Basic Real Analysis
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Lecture 61
Lp Spaces - Part III

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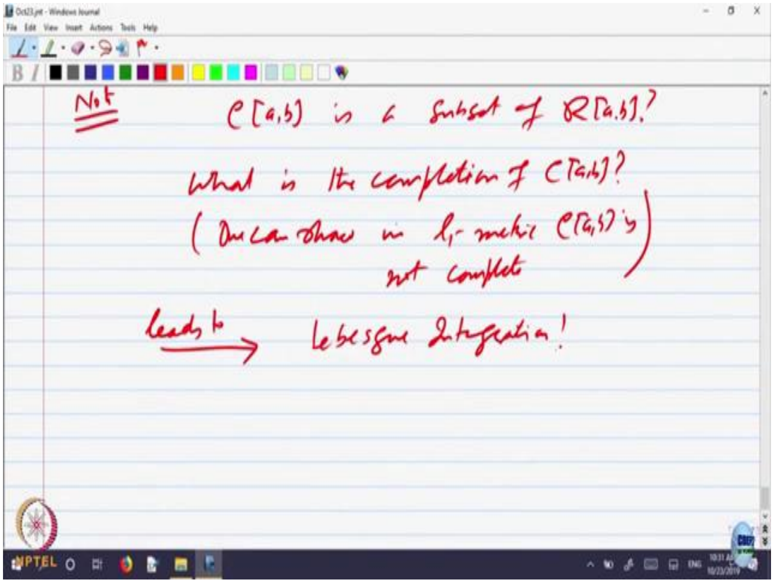

So, here is a note which is a beginning of some other topic. It says, C a, b is dense in R a, b, it is not dense in R a, b, it is difficult to state even that because it requires a lot of other things. So, how should I say? C a, b is a subset of R a, b, what is the completion of C a, b? One should ask first of all one can show in l 1 metric C a, b is not complete, C a, b is not complete in the l 1 metric. So we are looking for the completion of it, what should be the completion and that is given by what is called in l 1 this leads to what is called Lebesgue integration.

So, I will not say anything more about Lebesgue integration and so on, saying this is the beginning of another story called Lebesgue integration, measure theory and probability theory and that direction it goes somewhere else. So, analysis on R leads to many branches, on one side it is leading it to R 2, R n, norm linear spaces. If you do not have any structure, only the notion of distance it leads to metric spaces, study of metric spaces,. If you look at the L 1 metric on function spaces, then it leads to Lebesgue integration and so on. So, various sort of branches to which it goes to.

But in all this I have not, to study all these things you need to understand what does sequences mean on such spaces, on the metric space whether it is C a, b, R a, b, or any functions, what does the sequence mean and what is a convergence of a sequence mean. So, we are not looking at, so we are looking at sets whose elements are themselves functions. There is a metric on it, and we are looking at what does convergence of a sequence of functions mean under that metric, why that is important, why one should study that? Of course, mathematically on any metric space, we should be studying the notion of convergence of a sequence, but let me, because you are probably going to be doing something in probability and statistics, here is something that you should be interested in.

I think instead of writing, let me just say, imagine you are playing a game and what is the game? It is a gambling game, I am not encouraging gambling. The gambling game says that I toss a coin, I go to a gambling house with some money in my pocket and I want to gamble, and how do they gamble? There are many ways of doing it, I will not say about all, but the simplest game is that the person in the gambling house tossing a coin.

When he tosses a coin, if head comes you win 1 rupee and if tail comes, you lose 1 rupee, very simple game, tossing a coin, if head appears in the toss you get 1 rupee, if the tail comes you lose. So keep on, I start playing, after n tosses of the coin, n times the game have been played in the n tosses sometimes you might have won, sometimes you might have loss.

