Basic Real Analysis
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Lecture 63
Pointwise and Uniform Convergence - Part II

So, let us look at how we will be using this or how 1 can use it so let us look at some examples.

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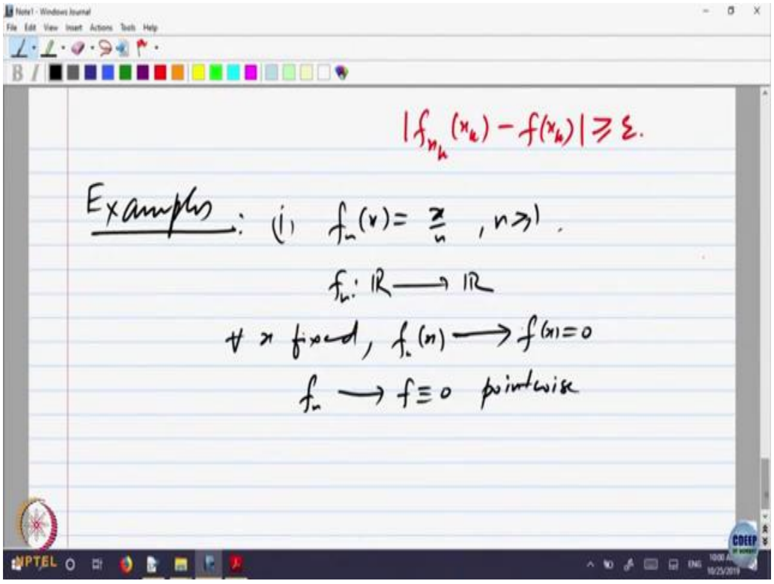

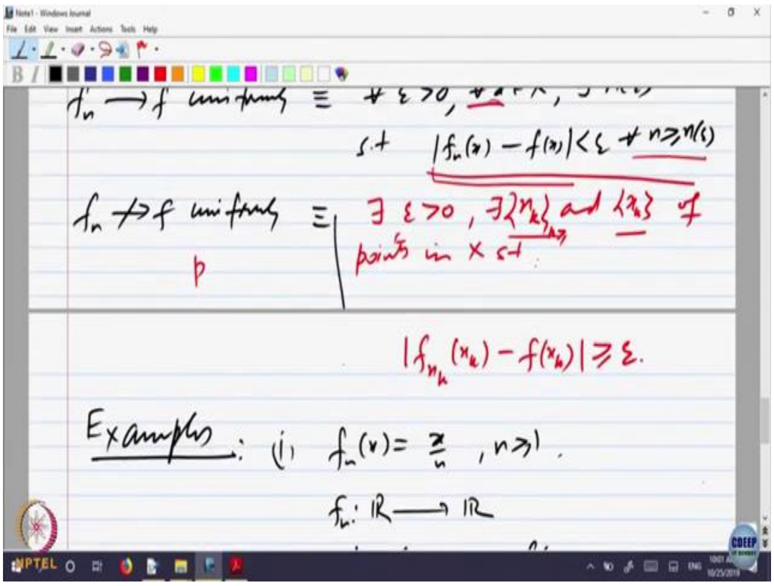

So, let us look at fn of x to be equal to x by n, n bigger than or equal to 1, so fn is a sequence of functions defined on X to R. Let us specialize this for the particular thing let us take X is equal to R so let us take so that we can choose special things. So, let us take R to R, so X is taken as the real line functions are defined on the real line hypothesis. So, for every x fixed fn of x is convergent, if f is fixed it is literally like 1 over n that converges to f of x which is equal to 0. So, fn converges to f which is identically 0 pointwise.

Let us consider, let us take n equal to, nk equal to k and x also equal to k, nk the numbers equal to k and the point x in the real line so I can take, so what is f of nk xk so that is equal to 1 for every k. That is x by n so it is equal to 1 for every n that does not converge to f of xk which is identically 0, f is the function which is identically 0, so f at each xk is 0. So, that means what implies f of nk xk does not converge to f of xk. So, what we have done? We have said for each there exist a sequence nk so I found nk, nk equal to k, xk equal to k. So, either this quantity is always 1 and this quantity we know it is 0 so what is the difference that is equal to 1. So, there is epsilon, epsilon equal to 1.

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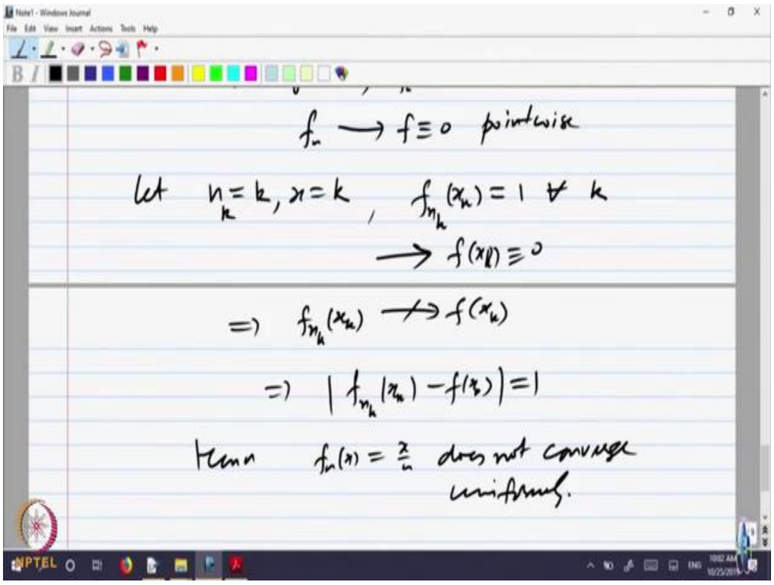

