Basic Real Analysis
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Lecture 6
Convergence of Sequence - Part III

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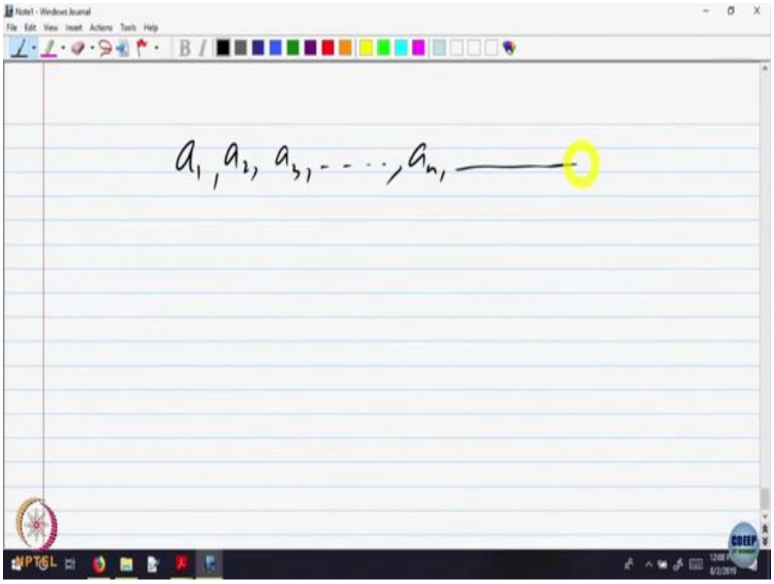

Okay, that is interesting let us probably say that also. Now, a n is a sequence which is given to you a 1, a 2, a 3 and so on, a n so on. Here is something which happens very often, you are standing in a queue, and suddenly somebody with a, some tag or something comes "tum bahar aajao, tum bahar aajao, tum bahar aajao, special queue tumhare liye bana diya, you are VIP's", a separate queue. So, what you are doing?

From that ordered collection, you are picking out some members, but he is not allowed that after he has picked up the tenth person, he can go back and look "koi aur bhi bacha hai ke nahi, woh nahi allow karenge, ek bar tumne nikaal liya toh nikaal liya", go ahead and pick up whichever you want. So, you are allowed to pick up elements of a sequence in a increasing order, whichever you like.

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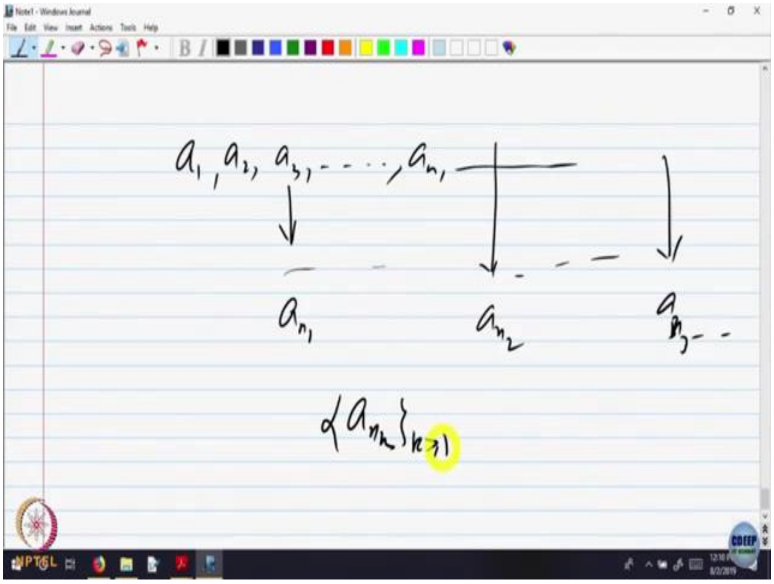

For example, you can pick up this one, now go on, you can pick up this one go on, you can pick up this one. So, this one you can call it as n 1, your selection, first selection, keep the hierarchy in the queue, do not come back, pick up the second one. This is a n second selection, third selection a n 3 and so on.

So, in general a n k you can form a sequence by picking up elements from the given sequence, but keeping in mind n 1 should be strictly less than n 2, n 2 should be strictly less than n 3, and go on, is it okay? So, such a thing is called a sub sequence. So, this is called a subsequence.