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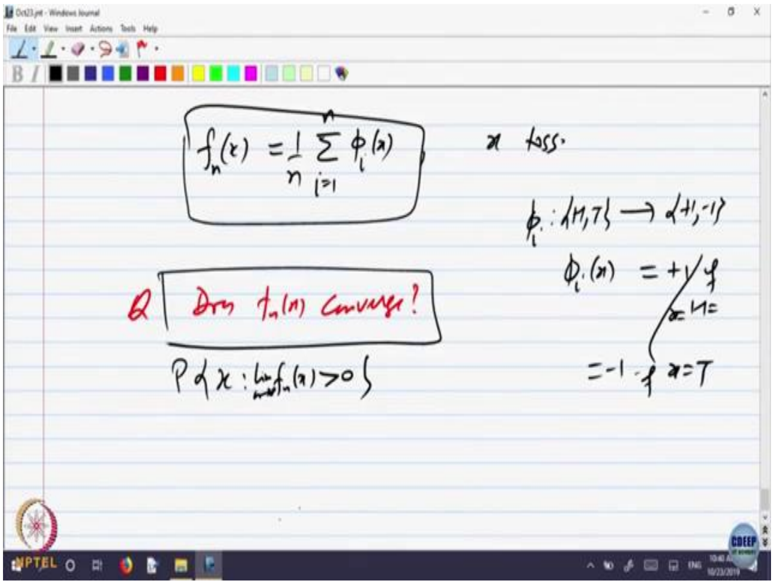

So after n tosses, f n x is the outcome of the toss, it is a function of the toss, after n tosses what happens. And let us say, what will this indicate? Let me instead of this, let me call it as P will be bad choice so let me write something, let me write Phi i of x, i equal to 1 to n x toss, outcome of the toss, what is Phi i? Phi i is the amount we have won or lost. When you toss x our outcome, if x is head, Phi i of x is equal to plus 1, if x is a tail, Phi i of x is minus 1, so phi is a function on, is a function on head or tail.

So how do you write head or tail, let us write H and T if you want write. Taking two values, what are the values, plus 1 and minus 1, these are the two values the function takes, Phi i of H is equal to plus 1, Phi i of x is equal to plus 1 if x is equal to H, minus 1 if x is equal to tail. Phi i x indicates the amount you have won on the ith toss. Toss takes two values; either plus 1 if you have won, if it has come as a head or tail. And what does f n indicate? It is up to n tosses your profit, it is your total profit. Loss also I will call it a profit, it is a negative profit.

So, what would we be interested in? If I keep on playing whether Phi n will converge to some value or not, eventually whether I will win or lose. So, what it by eventually mean? Of course, you are playing only 50 times you can calculate the profit, but normally the gamblers, they want to know if I keep on playing this game for a large number of times now keep in mind the sequences, for a sequence the limit is not that important as the behaviour for n very-very large, the trail of the sequence what happens for n very-very large. So, this will happen that f n is the total profit after n stages? What happens to f n for n very-very large?

So, question, mathematically question would be, does f n x converge? So, how do I make it more precise? For every x, it may converge, it may not converge, if it converges it can converge to a value. So, that will depend on x, so that will be function of x, so the question should be that does there exist a function f such that f n x converges to f of x for every x so, convergence of a sequence of functions.

You may be interested, you will see you are not interested really in this, you want to average out, what is my average profit if I keep on playing whether I will win or lose, these are kind of deterministic kind of models of analysis, but you may not be interested in deterministic models, you will like to know, what is that chance that if I play this game eventually I will win so a probability enters into picture.

So, what is the probability? I would define anything in x like that f 1 of x is bigger than 0, you will be interested in such kind of statements. What is the probability that if I keep on playing our limit n going to infinity f n x is bigger than 0, eventually what happens, what is the chance of winning, so that is not certain, it depends on a probability. For example, the probability will tossing a coin is not always sure, if it is the unbiased coin then it is half-half probability, each appearing on the toss, but it may not be unbiased. So, probability of head appearing maybe P, probability of tail appearing may be 1 minus P, where P is between 0 and 1, so probability enters into picture.

So, what one looks at what is the distribution of head or tail appearing or all such things. So, this kind of analysis you will be doing in probability and theory. So, what I am trying to tell you is the importance of questions like analysis of sequences of functions become important, how to analyse whether a sequence of functions on a space x converges or not converges, what is the meaning of that so, let us define that.

