Basic Real Analysis
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Lecture 60
Lp Spaces - Part II

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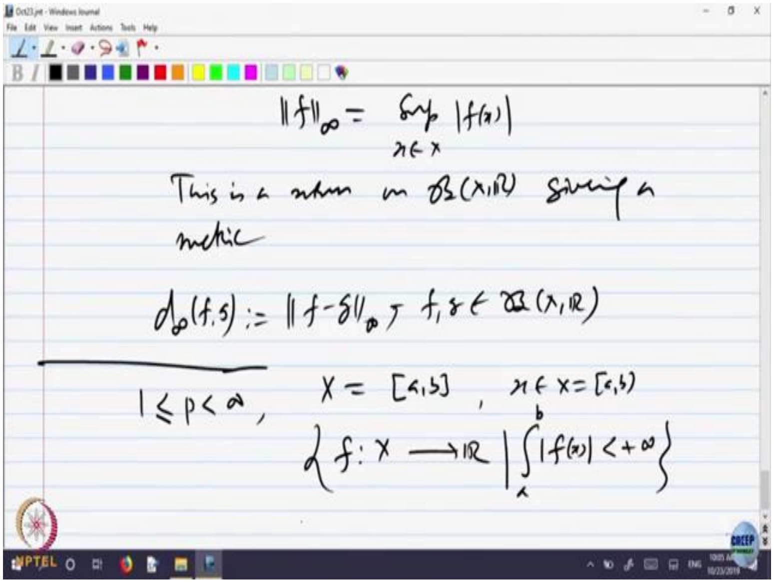

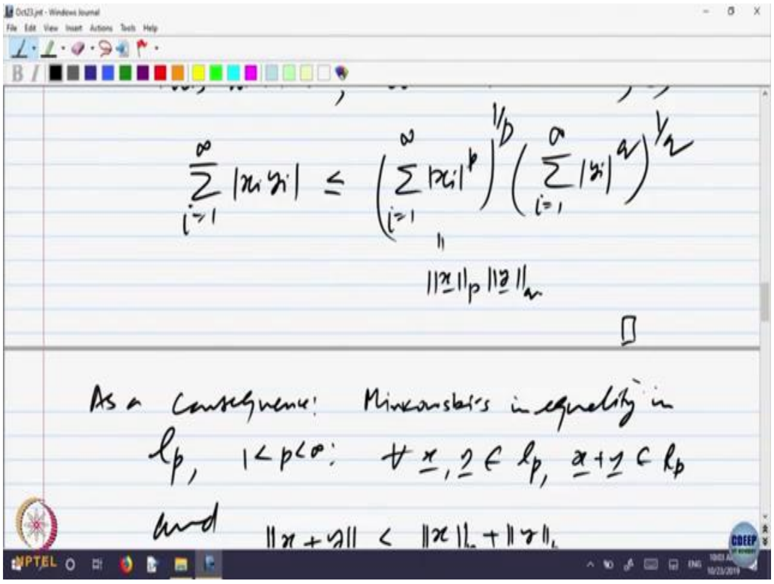

This is for L infinity. Supposing you have what. let us try to see what happens if you look at this, let us look at even 1 is ok, infinity we already defined. So, we want to look at all functions from X to R such that if I look at the component, so this is the xth component. What is Rn? Mod a 1 plus mod b 1 to the power p n they are finite. Some of the pth powers of the components are finite. In R infinity, it was some of the pth powers all the components is finite.

But if we do not know what is x, how do we sum it up? I want to sum of mod x, I want to add up all these things, we only know we can add up all these things when they are numbers, x may not be, x is arbitrary set. So, let us specialize x to be equal to for the time being let us look at it a, b just to understand special case, x is close bonded interval. Now, mod of f x is the real number, now x belongs to, x belongs to x which is a b.

Now we can add up all these things, so what should addition f of x when x belongs to a b should mean when x was a finite thing, 1, 2 up to n, I could write f of x I is submission 1 to n. When it is 1 to infinity we could say those which are finite, so what should summation of this mean when it is unbounded, a b, the number of elements x belonging to a b is not countable infinite, it is much more so it is natural to put integral a to b, integral is a sum on a b, so (()) (2:42) sum think of that, so we should put this to be finite. So, not all functions we can consider, we can only look at say functions whose integrals are finite.