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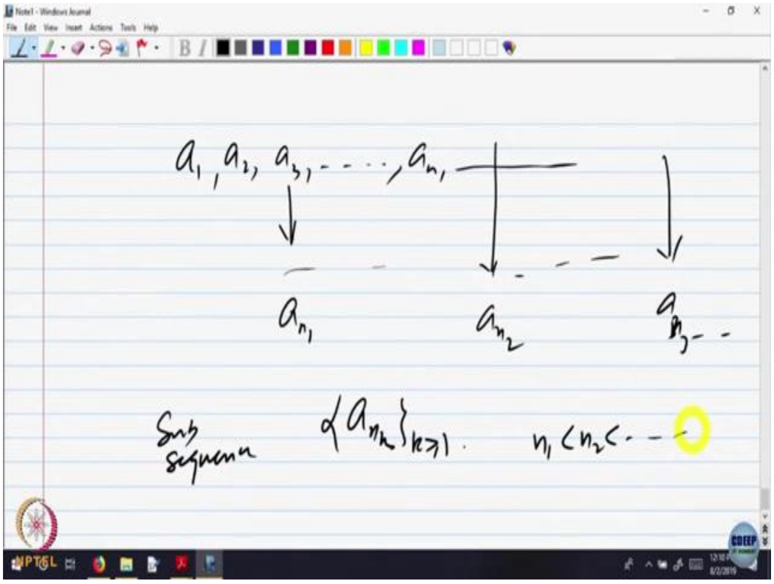

Where n 1 is less than n 2 less than and so on. So, what I am doing? So, pick up a strictly, monotonically increasing sequence of numbers, n 1, n 2, n 3 and so on. According to that pick up the elements of the given sequence that gives you a sub-sequence, is it okay? How to form a sub sequence? Now, if a sequence is convergent, what can you say about the subsequence?

Obviously it should be, because if those a n's are coming closer to a value L, part of that also should come inside, okay? So, first fact try to write a proof yourself if you like, okay, that if a sequence a n is convergent that implies every sub sequence must also converge to the same limit.

Can I say converse is true? Of course, if you will look at the sequence minus 1 to the power n, I pick up the odd terms, they converge to plus 1, even term converge to minus 1. But supposing I put a condition every sub sequence has to converge to the same limit then it is obvious it should be happen that your sequence itself should converge to that same value.

So, this is a theorem again we will not prove it, we will not ask you in the exam, but it is a simple obvious fact intuitively quite clear, a sequence converges to a limit L if and only if every subsequence converges to the same limit namely L.

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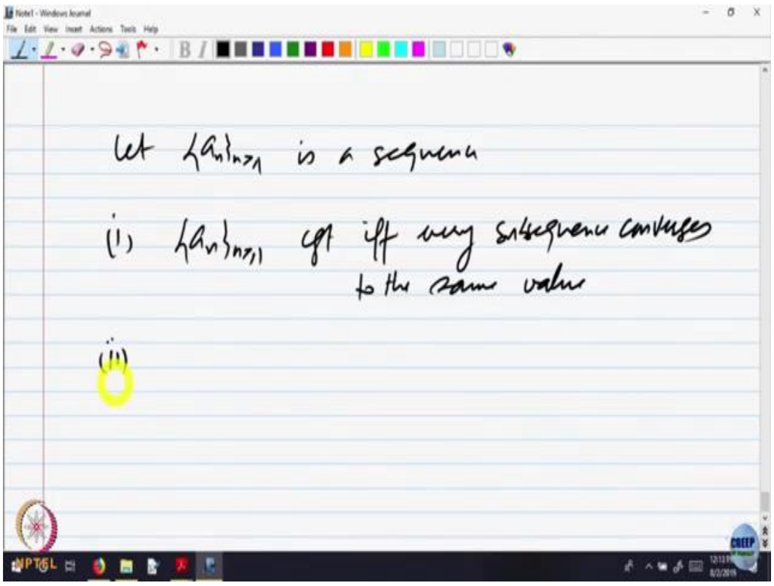

So, with that intuitive understanding let me go over and something. Okay, here is something, so let us assume a n is a sequence, so we have just now observed, a n convergent if and only if every subsequence converges to the same value, is it quite interesting, try to think of proof yourself, you will see how logic is playing a role there. Here is something which all of you should understand.