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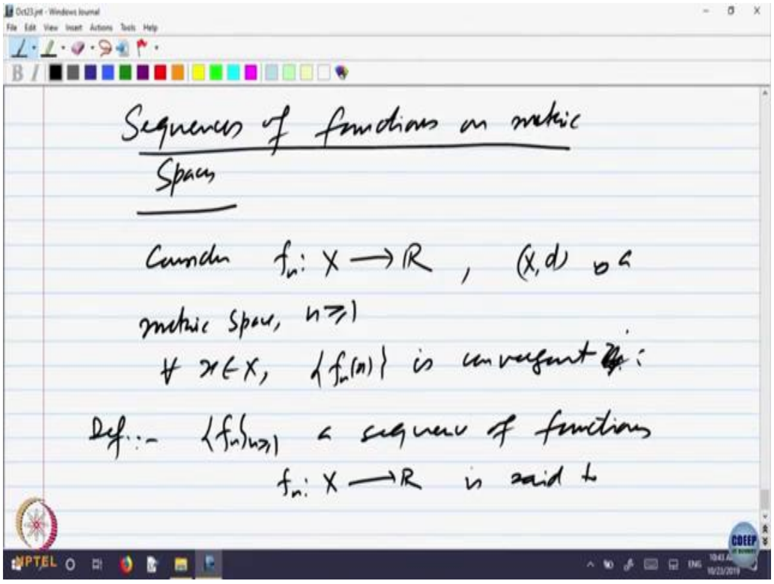

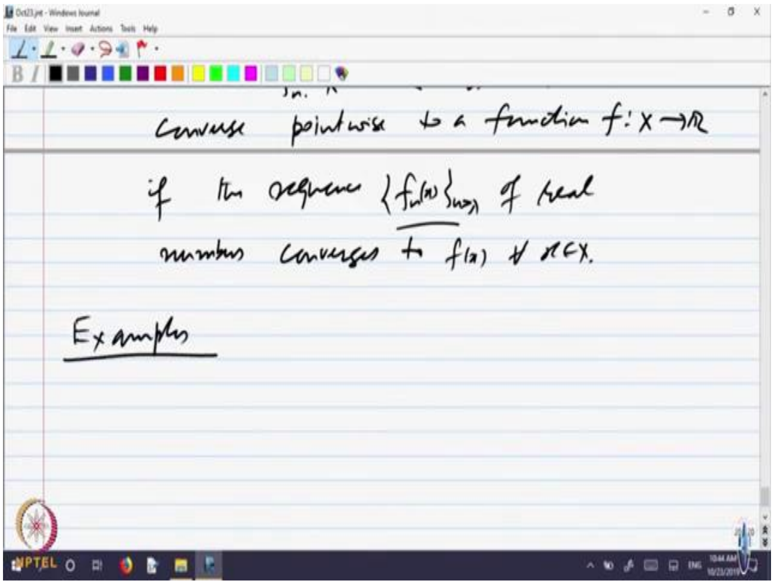

So, what we are going to look at is sequences of functions on metric spaces. So, consider f n say x to, it could be anything actually but let me write x to R x, d a metric space, so consider a sequence of functions. So one can analyse for every x belonging to x, can we say f n x is convergent. For this, one does not need a metric space, but let us keep this as a metric space is convergent or not convergent. Actually, we will be looking at more when x is the real line, but anyway, so that means what, so convergent means what?

So here is a definition, f n is a sequence of functions with domain x in R is said to converge pointwise, to a function f x to R if the sequence f n x of real numbers converges to f of x for every x belonging to x. For every point x in the domain look at the sequence f 1 of x, the value of the function f n at the point x that is a number R in the real line. So, look at the sequence of real numbers f n x, whether it converges as a sequence to the point f of x or not. Then you say, at every point f n converges to f that means, f n x converges to f of x, so let us look at examples.