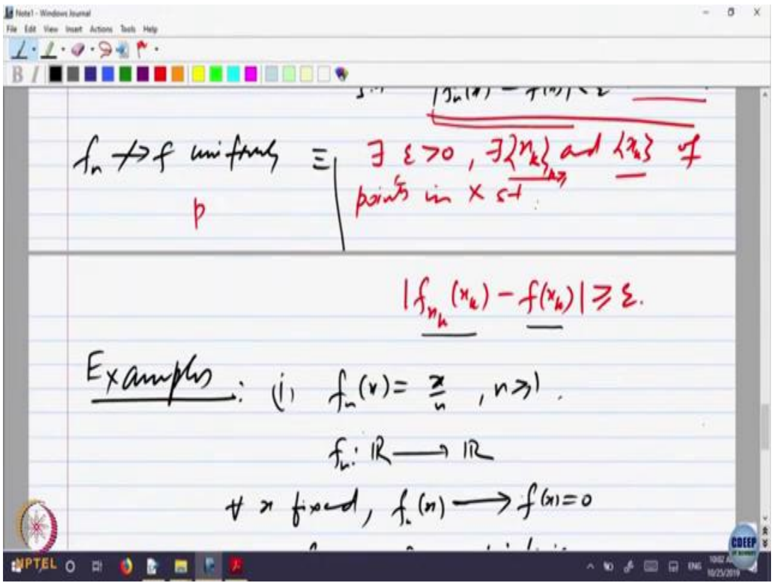

So, that implies mod of f of nk xk minus f of xk equal to 1, hence the sequence fn x which is x by n does not converge uniformly. So, criteria I gave for not uniform convergence that means we are able to find a sequence nk of natural numbers. And a sequence xk in the domain say that f of nk xk does not converge to f of xk or the difference always remains bigger, so that is what I have difference remains bigger than or equal to 1. So, this sequence so that means so we have applied this criteria, so this criteria, to the sequence so this sequence converges pointwise but not uniformly.

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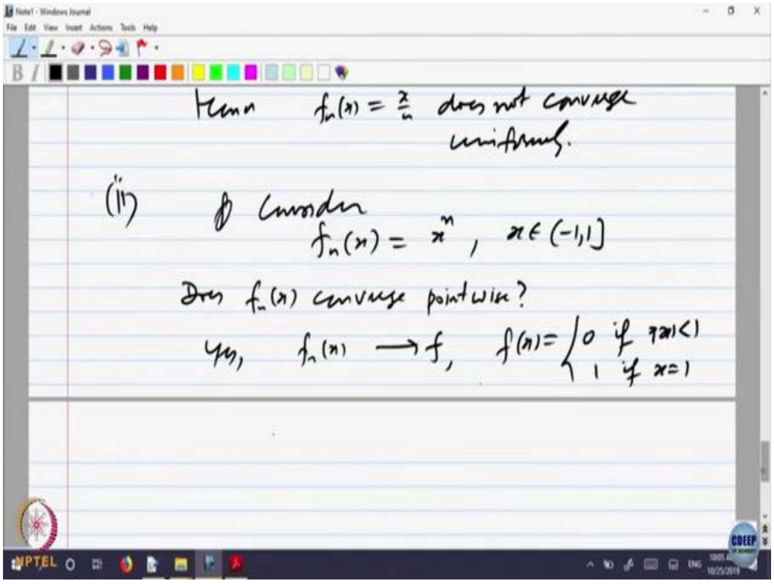

So, let us look at some more examples, let us look at consider fn of x to be equal to x to the power n for x belonging to minus 1 to 1. So, question is thus fn x converge thus fn x converge pointwise. Does fn x converge pointwise to anything? X is between minus 1 and 1. So, mod x is less than or equal to 1, if x is equal to 1 this is the constant function 1 so it converges to the value 1. If it is between minus 1 and 1 in the open interval, then this number x having mod strictly less than 1 raised to power n. So, goes on decreasing so converges to we have seen that converges to 0. So, fn converges yes fn x converges to f and what is f? f of x is equal to 0, if mod x is less than 1 is equal to 1 if x is equal to 1, so it converges pointwise.

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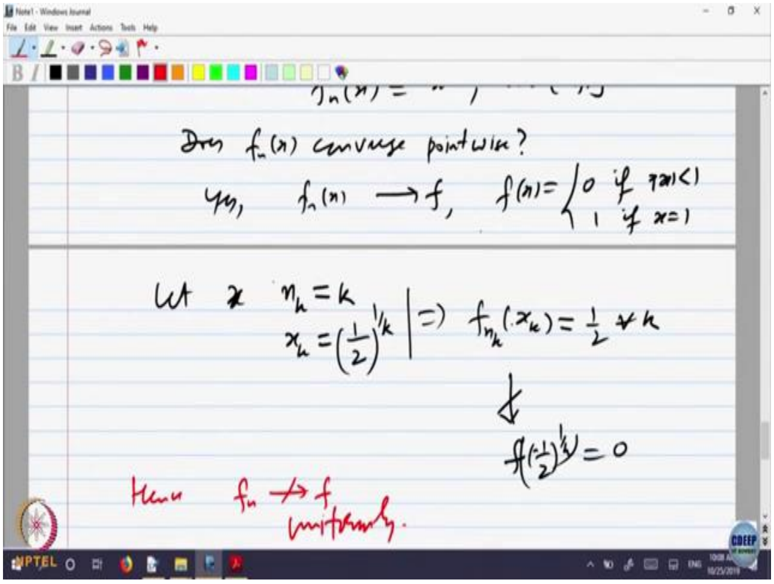

Now, x to the power n I can suitably choose x, suitably choose n so that this value becomes a, I can evaluate that easily. So, what do you think I should choose? So, let us choose let nk and xk I have to choose this 2 points so that, if I let us-let us choose the nk to be equal to k itself. So, it is x to the power k and x I can choose whatever I like. So, let us choose 1 over 2 to the power k, I have to choose a number between minus 1 and 1. Keep in mind in the domain is between minus 1 and 1 so let me choose this. So, let this be so this implies what is fnk xk sorry not x to the power k, sequence xk.

So, what is this value that does not help that is k by so let me choose slightly differently. So, this is minus, that does not a very good choice so let me choose it I want it to be constant. So, let me choose it to be, 2 to the power 1 by k will that help or 1 by 2 to the power 1 okay that was okay. I, this point goes out so I should not be choosing this point because this goes out. So, let us choose 1 over two raised to power 1 by k. That is point is still between minus 1 and 1.

Why I manipulating all this because then this comes out to be equal to 1 by 2 every k, then this value comes out to be 1 by 2 for every k. And this does not converge to f this value is not equal to 1, so f of 1 by 2 to the power 1 by k which is equal to 0 and f is 0. The pointwise limit is 0 everywhere except at the point 1, the value is 1 this is in between. So, implies so once again that implies so hence, f n does not converge to f uniformly. They are converging pointwise but they are not converging uniformly. So, the question comes this is one way of testing and something is not happening. Can we give a criteria for something when something happens? So we want to know can we give a necessary and sufficient condition saying that fn converges to f uniformly.

