is the limit, so here, this f is the limit. So, I can write as limit, that is same as, this converges so that is limit n going to infinity, integral a to b fn x dx.

Again, there is interchange of limit and integral here now. Integral of the limit is the limit of the integrals. So again, so it essentially says under uniform convergence integration is a continuous operation kind of, integration is continuity operation. Whenever there is interchange, something implies continuity of something.

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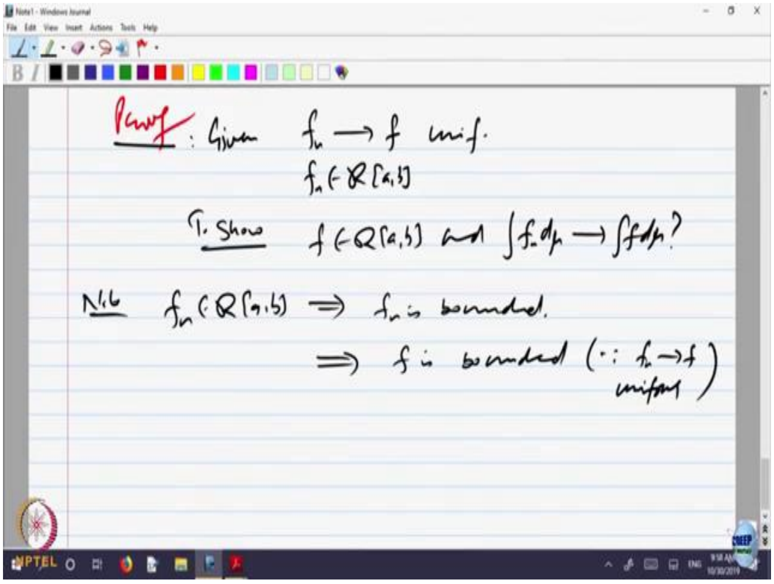

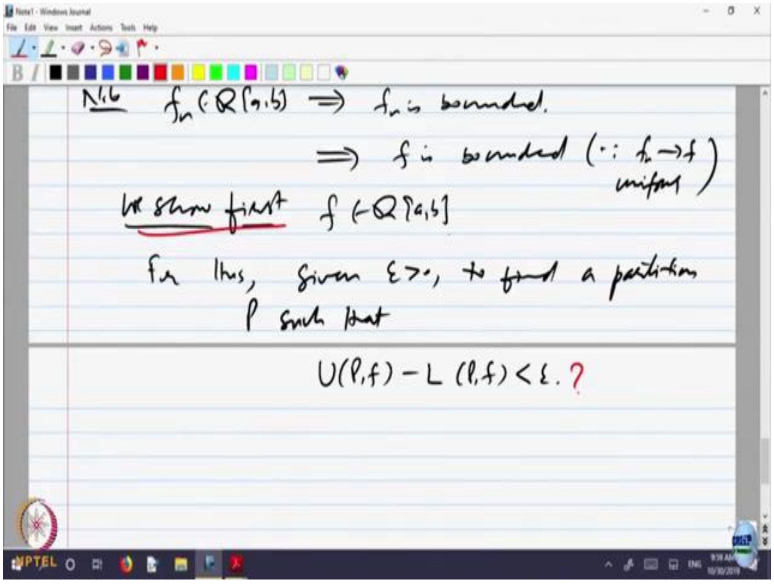

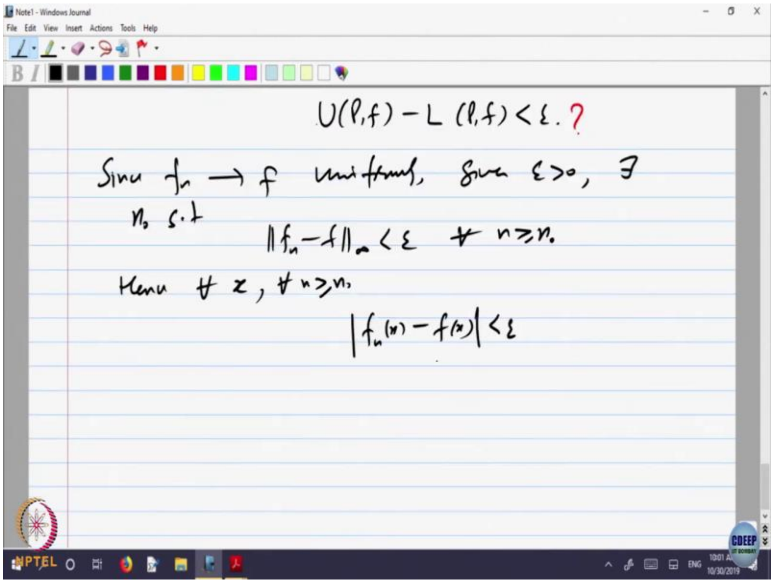

So, here it is saying that the limit of the integral is the integral of the limit. So, the 2 operations are interchanged. So, let us prove this, so proof by black. So, we are given fn converges to f uniformly. So, this is given and each fn belong to R a, b. We want to show that f belongs to R a, b, then only you can write it is integral. And integral fn d mu converges to integral f d mu. So that is what is to be shown.