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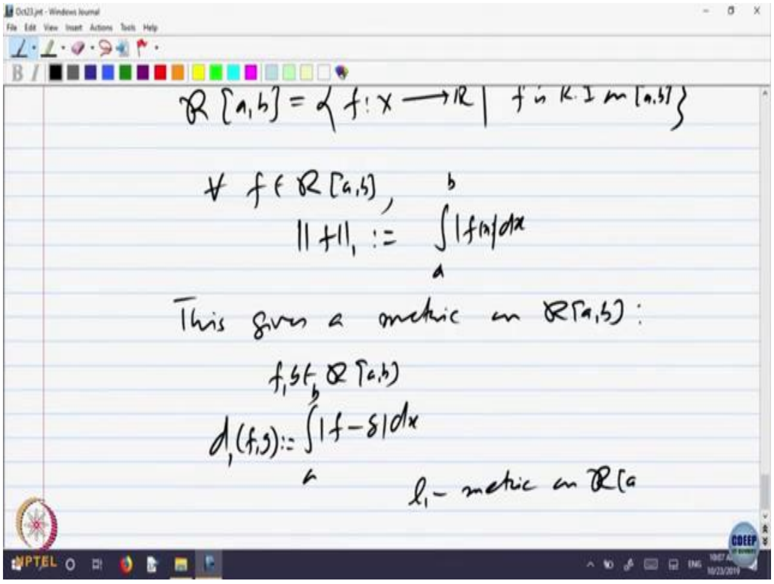

So, essentially what we are looking at is, we are looking at all, so let us look R a b. So, it is f x to R, what is the meaning of this is finite, so Riemann integral is finite. Riemann integral. So, all R a b is Riemann integrable functions, f is Riemann integrable on a b, if f is Riemann integrable on a b and mod f is also Riemann integrable we know that, so this quantity is defined and for every f belonging to R a b, I can define norm of f 1 to be equal to mod f x dx a to b. So, keep in mind it is same as adding up all the components, absolute values of all the component, notion of what is addition that depends on whether it is finite, whether it is infinite then it becomes a series.

If it is uncountable then you have to interpret it as a integral. But basically we are generalizing absolute value, absolute value on R, absolute value on R 2 summed up, absolute value on Rn summed up, absolute value for function summed up copying everything right. And one can show that this gives a metric, so what is a metric? So, f and g belonging to R a b, you can define integral mod f minus g dx a to b so, that is a distance d 1 f, g. So, this is what is called l 1 metric on Riemann integrable functions.

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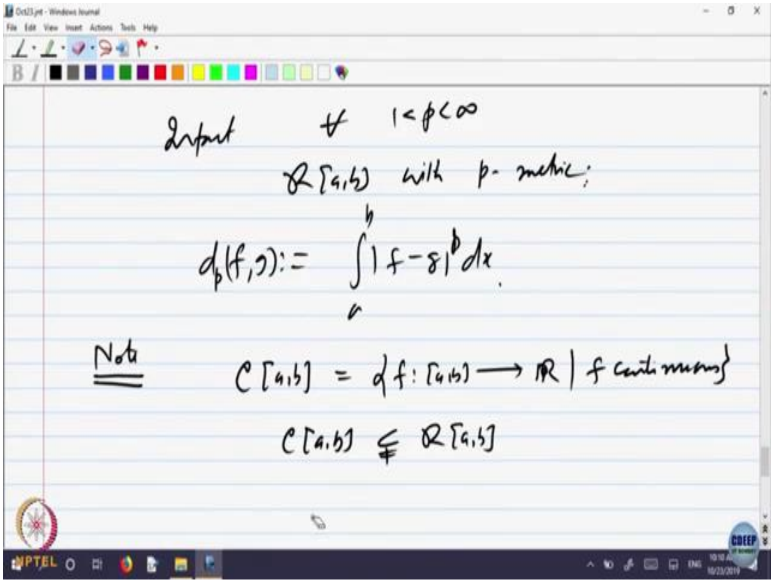

And proving it is a metric is because absolute value so d f g h does not matter? Absolute value of f of g is less than absolute value of f h plus h g, so integral will be less than or equal to, so that is not a difficult thing to prove it is a metric, same proof works., it is a metric. L infinity we have already seen it is a metric, l 1 l infinity, what is left in between is l p you can define, so I will not go very much detail into it.