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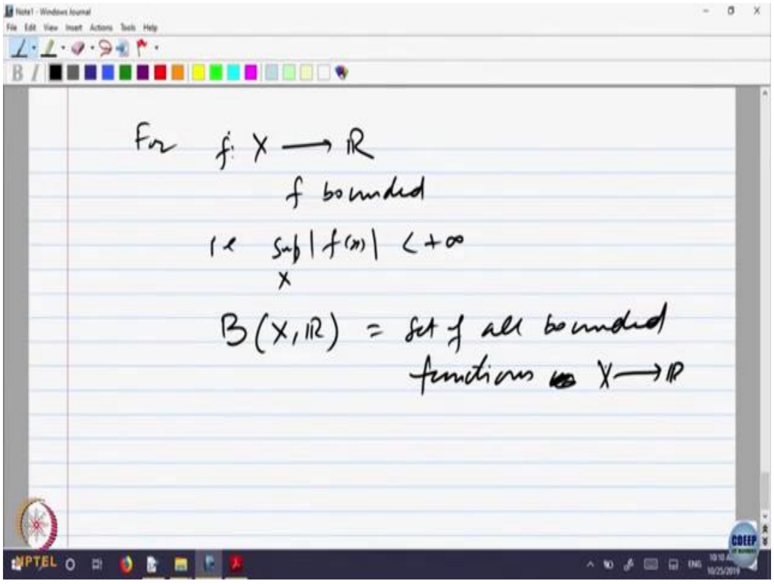

So, let us look at that, for functions fn belonging to, I have to now so okay so let me look at, so let us look at the space, so let us look at. Consider all functions f, X to R f bounded so we are going to look at bounded functions. So what does it mean? So, that is if I look at mod of fx and look at supremum over x that is finite, that is what a bounded function means. And you will call this as the bound for that function. I do not think I gave a name for this okay probably I give a (()) (11:33). So, let us write bounded X to R, so the space of all or the set of all, why space it is not a space, set of all bounded functions X to R.

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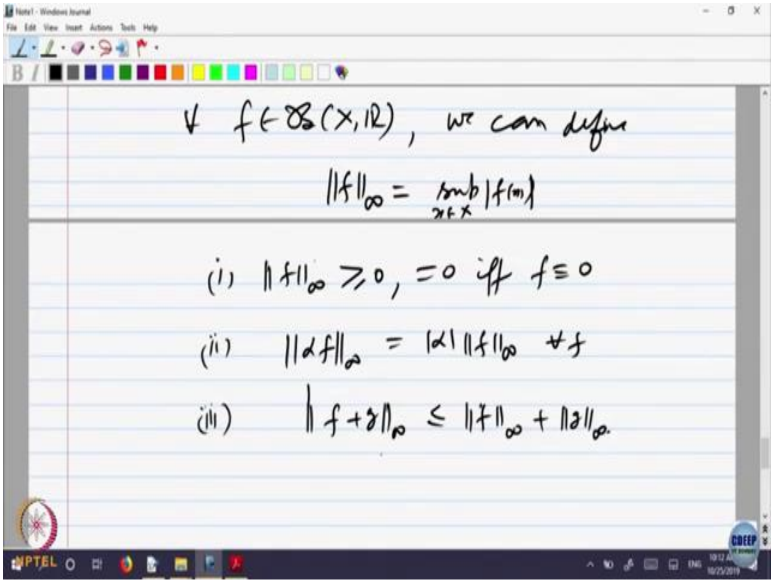

For, every f belonging to B X, R we can define we had already done it. The L infinity norm that is the supremum of mod fx, x belonging to X. This exist so call that as, so this obvious has those properties I am listing again is bigger than or equal to 0 equal to 0 if and only if f is identically 0. Because if a supremum is mod fx is 0 that means fx must be 0 so if and only if, second alpha f is equal to I am just repeating the infinity thing that we had done.

So, for every f because supremum model for, model for comes out and the third one that f plus g is less than plus mod g. That is also obvious triangle inequality because the supremum of mod f plus g will be less than or equal to mod f plus g is less than mod f plus mod g. So, supremum will be less than or equal to supremum of those things.

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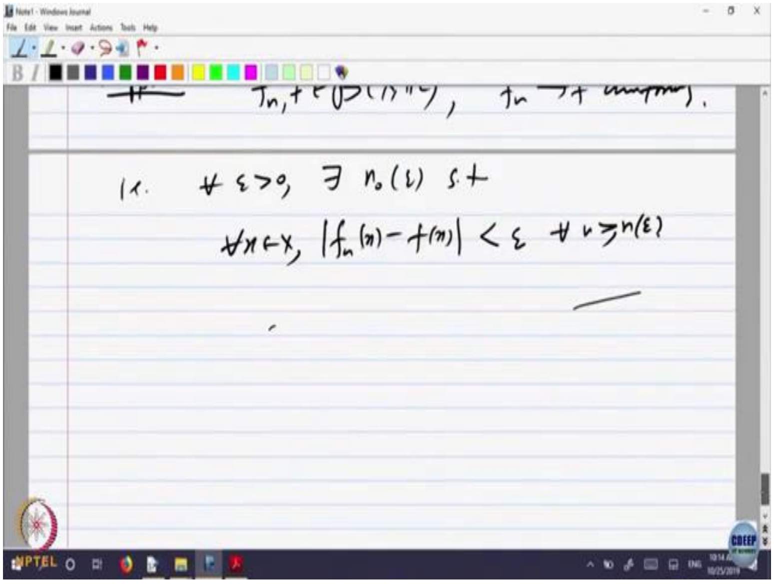

So, this gives a metric on B X, R which is a norm of f minus g. So, that is that is a metric d infinity f, g. So why all this being done, so, let us assume suppose fn f belong to B X, R I think I wrote script B X, R fn converges to f uniformly. So, assume these are can be bounded functions and they are converging uniformly. So, that will mean what? So that is for every epsilon bigger than 0 there is a n naught depends on epsilon such that, mod fn x minus f of x such that for every x belonging to X. This is less than, this is less than epsilon for every n bigger than n epsilon.