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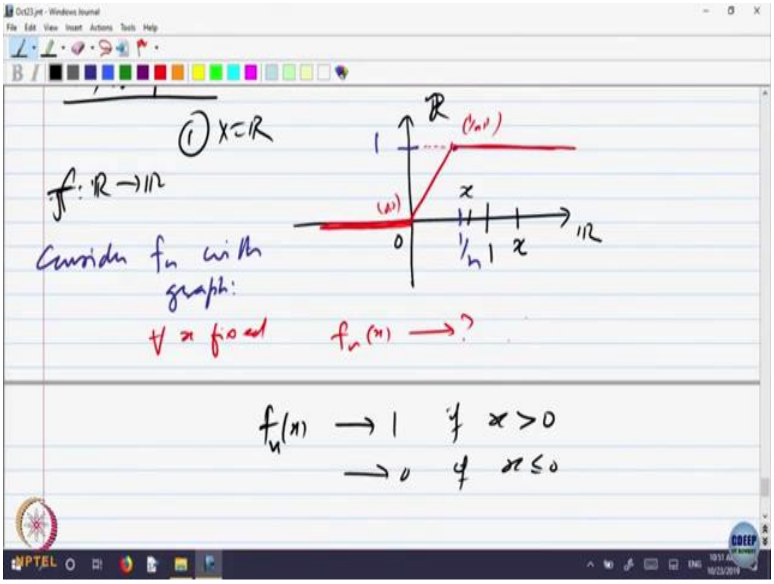

So let us look at example, let me draw a picture probably that may be better that is 0 that is R, that is R, so we are looking at a function f from R to R., so x is equal to R here. What does the function look like? I have to define what is f n, so let me take a point 1 by n. So I am going to draw the graph of f n, so consider f n with graph. So what is a graph? So let me call this as 1, so this is 1 over n, so the graph is, so this red, it is 0 on the left of the real line, take the point 1 over n from this point so 1 over n onwards the value is 1, in between what are the values? You join the line 0, 0 with that point, so it is a piecewise linear function, is it

You can write down the formula if you want to write down, that is not really very important as far as n is concerned at present, you can write down, this point is 0, 0 and this point is 1 over n. So, the line joining, you can write down the equation if you want to write down very particular. Now what I want to know is, for every x fixed what happens to f n x, does it go somewhere, what happens to f n of x, as n goes to infinity what will happen, can you make a guess what will happen? Where will be f of x?

Let us fix, let us fix x anywhere, if x is here somewhere so what is f n x for every x, x fixed for every n, f n x is equal to 1. If x is beyond the point 1, so f n x is constant sequence 1, so it will converge to the constant 1 so limit is 1. If x is less than 1, then what happens? Then 1 over n is coming closer and closer to 0, so somewhere x will come here, 1 over n will crossover x if x is less than 1, from that n onwards f n x will be equal to 1, is it. So my claim is f n of x converges to 1 if x is bigger than 0, less than 1, bigger than 0, y less than 1 anything is okay. If x is bigger than 0, it goes to if x is bigger than 0.

If x is bigger than 0, then either x is bigger than 1 or less than 1. If it is bigger than 1, then what is f n? f 1 1 at that point f 2 is also 1. So far every n, f n of x is equal to 1 if x is bigger than 1. If it is less than or equal to, you start 1 over n, n equal to 1, it starts at 1 then f 1 of x is 1. If x is somewhere else between 0 and 1 somewhere else, then for some n large enough, 1 over n will be on the left side of x. And what will be the value of f n there that is 1, so from that point onwards, because 1 over n is going to move to left and left only, it is converging to 0.

So each f n will be equal to 1 after some stage onwards, so f n x is a constant sequence after some space onwards, so it converges to 1 if x is bigger than 0. What happens if x is less than 0, less than or equal to 0? f n is always 0 so it is a constant sequence, so f n converges point wise, n converges to 0 if x is less than or equal to 0. Is it, this is the point wise limit, so the point wise limit of the sequence f n is nothing but the function which is 0 up to 0, x less than or equal to 0 it is 0, for x bigger than 0 it is 1. Now keep in mind, each f n is a continuous function.