So in fact, for every p between 1 and infinity, you can look at what are called R a, b with l p metric or one should not say l p, one should say p metric, so what is a p metric? For f and g the distance p, f and g is nothing but integral mod f minus g raise to power p dx a to b. And the proof is same as before, Holder's inequality, you use Holder's inequality for mod that a raise power p generalization of APGP mean right, a raise to power p, b raise to power q less than or equal to same, idea works everything. So, we will not go into it but this is for explanation I am telling you because you may come across these things later on in your other courses. So it is called a p metric.

Why we want to define metrics on such complicated spaces? One may think of R 1, R 2, R n good enough, why we should go to functions on defined on something, why what is a need for such things. So, the need arises, that may be a good way of looking at. So, let us note, let us look at C a, b so what is C a, b that is all functions on a b to R, f continuous, look at all continuous functions on the interval a b. Every continuous function is Riemann integrable right we had proved that so, C a, b is a subset of R a, b in fact, it is a proper subset of R a, b.

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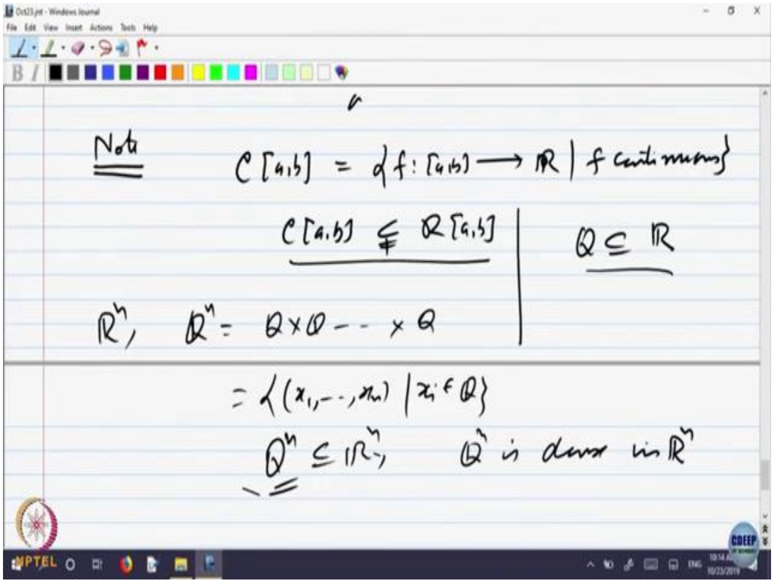

There are two things one we would like to ask here. I will keep analogy with real line and rationales form a subset of it. I am going to motivate this what I am going to do with these things, keeping in mind Q in R. Now, rationales we said in the very beginning, between any two real numbers there is a rational, and that property we call as a denseness of rationales, which is equivalent to saying if you take any interval that must have inside a rational, that is rationales intersect with every say open interval, that was denseness. Or another way of looking that was, if you look at this closure of Q in R, Q is a subset of R, if you look at the closure of Q that is whole of R that is same as saying that every open set intersects that means closure of Q is every real number is in the closure.

Keep in mind what we had done limit points, close sets. So, closure of Q is R so saying denseness, Q is dense in R you say my saying between any two real numbers (because) see if you say between any two real numbers that means there is a notion of order between any two. But if you take R 2 then what is the meaning of saying that between any two points there is a rational, it does not make sense because there is no order on R 2, there is no order on R 3. So, how do you write denseness, so there the denseness we saw it is interpreted as saying, every open ball must intersect that set then set is dense.

So, in R n if you look at say Q n, so, what is Q n that is Q cross Q cross Q that is same as all vectors x 1, x 2, x n so that each x i is Q. Then this is a subset in R n and Q n is dense in R n that means what? Given any vector, if we look at a ball around it of any radius then it must have an element of this inside it, so that is what denseness means. So, now this is on real line so on R n we saw what is denseness. So, you can ask on any metric space given any metric space, can you find dense subsets of it, are there subsets which are dense, can you define the notion of denseness on any metric space? So, you can define so let us come back to C a, b a bit later.