Now if you want to show that f is Riemann integrable, then you have to first show that it is a bounded function, f should be a bounded function.

But that follows from the fact that just now we have shown. So note, so let us note each fn is Riemann integrable, so that implies each fn is bounded. We have shown that every Riemann integrable function is also bounded. Implies that f is bounded because fn converges to f uniformly. So, all are bounded functions. The only thing that we do not know now is, whether f is Riemann integrable or not, f is a bounded function.

So, let us try to show, we show first f is Riemann integrable. We have got it is bounded, it is okay. Boundedness helps us look at upper and lower sums for a function. That way of defining integration by way of upper and lower sums is possible, only when f is given to be bounded. In the Riemann definition f is not assumed to be bounded. But anyway, we have got now boundedness.

For this what we to show, for this we want to show f is Riemann integrable. That means, for this, given epsilon bigger than 0, to find a partition P such that, when is a function integrable, when given any epsilon, upper and lower can be brought close to each other. Such that upper sum minus the lower sum is less than epsilon. So, this is what we want to show. So given epsilon, if we can find a partition P such that upper minus the lower is, difference is small then we are done.

And keep in mind, how is the upper sum defined, upper sum is, over a partition is by looking at the maximum value of the function in any subinterval multiplied by the length of the interval, sum it up.

And what is given to us, given to us is f is, each fn is Riemann integrable and fn’s are converging to f uniformly. So, close to f there is a fn, and fn is Riemann integrable. So, the idea would be the upper sums of f, we should try to approximate it by upper sums of fn. That is the route we should follow. So, let us do that.

So, by uniform convergence, since fn converges to f uniformly, given epsilon greater than 0, there is a stage n naught such that all are bounded. So let us write, mod fn minus f infinity is less than epsilon for every n bigger than n naught. Because all fn’s f are bounded, so I can now write the supremum less than, for every x and so supremum.

So that is same as saying, hence for every x, for every n bigger than n naught, let us write what is hidden here is that, fn x minus f of x is less than epsilon. That is same as saying that supremum and hence this also is less than.

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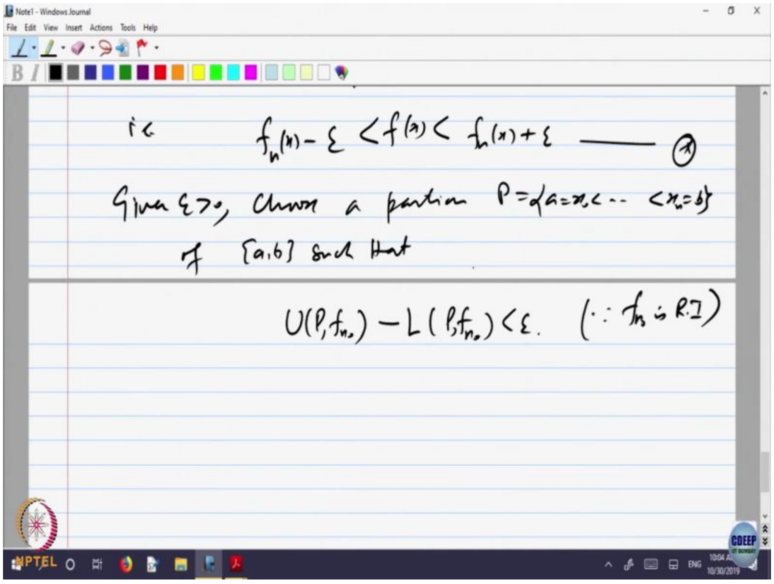