If you want to understand when a statement is true, you should understand when the statement is false, this is a fact of mathematics and of life, if you want to know what is true, it is as good as knowing what is false. If you do not know what is false, you will not know what is true, think about it. And the logic is played in mathematics also, if you want to say some statement, okay.

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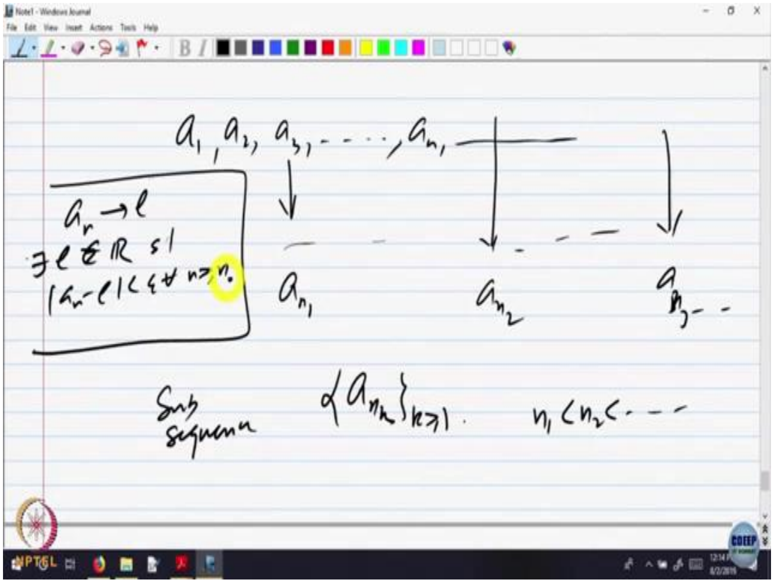

See, okay, I think let me elaborate that before writing it somewhere here, where do I write it let me write it here, a n converges to L, that is a statement, that is same as saying for there exists some L, I think we did that last time, L belonging to R such that a n minus L is less than epsilon for every n bigger than n naught, that is convergence.

So, truth of the statement that a n is convergent is equivalent to writing this that there is a number L which is going to be the limit and what does that mean? For every epsilon there should be a stage after which a n should come closer to. If this statement is not true, what does it mean?

So, there is a now the other way around now, this is, if you want to understand what is true, you should understand what is false. So, what is the falsehood of this statement? a n is not convergent, that means what? The first thing was there exist some number L, so, this should go bad.

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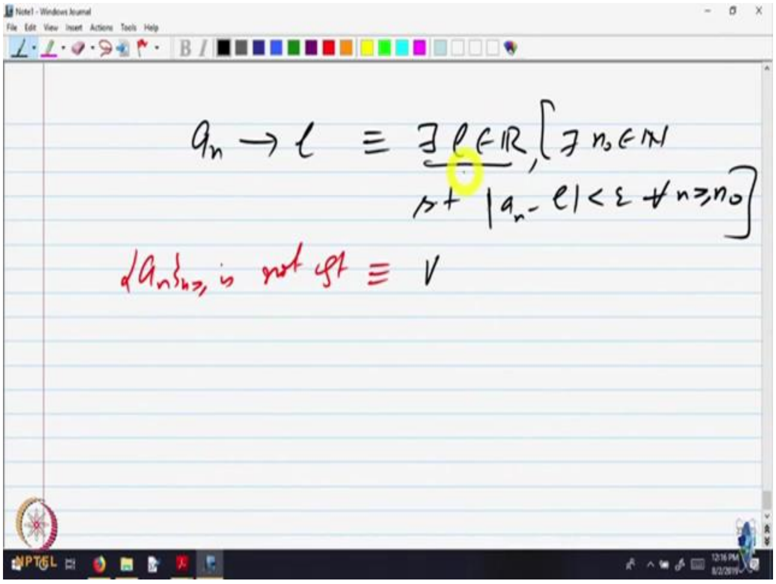

That means what? That means for every L, you may have problem later on using this slide, because something is coming somewhere. So, let me I think let me not do that let me go over to here. So, a n converges to L is equivalent to saying there exists L belonging to R such that such that R and there exists some n naught belonging to n such that mod of a n minus L is less than epsilon for every n bigger than n naught.

