Basic Real Analysis
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Lecture 62
Pointwise and Uniform Convergence Part I

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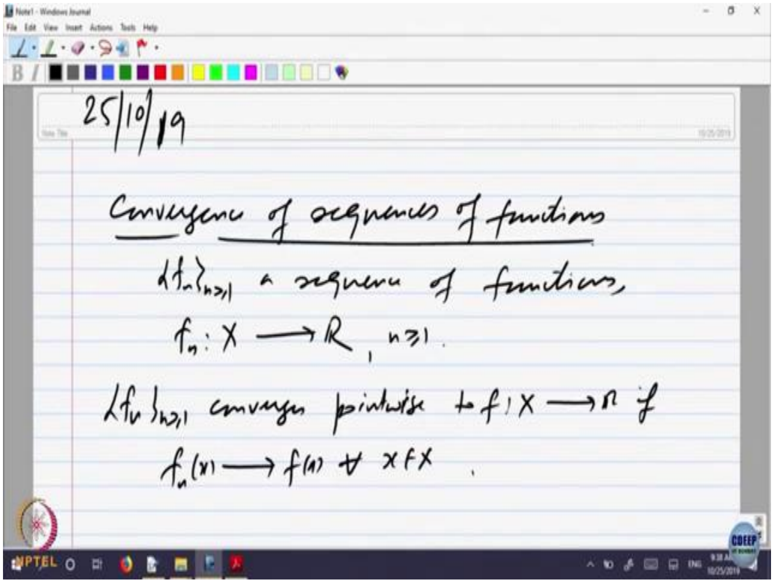

Alright, so let us start looking at Pointwise convergence, Convergence of sequences of functions. So, let us take a sequence fn, a sequence of functions. Each fn is defined on X to R, so we say that fn converges pointwise to a function f X to R, if fn x converges to f of x for every x belonging X. So, at every point you look at the value of the function fn that gives the sequence of real numbers and that converges to f of x.

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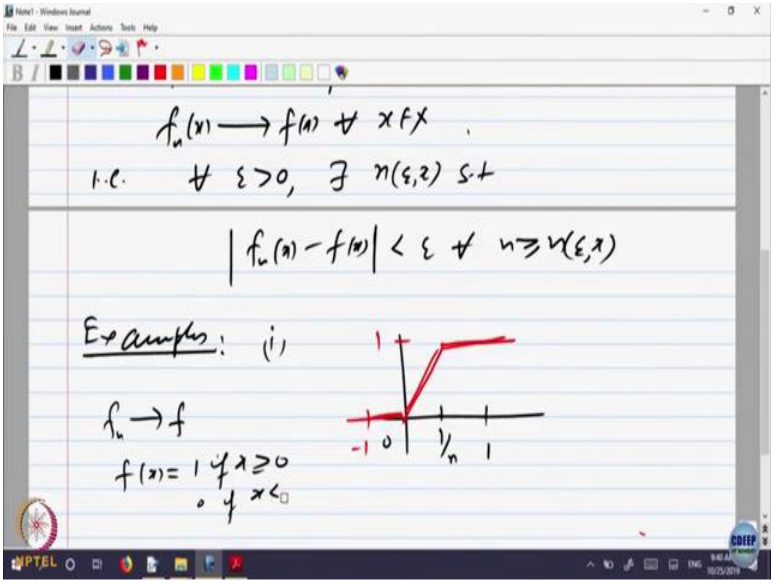

So, that is pointwise so that is same as saying in terms of definition of convergence of sequences, that is for every epsilon bigger than 0 there exist some stage n which will depend on epsilon as well as on x, such that fn x minus f of x is less than epsilon for every n bigger than this number n. So, this stage may depend upon epsilon of course, and it may depend on the point. So, at different points the stage may be a different. So, we had looked at some examples last time.

So we looked at one example was I gave you the graph of the function, 0 to 1 so here is 1 by n and you look at the graph to be say, you can start at 0 itself does not matter where you start. So you can go up to, so this is a point 1, if you like you can go to minus 1 does not matter. So, this is the graph of the function. So, if that is your fn, then this fn converges to f and what is that f, f of x is equal to 1 if x is bigger than or equal to so if you are including 1 over n then bigger than or equal to 0 and is equal to 0 if x is less than 0.