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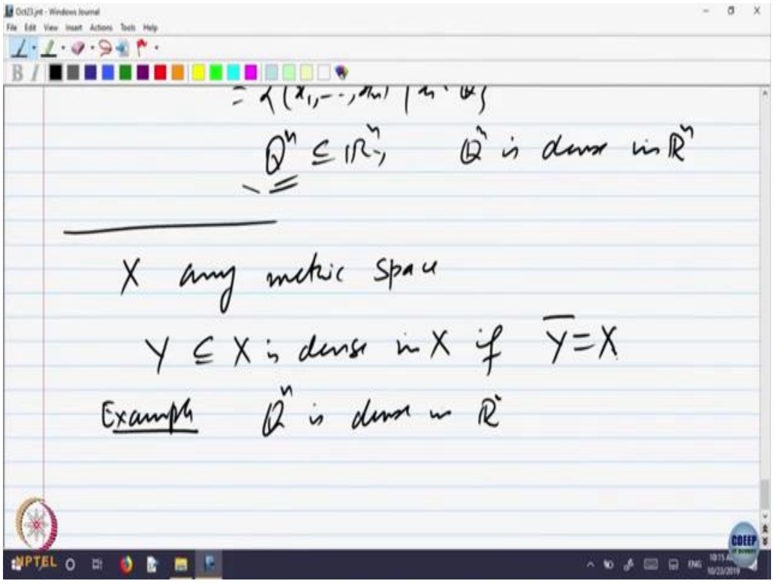

So X any metrics space, so we say Y subset of X is dense in X if Y closure is equal to X, that is a notion of denseness. So, you see that notion of denseness is defined in terms of closure. So closure can be defined in terms of sequences, every point of x is approachable by a sequence of elements of y or every open ball intersects y or the closure of y is equal to x, all those are equivalent ways of saying. So, example we said that Q n is dense in R n, there is one sort of trivial example of extending the fact that Q is dense in R n . So let us call it as 1 so let us go back to that.

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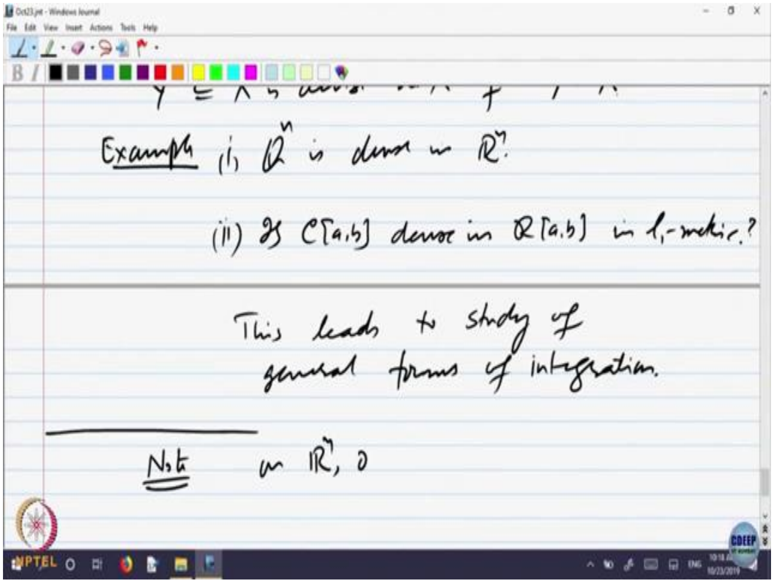