And I want to say that if this statement is not true, a n is not convergent. So, what is that equivalent to? So, first of all this statement should go bad. So, this is a combination of many statements, first one is there exist some L such that something happens, so, it should go back that means, if it is not going to be true for one particular L that means for every L it should be bad.

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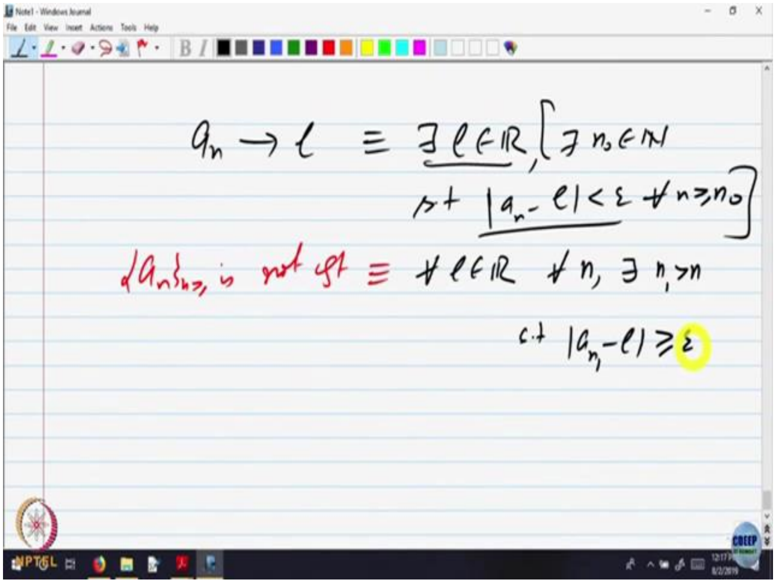

So, for every L belonging to R, what should happen? Here it says there exist a stage I should not be able to find this stage, that means for every n whatever I think n is the stage, there exists some stage after that. So, let us call it as n, n 1 bigger than n such that this is, this goes bad that means mod of a n 1 minus L is bigger than or equal to epsilon. Whatever, there is no number that means for every number. And what happened to that epsilon wherever what happened to the epsilon I did not write that?

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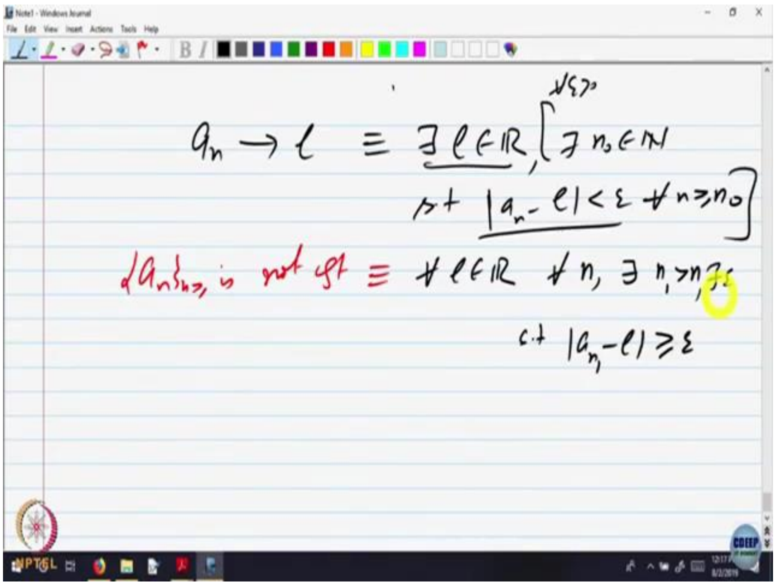

For such that for every epsilon was there. Because epsilon will really not come into picture. For every L given an epsilon, that means what? There exists at least one epsilon. So, there exists I should write there exist some epsilon such that this goes bad. At least there is one interval, so that you say that after that see everything is inside? No, at least one of them will go out and that is what is we are saying.

So, this is what I am saying trying to understand the negation of a given statement, a n is conversion, so try to do it every time whenever you find a theorem or something anywhere in your understanding, even in our life.

How do you compare something is true? You have to compare it with something which is false. In probability theory, you will all be doing probability theory. What is a chance of this fan falling down just now? Or in a particular date at a particular time? Chance is probability. That is equivalent to knowing the probability of this not falling down, both are equivalent statements, one is the truth that it will fall, other is the falseness it will not fall.