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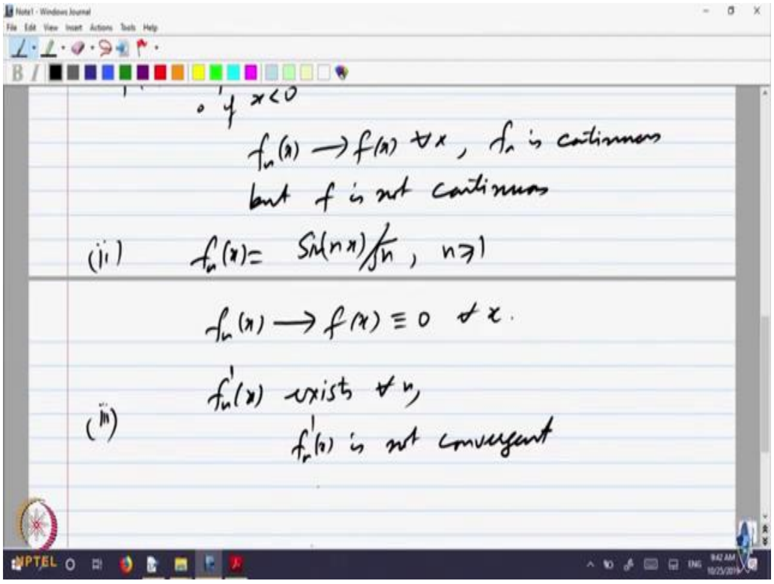

So, that meant that so fn converges to fx for every x. Each fn is continuous but f is not continuous. We look at some other examples also, so second example we looked that was I think fn x equal to Sin of nx divided by n, n bigger than or equal to 1. Then fn x converges to f of x which is equal to 0 identically 0 for every x. So, converges pointwise so the function because Sin is always bounded. And third but this example, this was an example particularly it is saying that, if I look at fn dash of x exist for every n. But, fn dash of x is not convergent. I think I should need to change that for the, if we want not convergent so let us put it square root of n here. Then the derivative will be square root of n into Cos nx which will be not be convergent.

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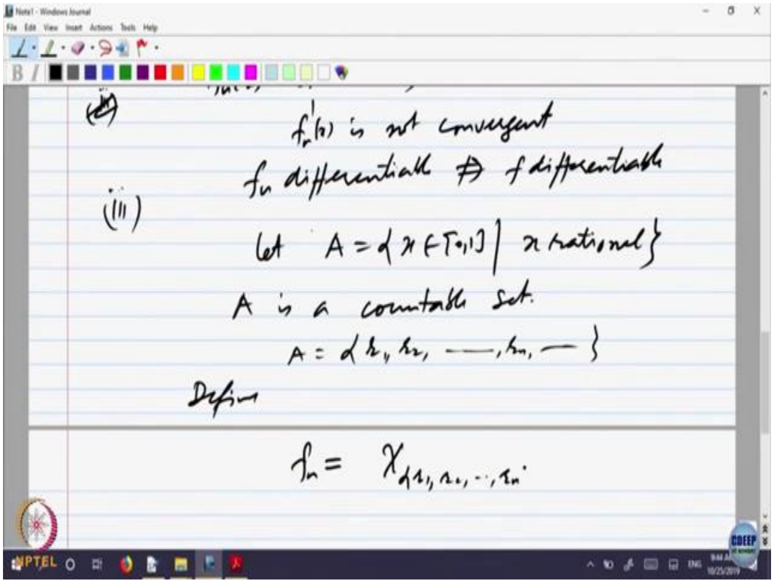

I am just repeating the examples that we have done last time. So, that essentially says that fn differentiable need not imply f differentiable. I think so let me give you one more example of, let us look at I gave one example last time let me give another one here. Let us look at rationales between 0 and 1, let what shall I write, let A be the set of x belonging to 0, 1 x rational. So, look at all the rational points between 0 and 1, A is a countable set, A is a countable set. So, let us write A to be equal to sum r1, r2, rn and so on. So, give an enumeration of the rationales, it is a countable set so call it r1, r2, rn we are not saying increasing or decreasing or anything, just enumeration of rationales.

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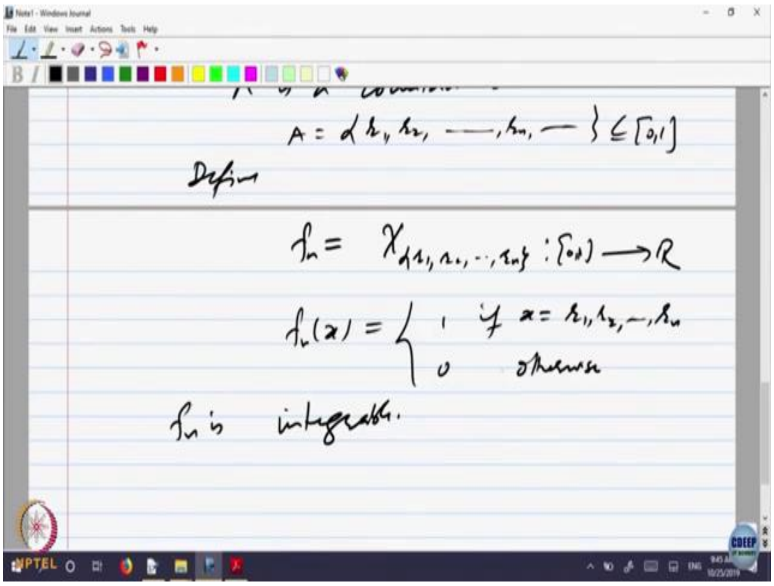