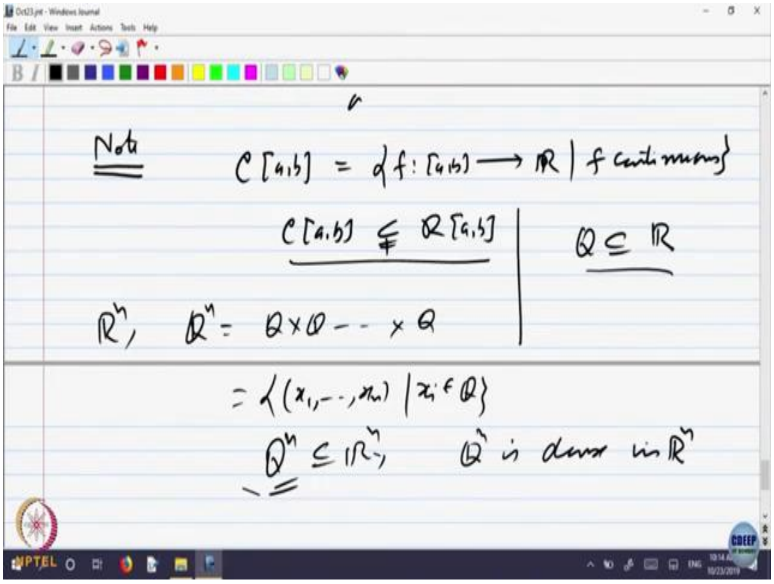

So the question is C a, b, so is C a, b dense in R a, b? Of course, denseness with respect to a notion of a metric because metric space so, let us write in l 1 metric to be precise, answer is yes and the proof is a bit difficult to give. And maybe in some of your courses later on you will find a proof of that so I will not go much into it, I just keeping that this leads to study of general forms of integration. There is some general theories of integration which will be required sort of to say analyse denseness, so we will not go into it, but I am just giving you a exposure which you may come across, you may come across in the future courses.

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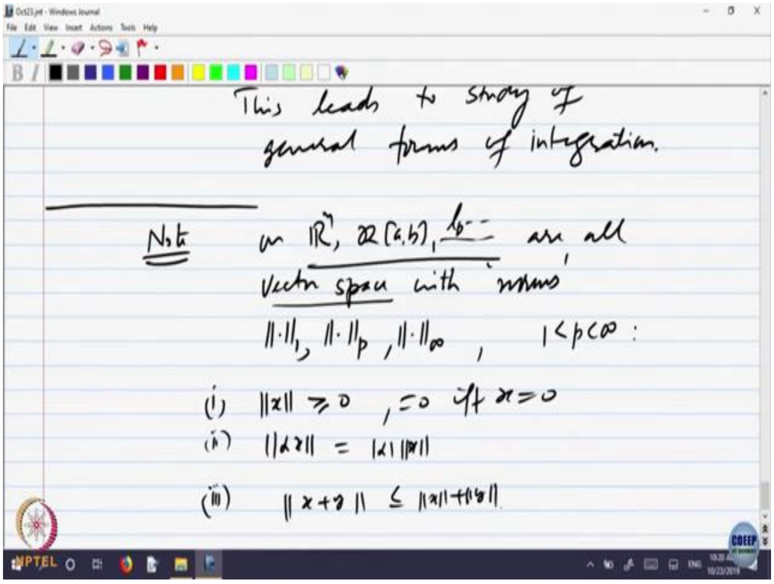

So, let us look at another question, there are many things I want to. And here is another thing so let us note, 1 R n on R a, b are all vector spaces or even lp and so on, are all vector spaces with norms, either it is this norm or this is p norm or we have basically looked at three types of norms. And all these places we have and l 1, lp and l infinity, they were norms, so meaning what? They are like the absolute values.

So norms meaning, so norm meaning, say norm of x is bigger than or equal to 0 where x could be any one of elements or anyone of this and equal to 0 if and only if x is equal to 0 and alpha x, because it is a vector space, so there is a scalar multiplication. And how does it behave with respect to scalar multiplication that is same as mod alpha times norm x. And the third x, if you look at two elements, you can add them because it is a vector space so triangle inequality holds.

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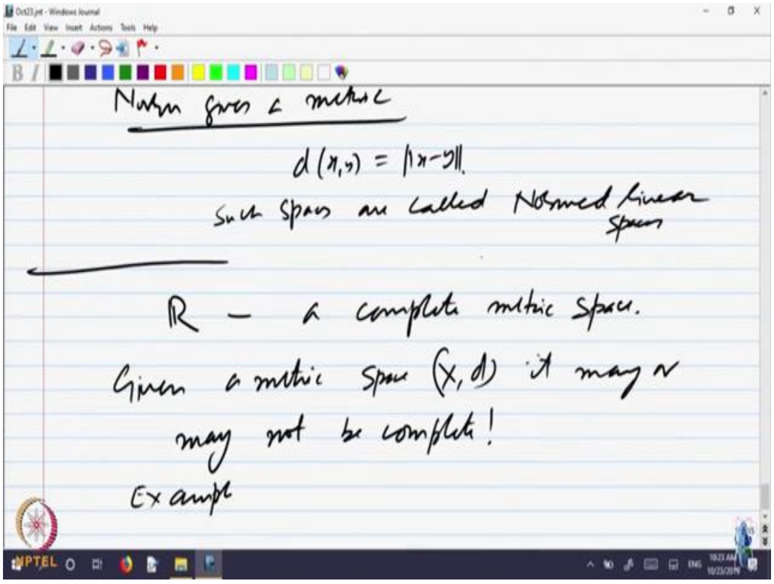