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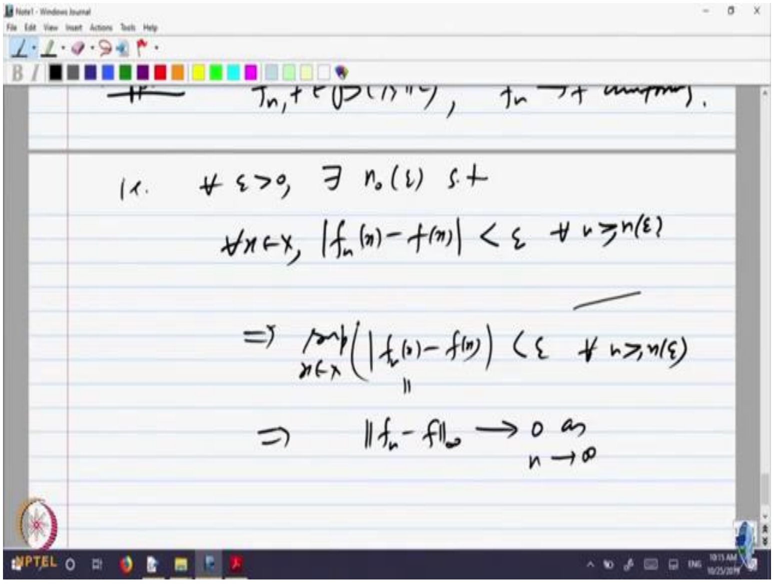

Now, this is happening for every x then I can take the supremum over x which exist. So, implies supremum x belonging to X of mod fn x minus f of x is also less than epsilon, for every n bigger than n epsilon. And because for every x something is happening so for the supremum that will also happen. So, what implies is norm of fn minus f goes to 0 as n goes to infinity. Saying that, what is this quantity? This is same as this supremum of mod fn minus f it is a norm. So, norm is less than epsilon after some says that means this goes to 0.

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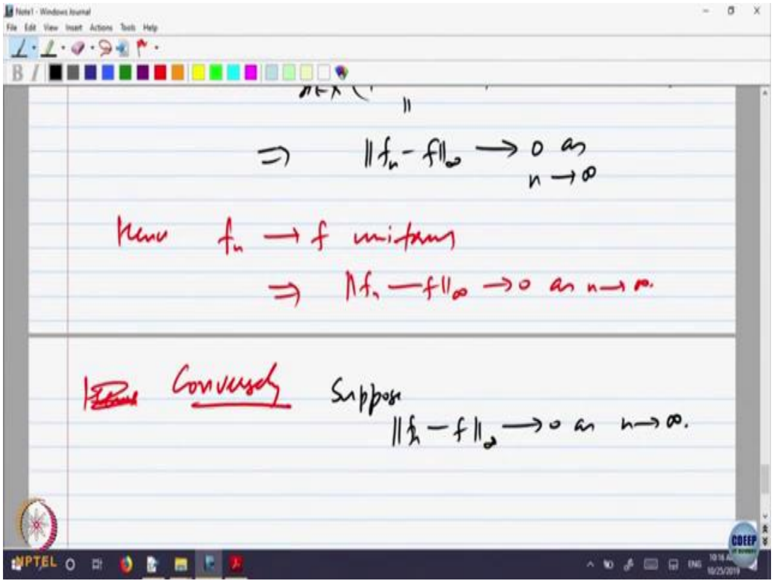

So, what we have shown is fn converges uniformly so hence so what we have shown so hence, fn converges to f uniformly implies norm of fn minus f all for bounded functions goes to 0 as n goes to infinity. So, let us look at can I say the converse also holds, so conversely.

So, what will suppose norm of fn minus f goes to 0 as n goes to infinity, suppose that happens. So, go back in the way we write epsilon delta if you like that does not matter actually.

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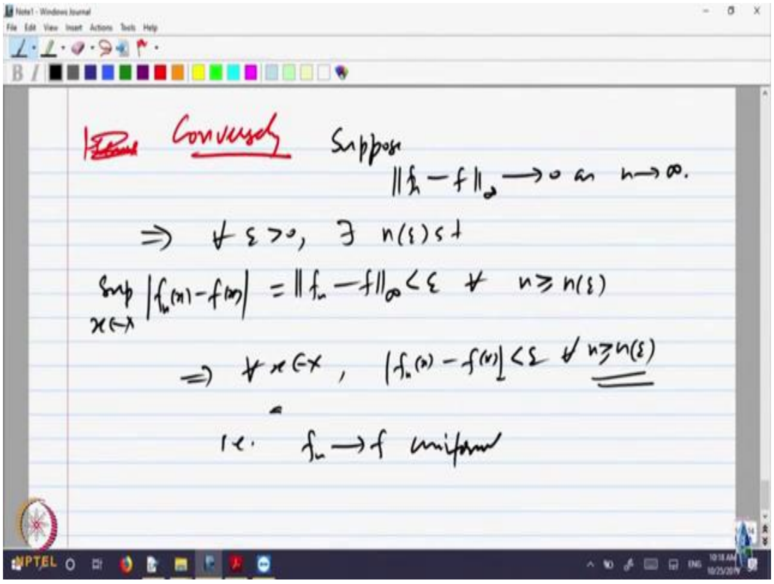

So, implies for every epsilon bigger than 0 there is a stage n epsilon such that, norm fn minus f is less than epsilon for every n bigger than n epsilon. That is a meaning of saying something goes to 0, but what is this quantity? This is nothing but supremum x belonging to X that obviously implies for every x belonging to X, if supremum is less than epsilon then for every term it should be also less than epsilon. So, implies mod of fn x minus f of x is less than epsilon for every n bigger than or equal to n epsilon. So, what is a meaning of that? That means a stage is not depending upon x at all. So, that means so that is fn converges to f uniformly.

Student: Sir it can be seen that supremum is less than epsilon, so every term should be less than epsilon.

Professor: Right.

Student: But sir before this converse part how did we say that if this thing is less than epsilon so supremum of this thing also less than epsilon?

Professor: Yeah, so basically what you are saying, if for every x something is small then the supremum over x also should be small. And conversely saying if supremum is small then every term must be small. So, that is the thing used if and only if that is all in both the things. We are not saying anything great, we are saying if supremum over somethings right is less than epsilon then each term must be less than epsilon. Conversely if each term is less than epsilon then the supremum must be less than epsilon. That is all nothing more we are not saying anything but what we are saying is, interpretation in terms of the metric.