So define fn to be equal to the indicator function of the set r1, r2, rn. So, what does that mean? What is the meaning of indicator function? If you recall that meant that fn of x is equal to 1 if x is equal to r1, r2, r up to rn and is 0s otherwise. Of course x is between so we are looking at this in the interval as a function 0, 1 to R. So, look at this is a subset of 0, 1 so given a subset A which is a countable set rationales in 0, 1 define fn to be the indicator function of the rationales r1 up to rn in that enumeration and then each fn is integrable. Because each fn is a function which has only discontinuity at the points r1, r2, rn everywhere else is a constant function. 0 only at this point it's the value is one so it has only jump discontinuities at this finite number of points. So, it is a finite number of discontinuities so f is integrable.

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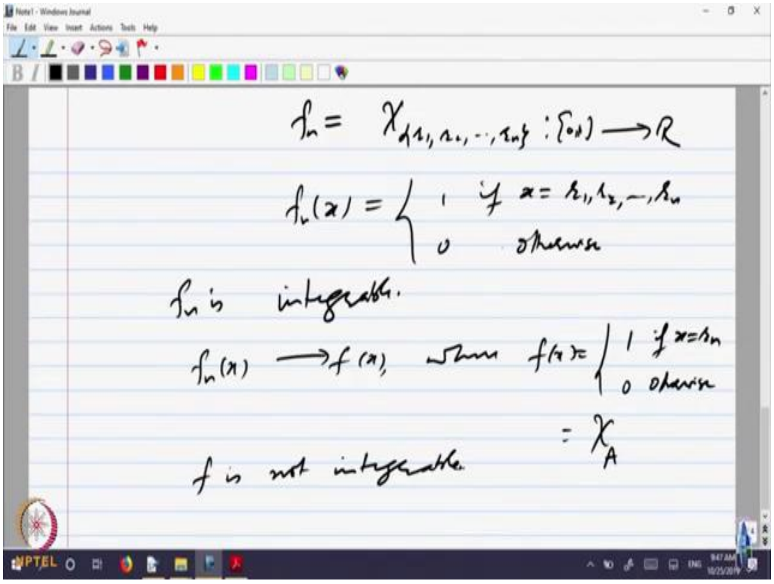

Where does fn x converge to, what is f of x it is nothing but the it will be 1 if x is any one of the rationales r1, r2, rn otherwise it will be 0. So, it is the indicator function of, so let me write it is 0, it is 1 if x is equal to rn and 0 otherwise. So, it is the indicator function of the set A. A is the set of all rationales in 0 and 1. So, what is the indicator function of rational mean? It is 1 at all rationales in 0, 1 and 0 at all irrational points. It is discontinuous everywhere and if you look at the upper sums or lower sums with-respect-to any partition, the upper sum will be equal to 1, the lower sum will be equal to 0.

So, this function is not f is, each fn is integrable f is not integrable. So, this very simple way of looking at convergence of sequences namely at every point, look at the sequence fn x and look at the limit if it exist that function. So, then you say fn converges pointwise. So what these examples illustrate that pointwise convergence does not preserve continuity, does not preserve differentiability and does not preserve integrability. So, is a reasonably bad way of looking at convergence of sequences.

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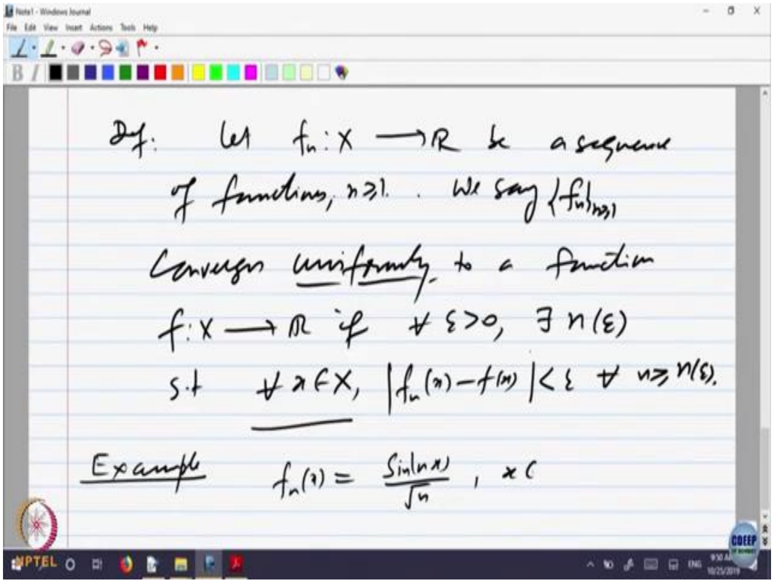