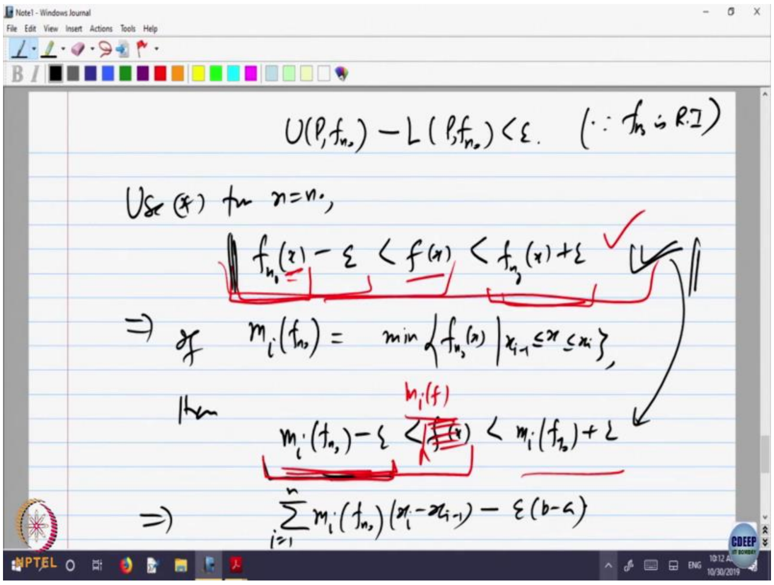

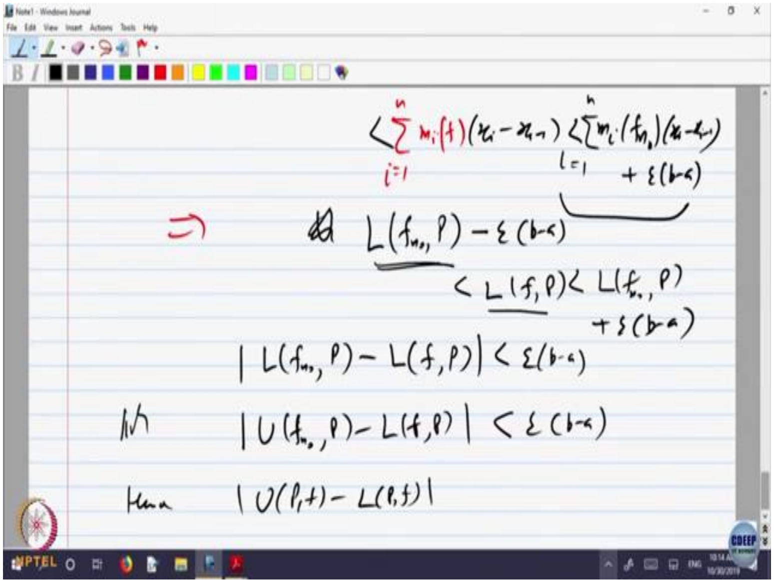

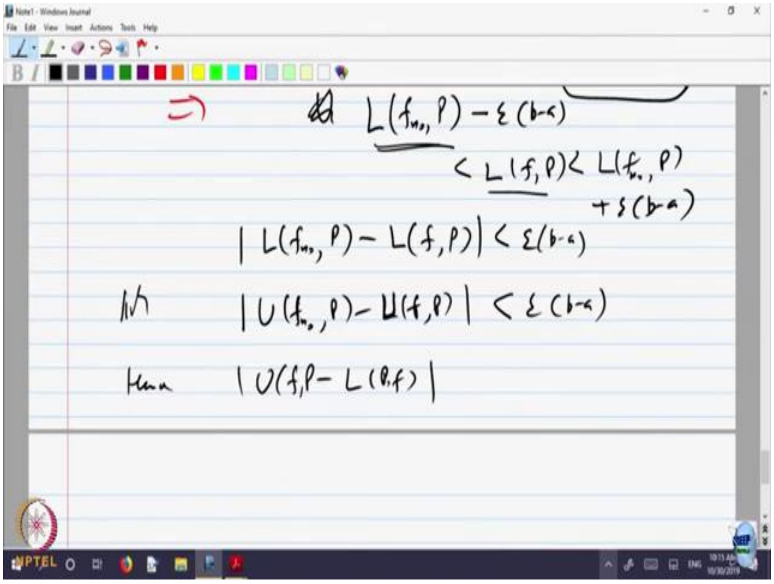

Sometimes expanding things help. What we want to do is, we want to relate f with fn, the maximum value of f with fn, the minimum value of f with fn. So this gives me, this is same as saying, if I look at fx it is less than, expand this inequality absolute value, so this is fn x minus epsilon is less than fn x plus epsilon. This is same as saying as this. The distance of fn from f is either this side epsilon or that side epsilon. That is same as saying this. Now this will give us the required thing.

So, let us choose the partition. So given, choose, so we have used uniform convergence, now we use the integrability. Choose a partition P, so let us call it as a equal to x, 0 less than x less than equal to b of a, b such that, so let us, we are not fix, so let us, n bigger than n naught. So, let us take n naught itself, such that U P fn naught minus L P fn naught is less than epsilon.