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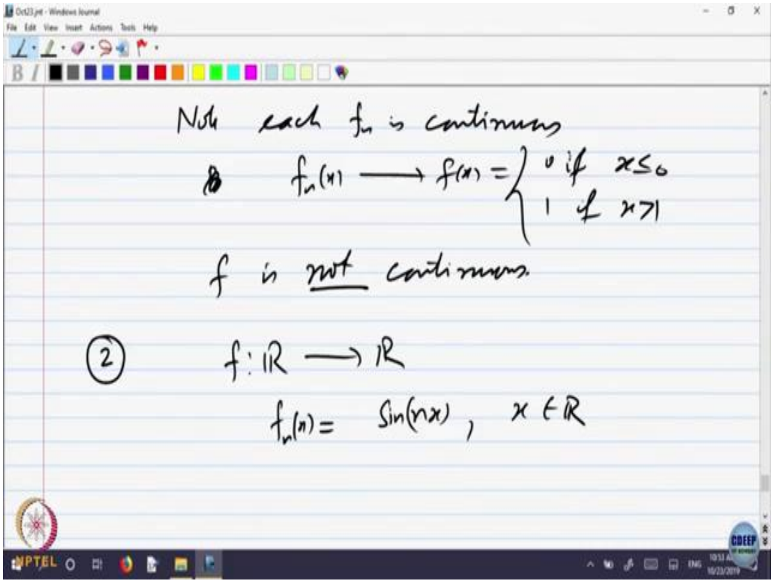

So keep in mind here, note each f n is continuous, f n x converges to f x, which is equal to 0 if x is less than or equal to 0, is 1 if x is bigger than 1, f is not continuous. The limit function f n x converges to f of x, so if you treat the limit as a function, so this function is not continuous. So let us look at some more examples. Two; let us look at example, f from R to R, f n of x is equal to Sin of n x. Do you think f n x converges point wise to something? Sin n x. Sin x we know, 2 x. So what is the meaning of sin of 2 x?

It is same as sin x, the period is now more or less minus, what is a period of? It is 0 at 0, when n x is equal to 2 Pi it will comeback so, n x equal to 2 Pi, Sin was a periodic function of period 2 Pi, n x what will happen n x is equal to 2 Pi. So, 2 Pi by n, the period becomes smaller and smaller. So, what do you think will happen? Will you think it converges point wise? No, it does not converge or it converges. So, think about it. I will put it as a exercise, think about whether it converges or not point wise to anything or not.

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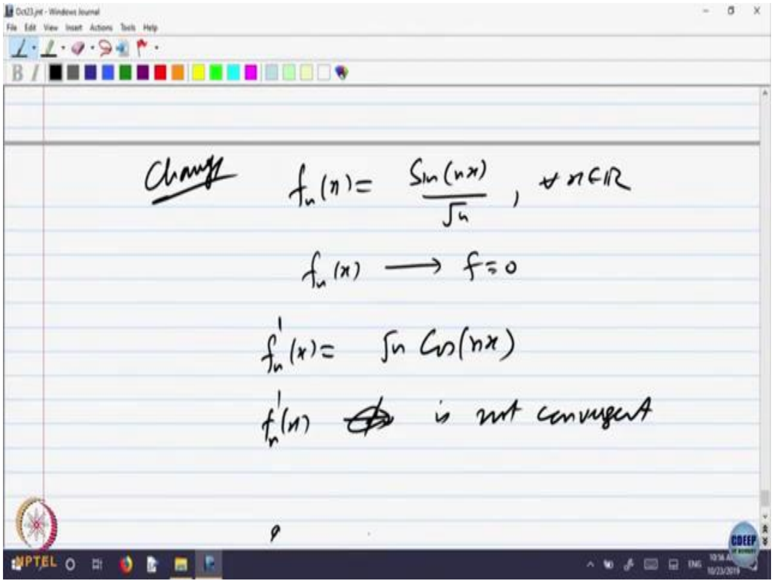

Let us modify this a bit so that things are easier to change. Let us look at f n of x let me look at it Sin, I want to make life simpler so let us take square root of n, see how things change drastically. Now look at this sequence, the value for n, earlier the value was plus minus 1, Sin function takes the value plus minus 1, now, value is 1 over of square root of n. So values are becoming smaller and smaller, Sin is bounded so now it is quite clear that f n of x converges to f which is identically 0, it converges point wise to the function which is identically 0. Yes, clear to everybody?

I want to look at something more, let us look at f n dash. Does f n dash exists, does a derivative exists of this function? Obviously, sin is differentiable, derivative is Cos so it is square root n times Cos of n x. Chain rule, can you see that derivative function converges point wise? Nowadays square root n coming in front, Cos is bounded so it goes on increasing or decreasing depending on n is positive. So Cos taking value minus or plus, it fluctuates but increases, it is becoming larger and larger fluctuation so, f n x is not convergent. Of course at points where, it is not convergent. Is it clear what we are doing? See I am just trying to manipulate so that you understand all the terms.