So, we would like to define a notion of convergence which is reasonably well behaved so that is called uniform convergence. So, let fn X to R be let fn X to R be a sequence of functions, I should say n bigger than or equal to 1. We say fn the sequence fn converges uniformly to a function f on X to R, if we want something stronger than pointwise convergence. So, it says if for every epsilon bigger than 0, there exist some stage n which depends only on epsilon such that for every x belonging to x, fn x minus f of x is less than epsilon for every n bigger than n epsilon.

So, what we are saying is convergence of fn x at any point brings it closer to f of x and that closeness is not affected by the point x. So, same stage works for all the points, so for every x so for every x this is true. Let us look at some examples, I click that we have one example ready here namely, that fn x which was defined as Sin nx divided by square root of n. So, x belonging to R.

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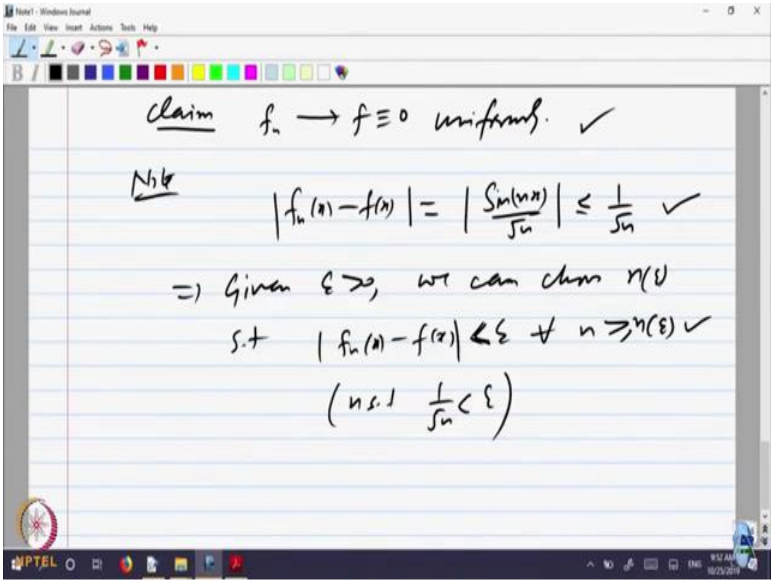

So, we want claim that fn converges to f which is identically 0 uniformly so that only we have note that, if I take fn x minus f of x. So, what is that? F is identically 0, so it is absolute value of Sin nx divided by square root of n which is less than or equal to 1 over square root of n. So, irrespective where is a point x, the distance between fn x and f of x is always less than 1 over square root n. So, that implies given we can choose say stage n epsilon such that fn x minus f of x is less than or equal to or is less than epsilon for every n bigger than n epsilon.

Because we can choose, so given epsilon n is such 1 over square root n is less than epsilon that is all we have to do. So, choose n larger than so that 1 over square root of n is less than epsilon. So, then this will imply that this is true so fn converges to f uniformly. There are many examples of functions which converge uniformly. Let us look at, will give more examples later on.

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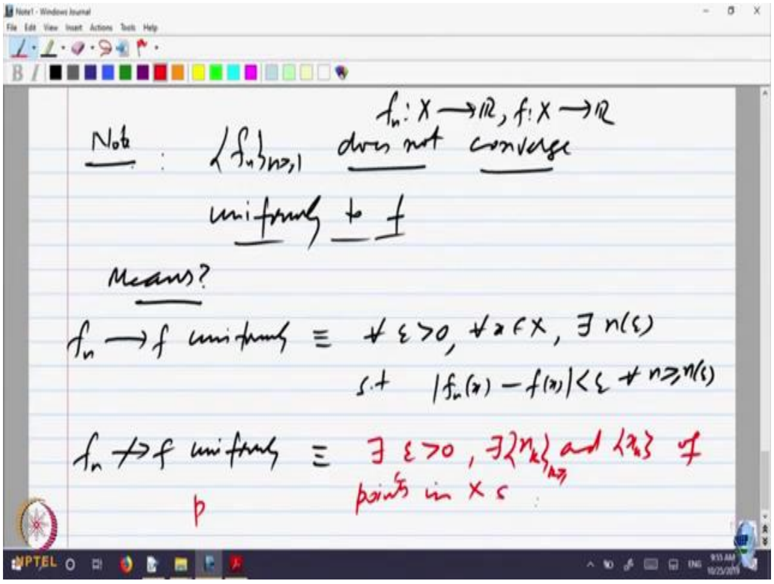