What I have used, I have used the fact that fn naught is Riemann integrable, because, is Riemann integrable. So, there must be, given epsilon there must be a partition such that things are close. Now let us, so this is, now that equation let us keep that in mind, so this was the equation when fn is close to, so star.

So, let us use star n equal to n naught, so what we have, f n naught x minus epsilon is less than f of x is less than fn naught of x, I am just specialized because this is true for all n bigger than n naught. So, let us fix n naught so that we do not have any.

See this is what we are saying, fn naught x is close to f of x by distance epsilon. So, I can take the minimum values, so implies, so let us look at the minimum value mi, let me write fn naught what is this mi of f n naught, that is the minimum value of the function f n naught in the interval xi minus 1. So if, let me write, it is equal to minimum of fn naught x, x belonging to xi minus 1 and xi then what does upper one give me? What will this equation give me?

This is f n naught x, so it will be bigger than the minimum value. So, mi of f n naught minus epsilon will be less than f of x will be less than mi fn naught plus epsilon. This equation, this true for all x. So, let us look x in the subinterval xi minus 1 to xi. So, this will be bigger than the minimum value, it is less than f n naught x, for every x. So, take the minimum over that. So, this equation gives me this.

Now from the minimum, value how do you go to the lower sums, by multiplying by the length and adding up. So, let us do that, so implies sigma mi of fn naught multiplied by the length, so xi minus xi minus 1, i equal to 1 to n. When I multiply epsilon by those lengths, those lengths will add up, summation, so it will be just epsilon times b minus a.

This, this part is this is less than f of x multiplied by the length is less than the corresponding thing, so that is mi fn naught xi minus xi minus 1 plus epsilon times b minus a. What happened to the sum, here is the sum. So, i equal to 1 to n.

The previous equation I have multiplied throughout by xi into xi minus 1 and summed up. So, the first term is mi, so this first term is, let me just underline it so this thing, if you multiply by xi minus 1 to xi minus 1, the length of the interval and sum it up, when you multiply by epsilon, that should be giving you epsilon times P minus a. In between, I should put the sum also, I forgot to put the sum here, 1 to n, there is sum everywhere.

So, what does this give me now? Why f of x? No. Actually, I can put here also the minimum of f of x. So, let me do that. Anyway this, I should have, I can do that. So, from here when I multiply mi less than, I can put here also the minimum of f.

See from this equation, let me just clarify, this is true for every x. Now in the first part here, take the minimum over fn naught. This is for every x, so minimum of the function fn naught in the interval xi minus 1 to xi, so that will be less than or equal to f of x for every x anyway. And there you can take the minimum of f also.

So, you can get here minimum, so you will get this part. Take first the minimum over x, you get this quantity. This is less than f of x. Take the minimum of fx over that interval. So, that gives you the minimum function of fi, mi, so what is that notation I am writing, we are not writing that, let me just write, what is that quantity, so that quantity is mi of f.

So, I am saying I can just write that, is f of x, this quantity is less than f of x, so it will be less than equal to minimum also. And that minimum will also be less than or equal to minimum of that. So, I can take minimum everywhere in this inequality, that is what I am saying.

One at a time, but you should do, one at a time to justify that. First take the minimum over this part, this is for every x, so minimum only in the left hand side, so you will get this quantity. Less than equal to f of x for every x, so take the minimum in that interval, so that is mi. This is less than equal to fn naught of x, so take the minimum over that, so as is mi. So, I should have, so this f of x I can replace it by mi of f.

Basically, the idea is fn is close to f by margin of epsilon this side or that side. So, I can take the minimum over the interval, this is happening for every x, so I can take the minimums.

So that means, so what is the meaning of this, this means, this is the lower sum, so lower sum of small mi, so lower sum of fn naught with respect to P minus epsilon times b minus a, this first part is less than or equal to the lower sum of f with respect to P, f with respect to P and the last term this one is less than or, why less than or equal to, is less than actually, it is strictly less than. It does not matter actually, is less than L fn naught, it is the minimum over f n naught, so fn naught of P plus epsilon times b minus a.

Basically, what I am saying is this equation, if I take the minimum over all x in the interval xi minus 1 to xi, so equation holds for minimum also. Multiply by the length of the interval and add up, that gives you the lower sums minus epsilon times this.

So similarly, the upper ones. So, that means what, the lower sum of f and the lower sum of this is less than, that means mod of L, we are writing fn first, fn naught P minus the lower sum f, P is less than epsilon. In the absolute value I can write it this way. This minus b minus a, epsilon times b minus a. I am writing that in equality back in terms of absolute value, nothing more than that.