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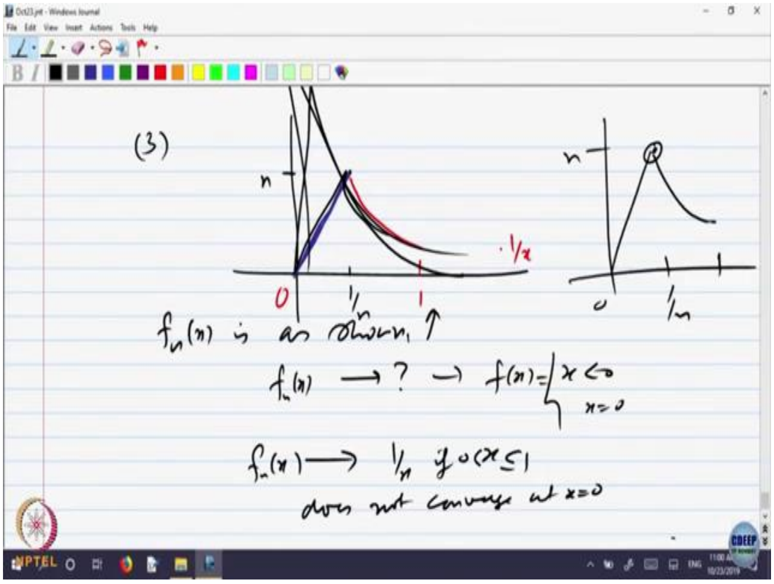

Let us look at one more example, let us look at third example. Let me look at this function is 0 to 1 and here is 1 over n, so here is the point n, so my function is going to be this line. And from this point onwards, it is a graph of 1 over x. Is it clear what I am doing? Up to the point 1 over n, look at the graph of the function 1 over x, 1 over n is coming closer and closer to 0. So up to the point 1 over n, keep the graph of 1 over x between 0 and 1. We are looking at so let me a bit of this part I am only looking at. This is a continuous function, graph obviously, it is a continue function.

So f n is as shown, as shown. Where does f n x converge, for every x where does it converge? As you move towards 0, more and more of 1 over x is coming into picture, so it converges to the function f of x, if x is less than 0 and limit at 0 for x is equal to 0. What is happening at x equals 0? It does not exist, so f n x converges to 1 over x if x is between 0 and 1 does not converge at x is equal to 0, it converges to infinity if you like to say.

Student: (()) (30:30)

Professor: It is a continuous function, the graph is this thing. Pardon.

Student: n equals to 2.

Professor: n equal to, so half, this point is half, what is the problem?

Student: (()) (30:49)

Professor: Yeah, height is increasing. It will go like this, it will go like this that is all. It will always remain, f n is a continuous function. What I am doing, how I am drawing the f n take 0 to 1, this is 1 by n. At 1 by n take the point n, so this point is located, join this by a straight line and this by the graph 1 over x that is all. Between 1 over n and 1 it is a graph of 1 over x. At 1 over n what is the value of 1 over x?

Student: 1 over n.

Professor: 1 over, x is equal to 1 over n, function is f of x equal to 1 over x so what is the value at 1 over n?

Student: n.

Professor: n, so that is this value. So, it is a continuous function and that is a limit. Now, the interesting thing is, each f n is integrable, is a continuous function on 0 to 1, where the limit is not integrable, limit is n bounded function, so limit is not integrable. So, all these examples not only tell you how the point wise limit behave, these example tell you that point wise limit behaves rather badly. If I take continuous first example says I take continuous functions, point wise limit of continuous functions need not be a continuous function. If I take differentiable functions, then the point wise limit exists, but the derivatives need not converge to the derivative. So, here is the derivative.

Point wise limit exists, but I cannot say because functions are differentiable, derivatives also will converge, they may not converge. And here is the next one which says that all functions are integrable, the limit exists point wise limit exists but the limit is not integrable, so in some sense these examples illustrate that point wise limit is not a very good way of analysing limits of sequences of functions, one has to devise some other way of analysing them, so we will look at it next time. So let us stop.