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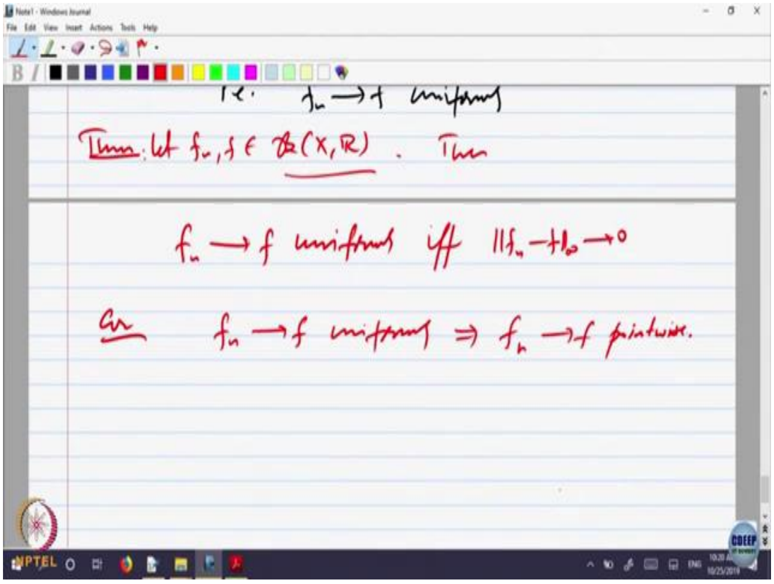

So, fn, so the theorem says so let us write the theorem, fn and f belong to bounded functions X to R. So, let fn converges to f uniformly if and only if norm of fn minus f goes to 0. So, essentially we are looking at sequences in the metric space B X, R under the L infinity norm, we are looking at sequences in, what is a meaning of convergence of sequence in the matrix space B X, R and that is given the name as uniform convergence. Which is so uniformly and obviously corollary of this fn converges to f uniformly implies fn converges to f pointwise. Obvious because saying fn converges to f uniformly means norm of fn minus f goes to 0 and saying pointwise as fn x minus f of x absolute value that goes to 0. So, that is 1 of the terms where supremum is being taken.

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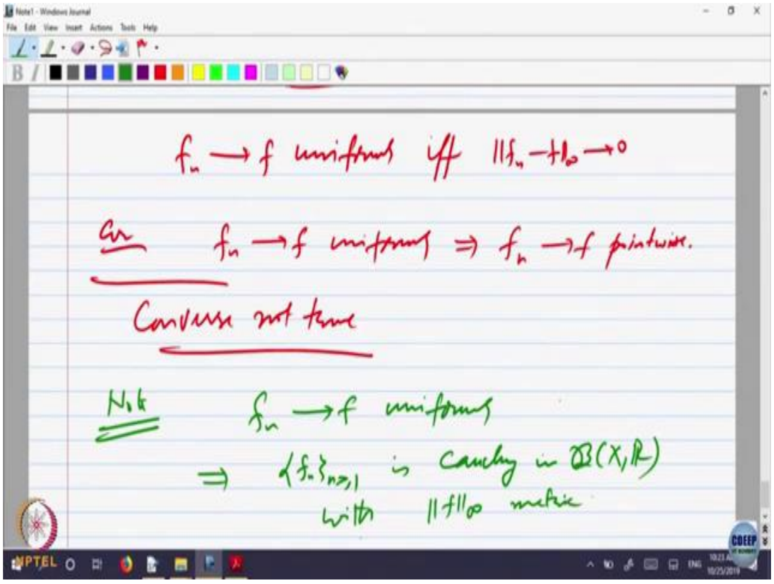

So, and converse we have already seen conversely not hold, there are pointwise not converse not true. We already seen many examples of sequences which are converging pointwise but not uniformly. Here is another way of interpreting this theorem. So, we are saying fn converges to f uniformly means in the metric space B X, R under the metric under the L infinity metric, the sequence fn converges to f.

And we have already seen that if a sequence converges in a metric space, then it is always Cauchy. Every convergent sequence is Cauchy; every convergent sequence is Cauchy. Can we say that in this metric space Cauchy also implies convergent? In the metric B X, R we want to know whether Cauchy is equivalent to saying, the sequence being Cauchy is equivalent to saying it is being convergent. So, let us write that so what we start with note, fn converges to f uniformly implies the sequence fn is Cauchy in B X, R with, did we give L infinity metric, okay, with metric.

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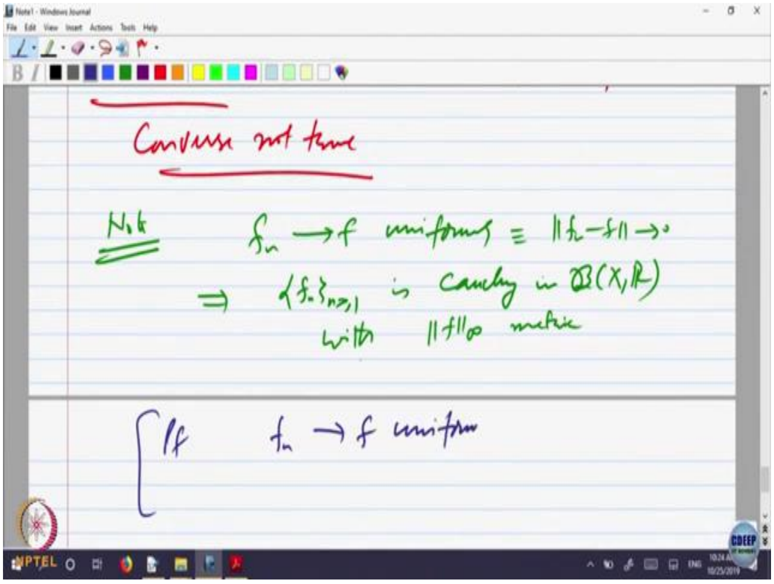

Because this is equivalent to saying that norm of fn minus f goes to 0. So, convergence implies Cauchyness in any metric space. What is convergence? a n converges to a, if an is converging to a, a n must come closer to a after some stage. So, if I take any two terms after that stage they should be close to each other anyway. We have proved that, every convergent sequence is Cauchy in the real line every Cauchy was also convergent. So, we are using the fact here that every convergent sequence in a metric space is also Cauchy. If you like you can write down the proof because if it is, let me write the proof here once again so that you feel little bit so implies Cauchy, so here is the proof.