So similarly, you will have upper sum fn naught P minus the lower sum of f, P will be less than epsilon times b minus a. Instead of taking minimum, you can take the supremum and the same thing. So hence, upper sum P, f minus the lower sum, I want to estimate this quantity, for f.

So that is why, so now add and subtract. So, less than equal or to absolute value triangle inequality, upper sum P, f, but that is close to upper sum of Pn, P, fn naught, I should interchange, because I am writing P first and then f, so be consistent so that is, so we wanted to estimate f...

Student: Sir the second equation is (())(39:12)

Professor: This one?

Student: Yes sir, it is U fn mod comma P minus U (())(39:19)

Professor: fn naught, no, that is U, sure, sure, thank you. That is upper sum. I said similarly, the lower and similarly the upper

Student: (())(39:32) maximum instead of minimum

Professor: Where are in that same thing here. Now this quantity is less than or equal to f of x, so maximum of that must be less than or equal to f of x, less than or equal to maximum of f x. I am saying same inequality gives you both, lower as well as the upper.

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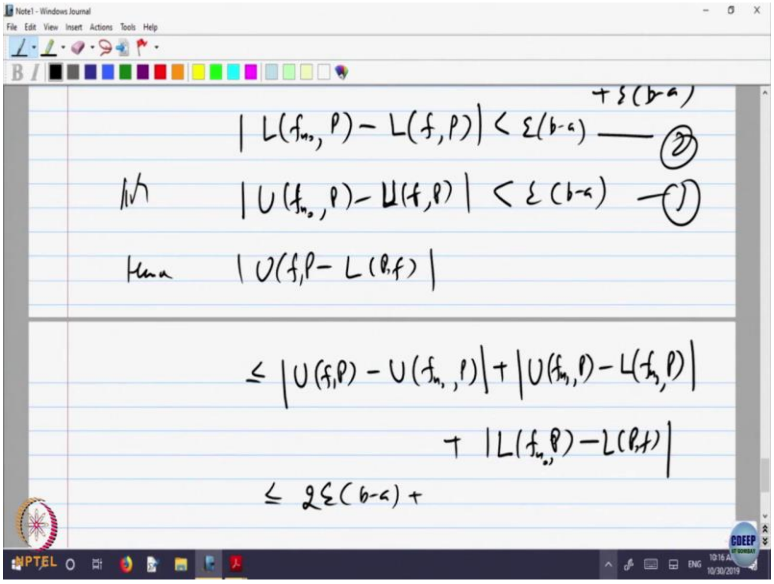