Let us just note one thing so note, what is meaning of saying fn does not converge, fn, the sequence fn does not converge uniformly to f, all functions are x to R. So, fn is X to R and f is from X to R saying that fn does not converge uniformly to f means what? So, once again I am trying to bring your attention to the point that, if you want to understand something is true, you should also understand when something is not true equally important to understand both ways.

So, let us write once again fn converges to f uniformly meant was equivalent to saying for every epsilon bigger than 0, for every x belonging to X, there exist a stage n epsilon such that fn x minus f of x is less than epsilon for every n bigger than n naught or than n epsilon. So, what is fn not converging to f uniformly? So that is equivalent to saying so for every, we should change it to, so there exist epsilon bigger than 0 so that for every x the statement is not true. And what is statement that is not true? There is a stage after which something is small, that should not happen that means after every stage I am able to find a point where this thing goes back.

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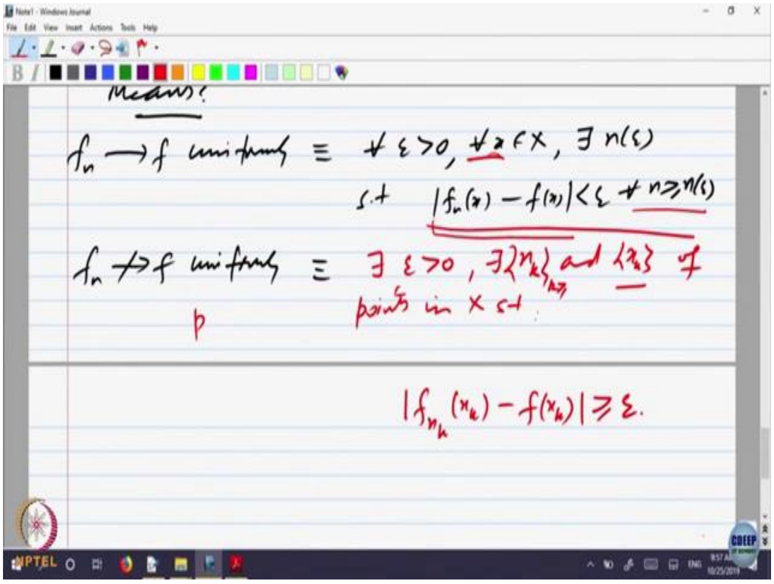

So, that means there exist a sequence of point nk and there is a sequence nk of a natural numbers and a sequence xk of points in X such that, mod of fn k places where the things are going bad, so that is same as nk xk minus f of xk is bigger than or equal to epsilon, is it okay? For every x the same stage works so it does not work for x the same stage does not work. That means there are points xk where this thing does not work. For every k there is a point where this does not work. That means there is a stage nk say that f of nk minus f of xk is not less than epsilon. It is bigger than or equal to epsilon.

Student: Sir… (()) (inaudible)

Professor: For every x it does not happen so there is a sequence xk for which things go bad. And what goes bad? Goes bad this is going bad right that means for every n, I am able to find some stage after which the things go bad. So, that stage is for every k nk so f of nk xk minus f of xk is not less than epsilon so, two things are going bad. Something was happening for every epsilon, for every epsilon that is taken care of for every x and for every here. When things go bad that means there should be points where things are going bad, so that a point xk. And for xk what is going bad? fn of xk minus f of xk is not less than epsilon for what n.

At least, every given any stage I can find something after which. So, that is same as a finding a subsequence xn k and k. So, this is what is it means saying that, the sequence does not converge uniformly. So, this gives us a way of testing many something not converging uniformly. For example, I will take ordinary sequences of numbers saying that a sequence a n converges to a that means what, a n comes closer to a after some stage. If a n, is not converging to a that means what?

That means there is a epsilon such that whatever stage you give me, there is a something stage after which, so there is a subsequence so a and k. So, sequence not converging to a means there is at least one subsequence it is not converge. But, here this was happening for every point also so there is a sequence of points where things are also going bad.