So, mathematically also you will find these things coming back to you because you are all ASI students, that is same. Okay, so, let me not spend much time on it and go back to what I was saying is corollary we approved. So, I said every subsequence converges for a given sequence.

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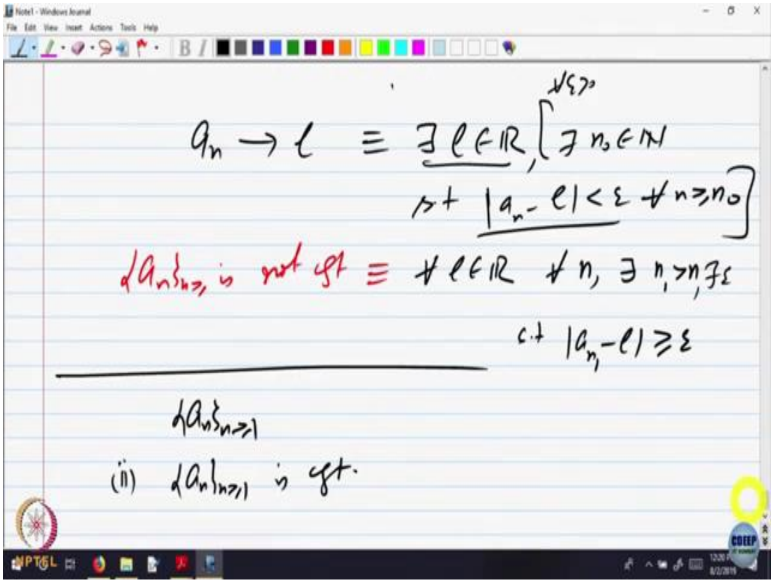

I think the more things let me right here now. So, a n is a sequence okay, so, one we have already said, every sub sequence converges if and only if the sequence converges. Now, suppose a n is convergent, why limit is not going to be important, so, a n is convergent, again convergence means, elements are going to come closer to it.

Let us not bother about to what it comes closer let us see the effect of this, if L is the limit it should come inside this, I can make it smaller, is for every epsilon still the a tail should come inside, still a tail should came inside, so, what should be happening to the terms of the sequence itself?

If and if I do not know L the limit, but I know it is convergent, that means the terms of the given sequence must come closer to each other as you progress, as n becomes larger and larger a n should become closer and closer to each other, because if they remain away, they are not going to converge.

So, it looks like a intrinsic property of the sequence convergence, a sequence convergence should imply that the terms are going to come closer to each other, because if I look at the sequence whether I know the limit or not does not matter, I can just say that the terms are coming closer to you, I look at that person and say he is honest, so that is an intrinsic property, without verifying whether he has a criminal case against him or not that is an extrinsic property. Limit is something outside which is not given to you, but saying terms are coming closer is a property of the sequence I can verify, so, let us give it a name.

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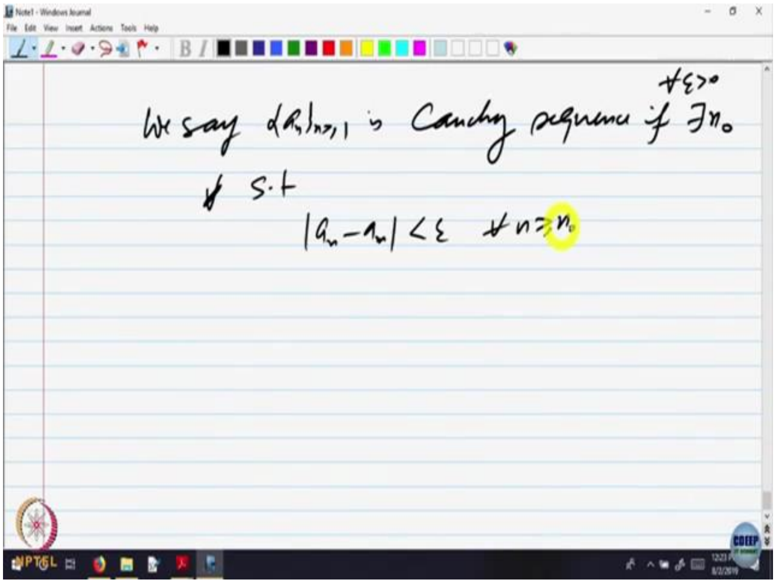

So, let us call a sequence, so, we say a sequence a n is Cauchy, Cauchy was a mathematician and his name I think you will find he will come in your course also somewhere or the else, in mathematics he comes left and right, in mathematics courses and statistics also he will come somewhere, okay.