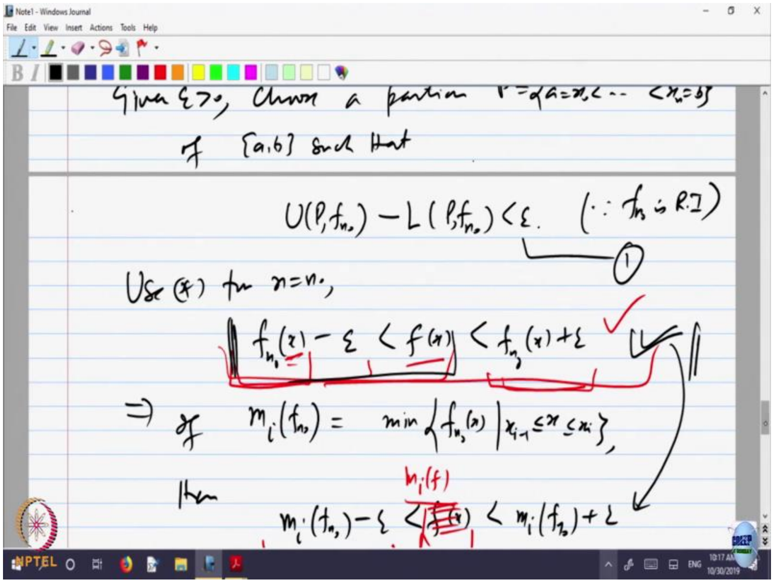

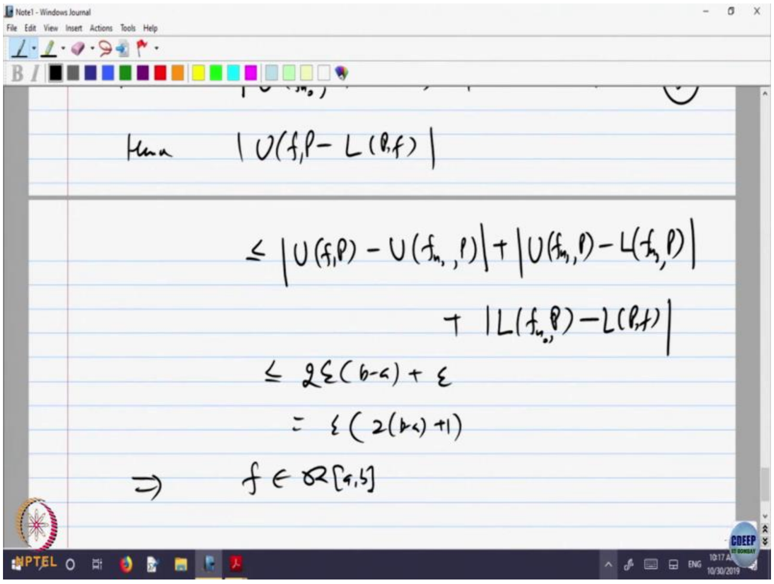

So, now I want to estimate this quantity. For, so add and subtract, so this is less than or equal to upper sum f, P is close to upper sum of fn naught P plus, so I should add upper sum of fn naught P minus lower sum of fn naught P plus, the last term now left would be lower sum of fn naught P minus lower sum of L P, add and subtract. This technique by now we are very familiar.

So, less than or equal to upper f and fn so this is this equation now. Call it 2, call it 3. Using 2 and 3, the first one is less than epsilon times b minus a. The last also is less than same. So, 2 times this, upper minus the lower is less than epsilon. So, that we have already seen, would where was that, upper minus the lower, here.

So, you can call that as 1 if you like, where fn was, fn naught was integrable. So, upper minus lower is small, that is what we started.

And 2 and 3 are giving corresponding things between fn and f, so, plus, so 1,2 and 3, using 1,2 and 3 I get this, no problem? So, if you like, this is epsilon times 2 of b minus a plus 1. Because epsilon I can always modify, go back and change wherever required, so implies f belongs to R ab.

So, the basic idea is because of uniform convergence fn is close to f for all x, that is important thing. So, you can take the minimum over that subinterval, as well take the maximum over subinterval and then multiply by the lengths and add up to get the corresponding.

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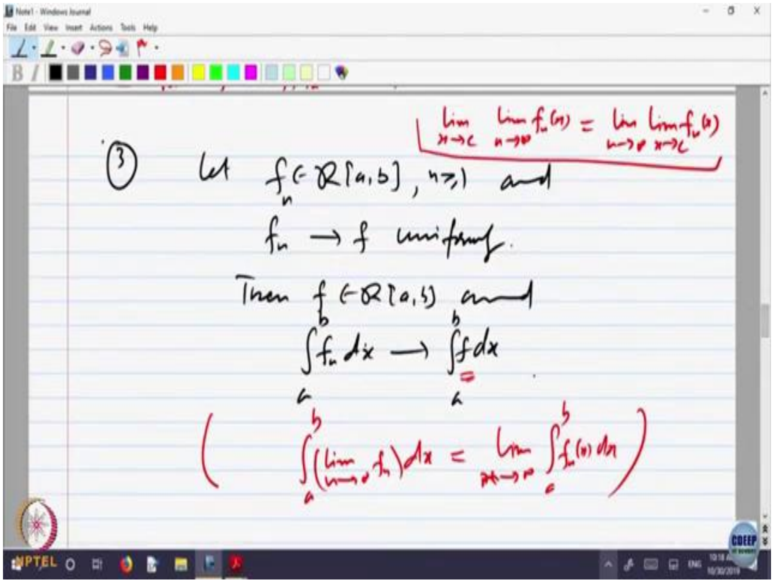

Student: (())(42:14) to see that if fn converges uniformly and f does not, f is Riemann integral then fn must be Riemann integral.

Professor: fn's are given to be Riemann integrable.

Student: Here fn's are given to be Riemann integrable, we prove that f is Riemann integrable.

Professor: But why in general, given that fn is bounded why it should be Riemann integrable?

Student: I am not saying that if it is bounded, it must be Riemann integrable. I am saying that…

Professor: What are you saying? So, here is the theorem. So what modification you are suggesting?

Student: What I am saying that, fn converges uniformly f are bounded, f are Riemann integrable. Small f's are Riemann integrable.

Professor: Small fs meaning what? fns or what?

Student: fns, only fns.

Professor: If the limit is Riemann integral why you should say all corresponding sequence is Riemann integrable? Why should it say that? All fn's can come to a singleton, all fn's may not be Riemann integrable, but they converge to a constant function. Then the limit is Riemann integrable. Why should the function be Riemann integrable, each fn? That is expecting too much, giving nothing and you are expecting too much.

So basic property we are saying is if fns are Riemann integrable, and fn's converge to f uniformly, then f also becomes Riemann integrable and integrals converge. We have not proved integrals converge yet. We have only proved f is Riemann integrable.