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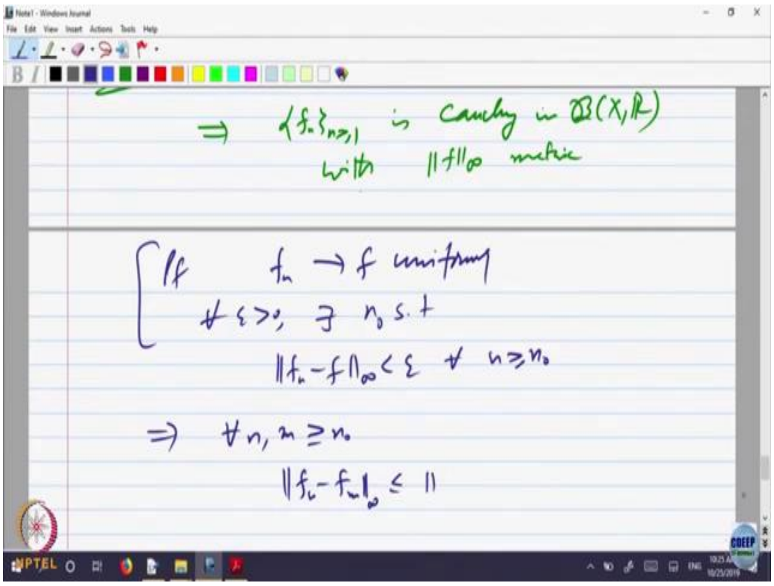

So, let us say fn converges to f uniformly, so that is same as saying for every epsilon bigger than 0 there is a stage n naught, such that norm of fn minus f is less than epsilon for every n bigger than n naught. So, that implies for every n and m bigger than or equal to n naught. Let us look at norm of fn minus fm for Cauchyness we have to say that two are closed. But this is less than or equal to norm of by triangle inequality prove of the norm.

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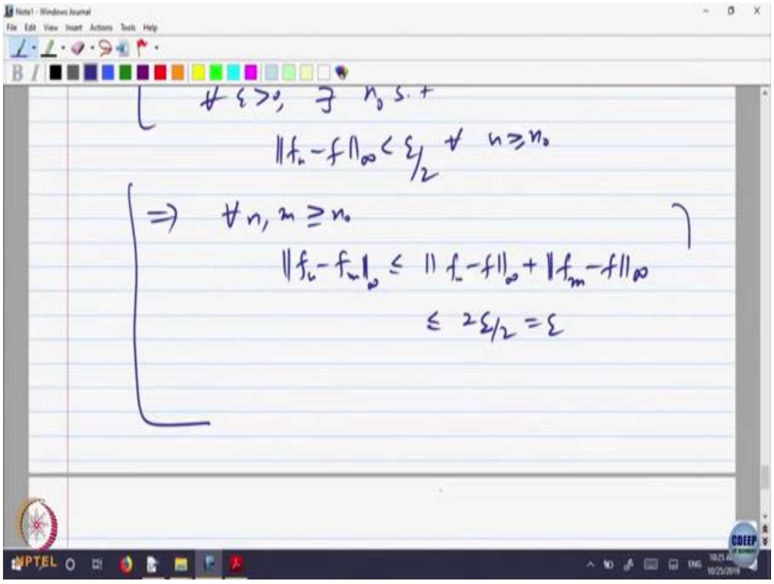

So, this is less than fn minus f plus fm minus f so that is less than 2 epsilon so if we want to be (()) (26:50) you can make it epsilon by 2 and is equal to epsilon. So, this is a proof every convergent sequence is Cauchy I am repeating the proof that is all nothing more than that.

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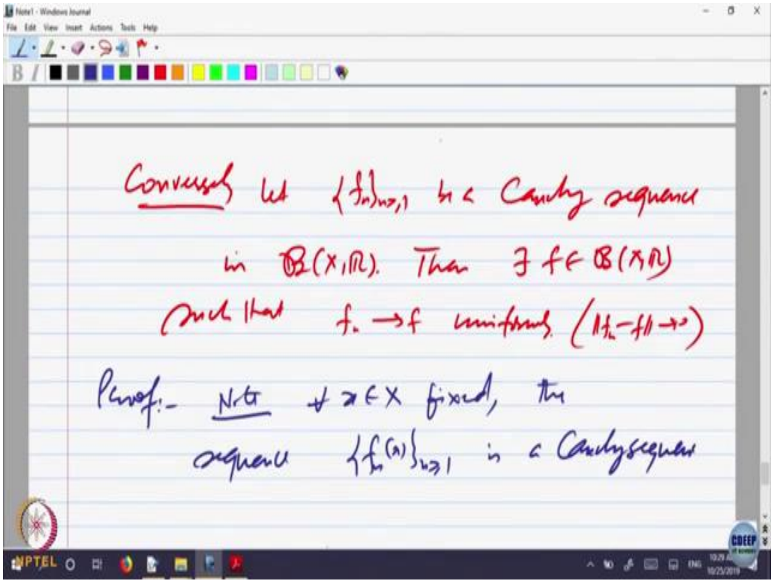

Let us prove conversely every so converse is also true so conversely, let fn be a Cauchy sequence in B X, R. We want to say it is convergent then there exist a function f belonging to B X, R such that, fn converges to f uniformly. That is same as norm of fn minus f converges to 0 so every Cauchy sequence in B X, R is convergent, so let us prove that. If I want to prove every Cauchy sequence is convergent uniformly, if it is going to be uniform convergence I know uniform convergence implies pointwise. So, first of all I should be able to say that, this there is a f such that fn converges to a pointwise. See here the problem is given something is Cauchy I do not know what is f.

What is going to be the limit, I have to locate a function and make a guess, make a conjecture that also is the limit in the L infinity norm. So, how do I get hold of that? So clue is fn converges to f uniformly. If I am able to find such an f then uniform convergence implies pointwise convergence. So, fn should converge to a pointwise also whatever that f maybe. So, that gives me the clue that, I should try to show that fn is pointwise Cauchy. Once it is pointwise Cauchy by the property of real line being complete it will converge somewhere. And that I will call as the function f and then prove fn converges to f uniformly.

So, to make a guess we will look at the known properties. So, the first thing is note, for every x belonging to X fixed, the sequence fn of x is a Cauchy sequence that is a Cauchy 0 for X fixed look at the values.