And we had mentioned that norm induces a metric, norm gives a metric and what is that metric? So, d x, y is normal x minus y. So, it is a general process, given a vector space for every vector there is a notion of absolute value you can think of and it has these three properties and every notion of absolute value gives rise to a notion of distance. Study of such spaces are called normed linear spaces. Such spaces are called normed linear spaces, it is a linear space, it is a vector space and there is notion of norm defined on it and norm behaves very nicely with respect to scalar multiplication and addition, so these are the properties.

So, such as space is called a norm linear space and study of norm linear spaces goes into a topic called functional analysis. So that is a separate subject normally called functional analysis. So examples of norm linear, so we have a lot of examples of norm linear spaces, l p spaces, what are its properties, what are its uses and so on, that is separate topic called norm linear spaces, we will not go into that but what we will want to look at is the following.

So, we have real line, the starting thing which motivated us. Real line was a complete metric space, it was a metric space which was complete that was a big advantage that we could do everything, so completeness in various forms. One way of completeness was every monotonically increasing sequence which is bounded above is convergent that is sequential completeness.

Another form was a Cauchy completeness, a sequence is convergent if and only if it is Cauchy that was equivalent to it. So, these are various forms of completeness so complete metric space and gave us nice results, for example, that nested interval property was a consequence of completeness we use completeness to prove nested interval property. Now, given a matrix space x, d, it may or may not be complete, given a set x, given a metric for example, let us look at example.

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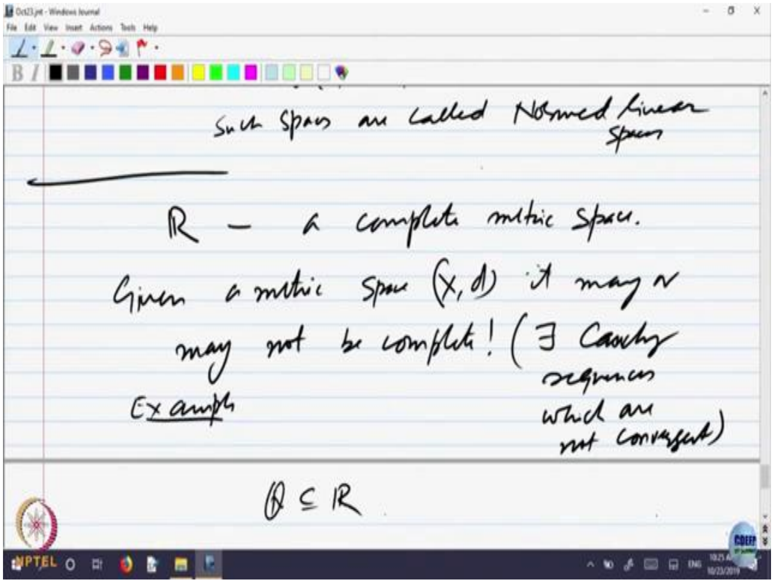

So, what is the meaning of it may not be complete, a metric space may not be complete means what? So, that means there exists Cauchy sequences which are not converging. In a metric space we have the notion of sequence; sequence being convergent, sequence being Cauchy because we will only need the notion of distance, convergent means closeness x n converges to x so distance of x n, x goes to 0. Cauchy, distance between x n and x m becomes small as n and m go to infinity. Absolute value to be replaced by the notion of distance that is all closeness.

So, whenever you have the notion of closeness metric, you can do all that you do on sequences on the real line R n and so on. So, there are metric spaces, when you generalize the notion of distance on a metric space, there are notions which are saying that the metrics space under that metric may not be complete. So a very simple example is, just look at Q as a subset of R, so what does that mean?