We say is a Cauchy sequence if I can find there exist some stage n naught, everything is a property of the tail in sequences and now it such that a n minus a m the distance such that if I should write that Cauchy if for every epsilon bigger than 0 the distance is epsilon for every n bigger than. How close? As close as you want it, but for a tail. Given epsilon you want this close that will be a stage, much closer you want some other stage will be there, but this is a property of saying it is Cauchy. And I said it is an intrinsic property.

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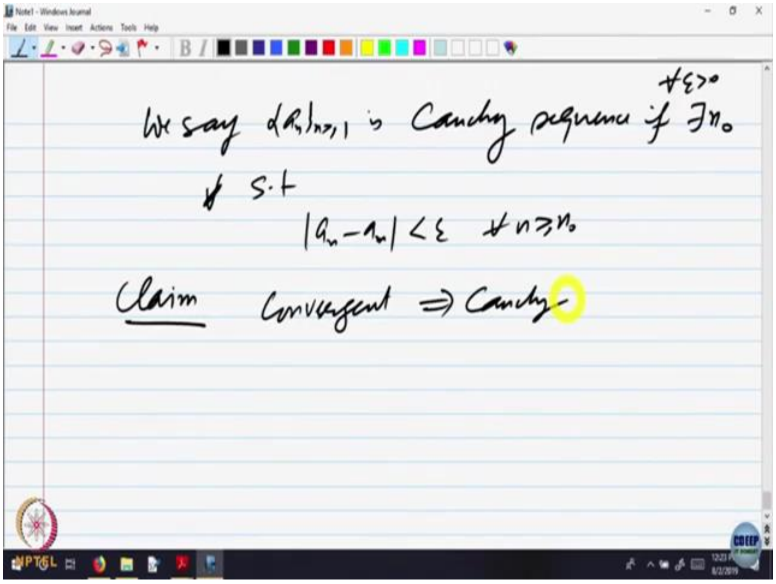

So, claim, here is a claim convergent, I should write better, convergent implies Cauchy. Cauchy as human being, convergent is a property, looks very odd writing convergent implies Cauchy, means every sequence which is now we are using them as adjective, if a sequence is Cauchy, convergent then it is also Cauchy. So, we are not using it as a noun, we are using it as a adjective, okay.

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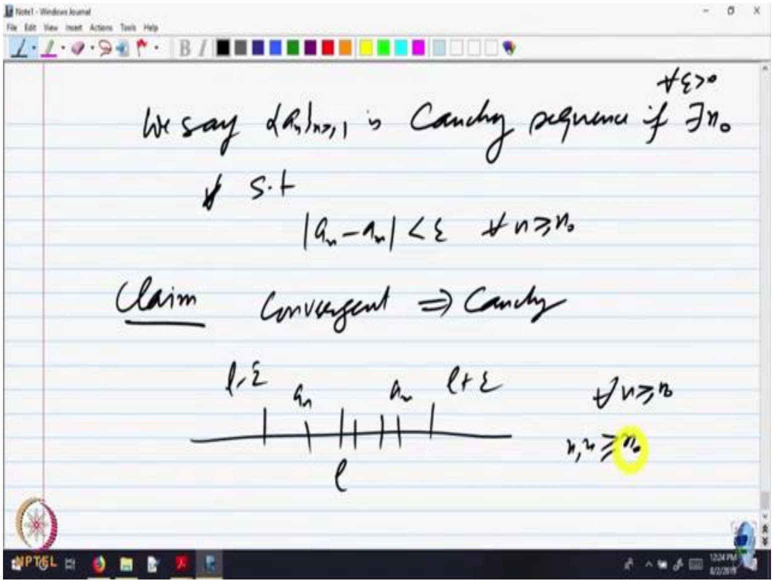

So, let us see how is it true? Now, here is convergent means there is a L, there is L minus epsilon and there is that is a neighborhood that is a interval such that for every n bigger than n naught, a n's are here, a n is here, a m is here when n and m are bigger than n naught what that obviously says the distance between a n and a m is small.