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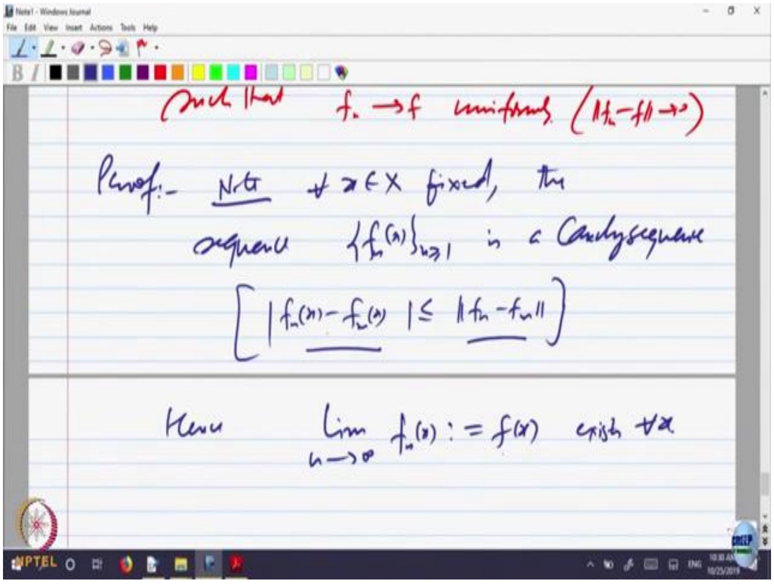

Why obviously because if you like fn of x minus fm of x is less than or equal to norm of fn minus fm. For, every x that is true because right hand side is a supremum over all x, left hand side is some x is fixed. And if this is going to 0 then this is going to 0, so that prove that because fn is given to be Cauchy in L infinity norm that implies pointwise Cauchy and hence limit n going to infinity fn x equal to fx so I define exist for every x. Rfn x for every x fixed is a Cauchy sequence so it must converge that limit I call it as f of x. So, for every x that is convergent so it has a limit, so limit is given a name it depends on x, limit will depend on x. So, it is a function of x so let us call it as f of x for exist for every x.

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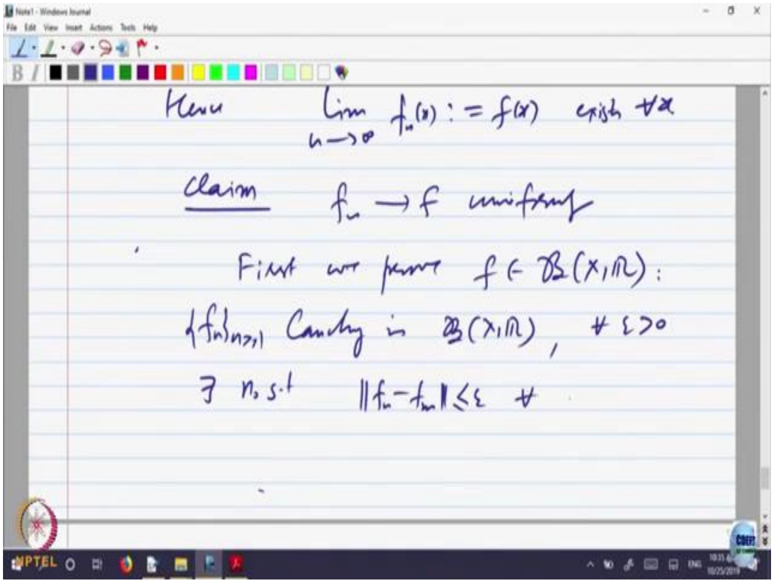

So, now I claim fn converges to this f uniformly this converges to f uniformly. If I want to prove fn converges to f uniformly all the fn's are given to be in the space B X, R we do not know where is f. Whether f is bounded or not. So, let us look at note we prove f belongs to B X, R that it is bounded so to do that I have to estimate mod of f of x. I know that fn converges to f of x, so for that let us look at, fn Cauchy in B X, R. We already seen for every epsilon bigger than 0 there is a stage n naught, such that norm of fn minus fm is less than epsilon for every n and m bigger than n naught.

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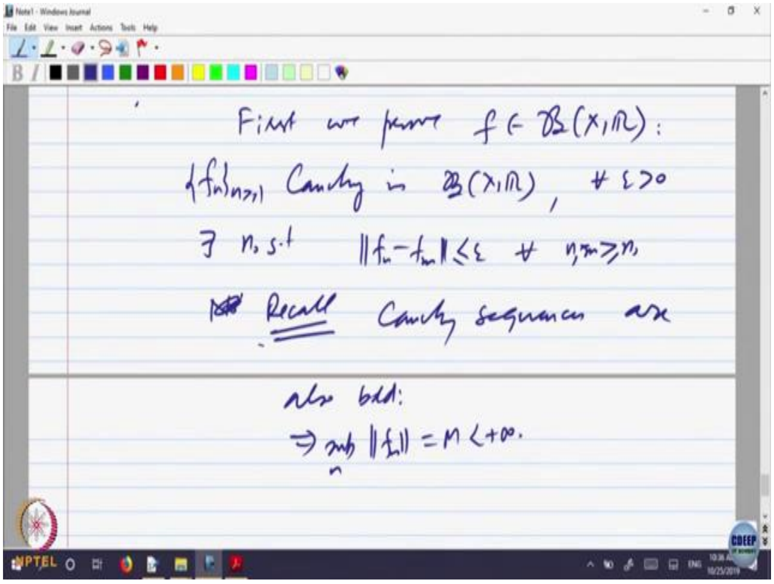

So, this is another thing I should recall we also proved a theorem for sequences, that if sequence of real numbers if it is Cauchy it must be bounded. So, recall Cauchy sequences are also bounded. So, implies fn is Cauchy so it must be bounded implies mod of fn supremum over n equal to some number say M is finite. What is a metric? What is a metric? L infinity metric so it is the Cauchy sequence so it must be bounded so this is bounded and now that is okay.