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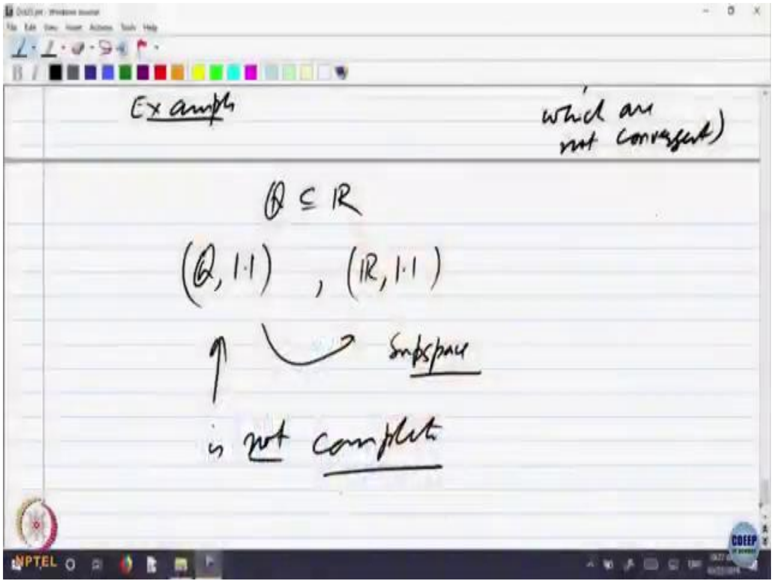

We are looking at Q with absolute value as a metric space, look at the set of rationales, look at the notion of distance given by absolute value of that number, rational is a real number also. So, we are restricting the notion of absolute value function from real line to rationales. And you can real line with the notion of absolute value. So, these are two different metrics spaces, Q is a subset of R, so you can say this is a subspace, you can say it is sub space does not mean, it only says that Q is a subset of R and the metric of real numbers is the restricted to metric on rational numbers.

And we had example of sequences of rational numbers, which are Cauchy, but which are not convergent, we had given such examples when we were looking at sequences. So, there are sequences of rational numbers, which are Cauchy which are coming closer and closer, but they do not converge to a rational number of course, they will converge to a real number by the completeness property so this is not complete. But what is happening? Rationales are a subset of real numbers, as a metric space they are not themselves not complete but they sit inside R as a dense set. If you look at this closure, there is whole of real line.

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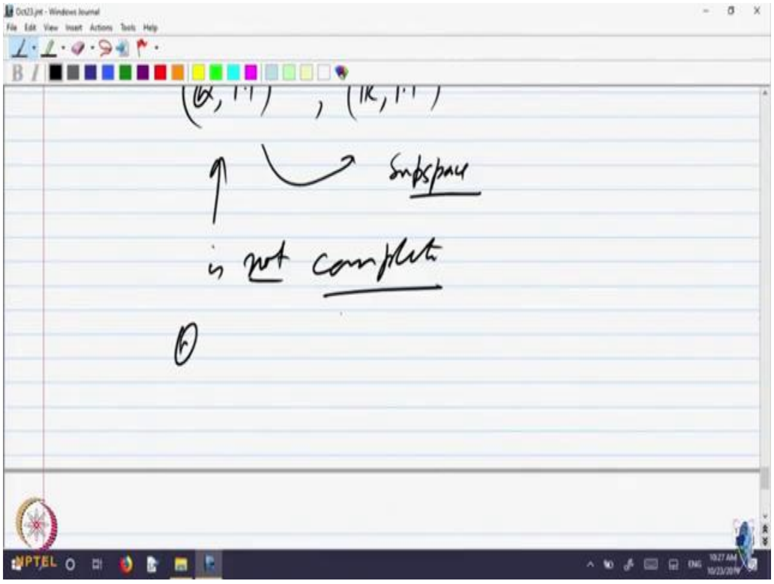

So, one says so there is something which we should look at. So one says, Q dense in R is interpreted as that R with the usual metric absolute value is the completion of Q with absolute value. So what I was saying is, there is a bigger metrics space example R d, there is a subspace metric Q inside it. Q is sitting as a dense set inside R, Q is not complete but R is complete. So one says, that real numbers is the completion of the rationales. Real number are obtained from rationales via completion, all limits are put inside it, they are completed. So one says that the metric space Rs with the usual metric is the completion of metric space Q.