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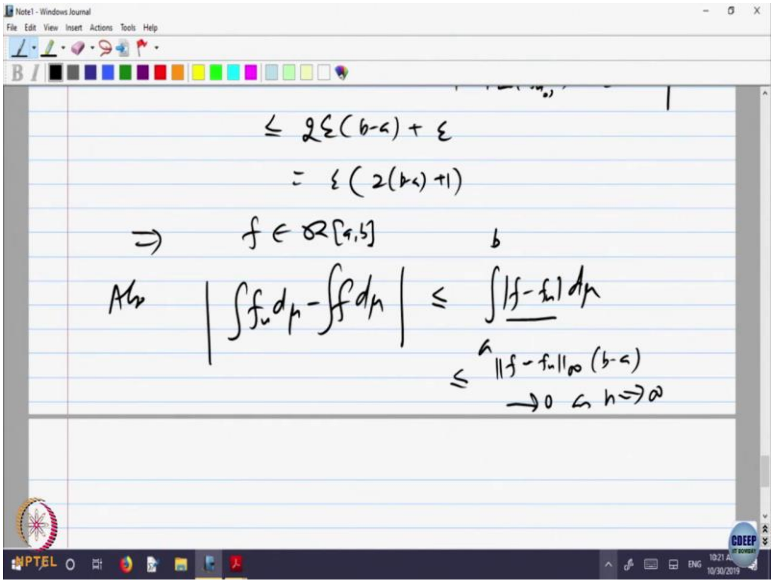

But proving integrability is, so also look at integral of fn d mu minus integral of f d mu. You want to show this converges, so you want to show that this quantity goes to 0, we want to show that this quantity goes to 0.

But we know something about, this is less, absolute value of, this is less than or equal to integral of f minus fn. Using the property at absolute value of the integral, Riemann integral is less than equal to integral of the absolute value and which is less than or equal to, now each function fn x is less than equal to norm of, into the length of the interval b minus a.

The right hand side, because the function f minus fn is bounded by the constant, norm of f minus fn. So, I can take it out, less than and this goes to 0, as n goes to infinity. Because of uniform convergence norm goes to 0. So, that is not much of a problem.

Once you know that limit is integrable, convergence of integrals is not a big issue. But the important thing is that you need uniform convergence, fn's converging uniformly to f to say that you can interchange limit and the integral sign.

This is the beginning of a, I have already mentioned I think that the Riemann integral is not very well behaved with respect to limiting operations, namely pointwise convergence need not imply integrability, limit of integrals is equal to limit of integrals.

So, that started the research for looking for a integral which has better properties compared to this. So, that is the another point for the beginning of what is called Lebesgue integration, whether this is much better behaved. You do not need uniform convergence, you need a slightly milder conditions to be true.

So this is, now the question comes, see what we are doing today is very important in mathematics to relate ideas. Continuity is an idea about how fn x, f x converges to something, f of x, as x goes to c. And if you take limiting operations, we know algebra of limits. If f plus g are continuous then f, f and g are continuous then f plus g is continuous, product is continuous.

So, it says if you go on taking limits, so, see this is a very general thing one consider. You are looking at the class of functions, real valued functions, say R to R. What are the properties you would like to analyze, you would like to analyze, you can add functions, so you like to know under addition what is preserved.

I can multiply, under multiplication what is preserved, I can scalar multiply, given a function f I can multiply by c times f, what properties are preserved under that? Because, on R these are the structures available, addition, scalar multiplication, product, you can take limits.

Under all these what are the properties which are preserved? So, what we have shown is when we did continuity, differentiability and so on, if f is continuous, g is continuous, f plus g is continuous. Differentiability we saw that, f plus g is differentiable if f and g are differentiable, product was differentiable.

If fn’s are differentiable, can you say the limit is differentiable, that we saw it is not necessarily true under pointwise. So that is why, there is a need to put some stronger conditions which allow us to pass over limit and that operation, interchange those 2 operations, uniform convergence is one.

Unfortunately, differentiability does not even, is not preserved under uniform convergence. One has to put slightly more stronger conditions. So, we will not prove that theorem. Let me just show you that theorem, so that you understand.