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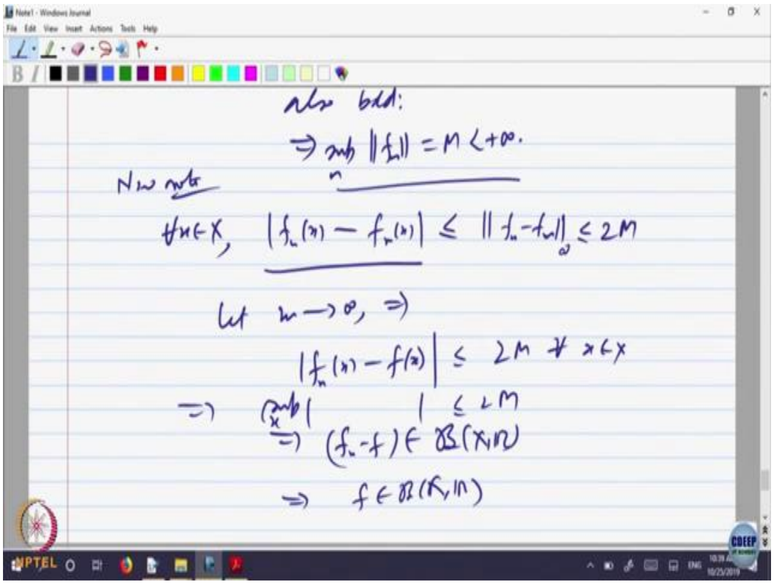

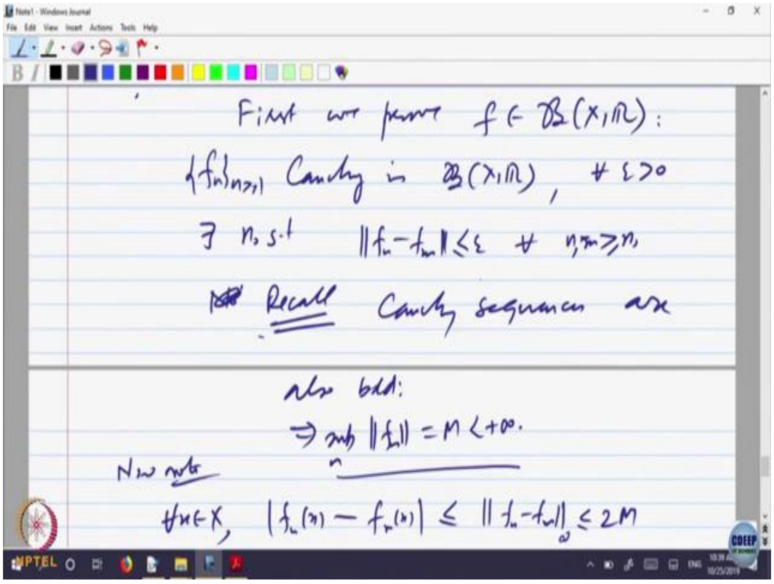

Now, I can prove so now mod of fn x note minus fm of x is less than or equal to norm of fn minus fm that is okay. Note for every x belonging to X this is true and this I can make it less than or equal to if it is M it is 2 times M. If this is true supremum over n so this is less than mod f norm of fn minus plus norm of fm so 2 times. So, this is true for every n and m bigger than something. So, let m go to infinity then you get implying fn of x minus f because it is converging pointwise fm x converges to f of x pointwise. So, this is less than or equal to 2M for every x belonging to X.

And that implies that fn minus f belongs to B X, R you can take the supremum so implies supremum over x of this quantity is less than 2M. So, that means fn minus f belongs to X and that implies f also belongs to B X, R. But, that is good enough we can conclude x also belongs, so it has to belong actually because fn is Cauchy and we are trying to prove it is convergent. So, is that okay for if you because I was trying to work it out? So, basically this being a Cauchy sequence it is bounded so that means all the fn’s norm of fn’s must be less than some number, so that is this quantity.

And that using the fact that fn x minus fm of x is less than norm because that is a supremum that is remains bounded. So, for every x this is bounded so you can let m go to infinity so that means, fn x this goes to f of x pointwise convergence is already there. So, this is less than or equal to 2M and this is happening for every x so I can take the supremum now. So, supremum over x of this quantity means this is finite less than 2M so supremum of this. And that means fn minus f belongs to B X, R and is a vector space, B X, R is a vector space so f also belongs to that.

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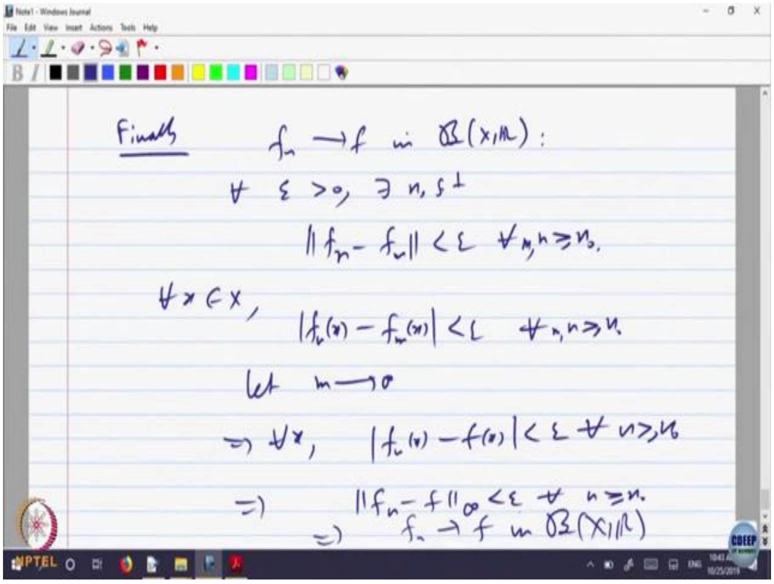

So, finally so fn converges to f in B X, R actually that is already there, so let us so proof of that. For every epsilon bigger than 0 there is a stage and note such that, fn x okay the same idea repeated again. Norm of fn minus fm less than epsilon for every n bigger than n naught. So, basically what we want to do is we want to show it is in this so let us write, for every f belonging to X mod fn x minus fm x is less than epsilon for every n bigger than n naught. And now let m go to infinity for every n and m bigger than n and m Cauchyness.

Let m go to infinity implying for every x mod fn x minus f of x is less than epsilon for every n bigger than n naught. And this happening for every x implies norm of fn minus f is less than epsilon for every n bigger than n naught. So, basically in Cauchyness n and m so let 1 of that things go to infinity. So that f comes into the picture in at every stage you are doing that, so implies fn converges to f in B X, R. So, what we are saying is if you are looking at the space of bounded functions proving something converges uniformly is equivalent to proving, that it is Cauchy in the supremum norm.