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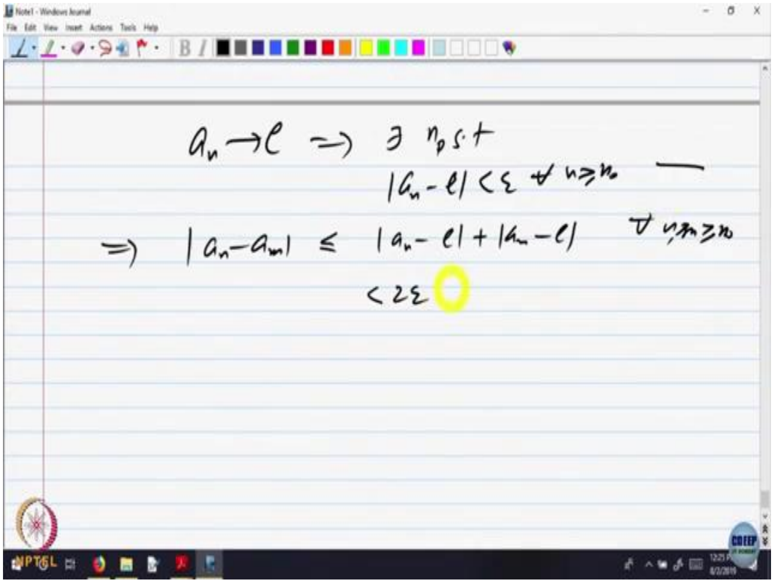

So, let us write it mathematically. How do I write it mathematically? It will like, we can write it mathematically as a n converges to L implies there exists some stage n 1 such that or n 0 such that mod a n minus L is less than epsilon for every n bigger than n naught. What I want? a n minus a m, that is my target, but I know something about a n minus L and a m minus L.

So, let us bring in add and subtract, so less than or equal to for every n bigger than this is true for all L and m. So, in particular for n and m bigger than or equal to n naught and that says is less than 2 epsilon, because each is less than epsilon, because of convergence, is it okay? Because of this property each is less than epsilon.

Now, this is a (())(17:17) what I call at cosmetic surgery, I do not want two epsilon, I want only epsilon. To look it nicely, so, I will go back and say it epsilon by 2, so it will be epsilon by 2 by 2 that is epsilon. That is a minor thing, because epsilon is arbitrary, so, I can change it to anything I like to start with okay.

So, that says every convergent sequence is Cauchy. So, Cauchyness is a necessary condition for a sequence to be convergent, like boundedness. Question asked, now, can I say if a sequence is if the terms are coming closer and closer, will the sequence converge? For rationals, it is not true, for reals it is true.

So, we want to prove a theorem that every Cauchy sequence is also convergent. So, it is a equivalent way of saying convergence, sometimes you are not interested in knowing the limit of the sequence, you are only interested in knowing the property of the sequence, there Cauchyness is very useful because you do not have to bother about the limit you have to only bother about, given the sequence, intrinsically look at whether the terms are coming closer or not.

So, we want to prove a statement that every Cauchy sequence is also convergent. So, we will prove that probably next lecture because there are only 3 minutes and we cannot prove it, but here is something I want you to sort of have a look at it.

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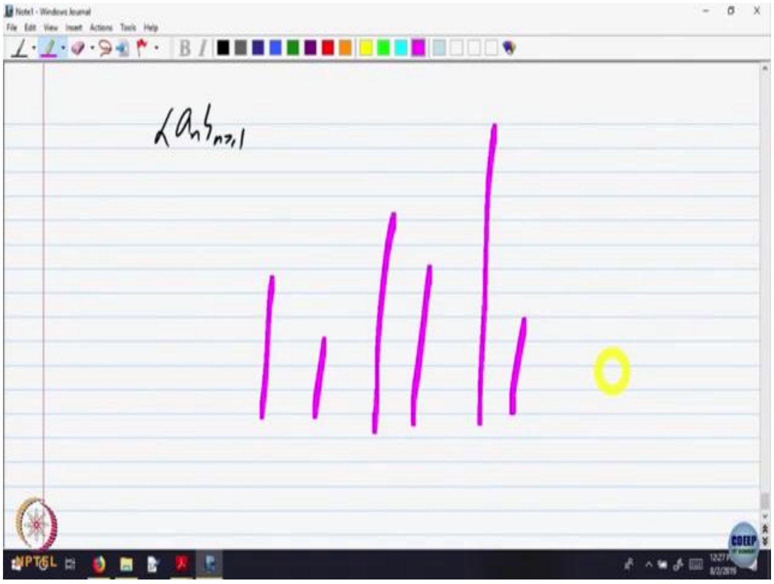

Given a sequence a n there is one way of visualizing it, I want to visualize this sequence, not as points on the line, but as if they are heights of poles, a 1 is some height, a 2 is some height, so, a 1, a 2, a 3, a 4 may be a 5 and such things. Imagine a sequence to be saying the height of something, a building probably or something.