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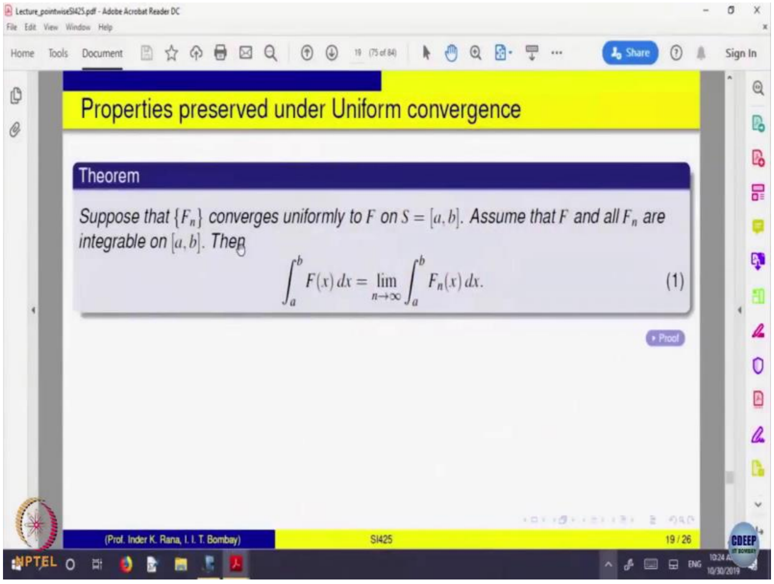

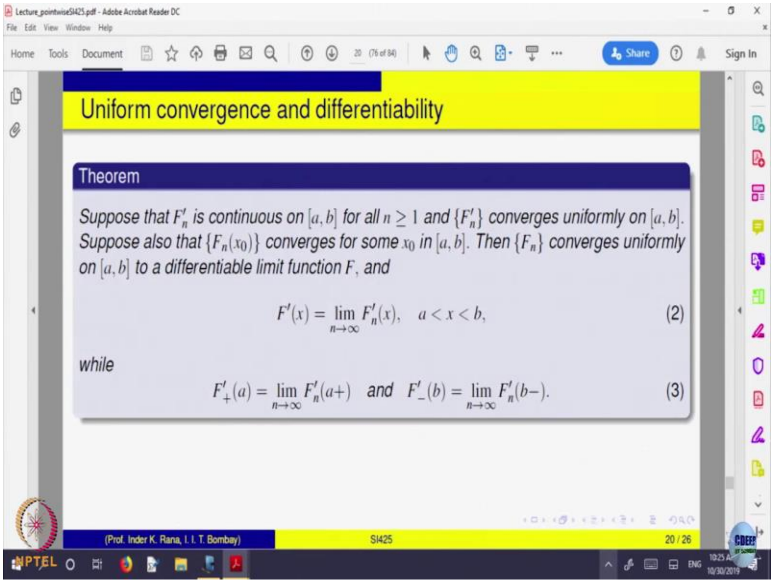

So, this is integrability fn converges to f uniformly, then f is integrable. And, so here is the, it says suppose not only fn is converging uniformly, fn dash each function is differentiable and the derivatives also converge uniformly, you need something more. Also suppose that, it converges at some point then fn converges uniformly to a differentiable function and derivative.

So, it is slightly more convoluted kind of a thing. You have to put conditions on the derivative itself to ensure that the limit function is differentiable. So, we will not go into that because it is not a very useful result, in many situations. It just, simply does not say fn's are differentiable, converging uniformly then limit is differentiable. So, we will not do that.

So, let me just summarize what we have done till now. We had looked at sequences of functions, and tried to analyze what are the limits of, under what is the meaning of pointwise convergence at every point it converges. And we observe that, at pointwise convergence is not very good behaved, they are not very well behaved.

It does not have any one of the properties that we can think of, continuity, differentiability and so on. So for that, one looks at what is called uniform convergence and it preserves continuity, it preserves integrability, it preserves boundedness but in some way it preserves differentiability, not exactly straightforward way.

So, next time what we want to do is, I will stop here today; there is a reason for that. But what we want to do is, start with sequences and see what is other use of sequences. For example, the algebraic operation of addition of numbers, given 2 numbers you can add them, a plus b.

Given 3 numbers a1, a2, a3 you can add a1 plus a2, any finite number of real numbers you can add them, because there is operation of addition, and inductively given any n, you can add them. Can you add infinite number of numbers? That is a question.

How does one find a way of adding, so what does it mean, given a sequence a1, a2, a3 and so on. How do I say something like a1 plus a2 plus a3 plus dot, dot, dot? It makes sense or not. In what way I can give a sense to that operation, infinite addition of numbers. And that, very natural way of doing that, that is how do you add? a1 you add a2, so a1 plus a2, you have got another one, add a3. So up to an.

So, you can go on doing it but what eventually you want is when you go on adding a1 plus, plus an, whether they become stabilized somewhere. So, that is limit of a1 plus a2 plus an, you will like to consider. So, that gives you a notion of what is called series of numbers. So, we will do it next time. So, we will stop here today.