So, when will you say a sequence is increasing? Imagine if they are buildings for example, you are able to see the top of this building, increasingly you are able to see that top of the next building also, this building does not obstruct the view of the next one everything is visible and you want to say it is monotonically decreasing then everything below is visible.

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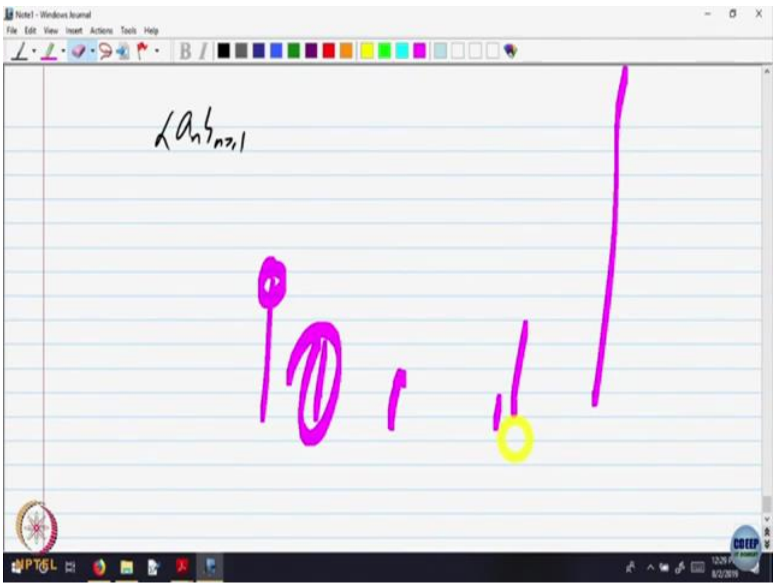

If it is something like this, you cannot say that this is increasing because something is obstructed in between, but at least it looks possible. Let us start with something and let us see the next building which is visible to you and the next building which is visible to you and then the next.

If I pick up these buildings, what I will get? I will get a sub sequence which is monotonically increasing or sometimes I can see this and then probably I forget about this building then I can see if this building is not there if this is not there, then I can see the next one here which is smaller than this, probably there is something smaller than this, then if I (obs) if I do not forget about these ones, then I can see that also. So, then I will be picking up something a subsequence is monotonically decreasing.

So, it seems to be a fact that every sequence has either a monotonically increasing subsequence or a monotonically decreasing subsequence this picture seems to seems to says through me that. So, we will prove that next time more mathematically that every sequence has got a subsequence, which is either monotonically increasing or monotonically decreasing.

Now for a convergence, I need a bounded. Suppose, I am given a bounded sequence then combine it with this result now there is a subsequence, which is monotonically increasing or decreasing and the original sequence is bounded, then this must be bounded. So, every sequence which is bounded will have a convergent subsequence that is again an important theorem is called Bolzano Weierstrass property.

Now, look at a Cauchy sequence. To say that a Cauchy sequence is convergent, it is enough to prove a sub sequence is convergent, because terms are coming closer, if a sub sequence is converging, the sequence itself must be coming closer to that value because they are also coming closer to each other.

So, another fact that a Cauchy sequence is convergent to prove this it is enough to produce a subsequence which is convergent. Now, a Cauchy sequence has to be bounded also because they are coming closer and closer given epsilon only finitely many can be outside.

Again does not matter where, again a Cauchy sequence is going to be bounded, Cauchy sequence is bounded, Cauchy sequence has got a monotonically increasing subsequence, by the Bolzano Weierstrass property it must converge. So, a Cauchy sequence has got a convergent subsequence, hence the sequence itself must converge that will proof Cauchyness is equivalent to convergence. So, I have already given you a trailer of the next class, so, we will do it next time. Okay. Thank you.