Basic Real Analysis
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Lecture 58
Metric Spaces - Part III

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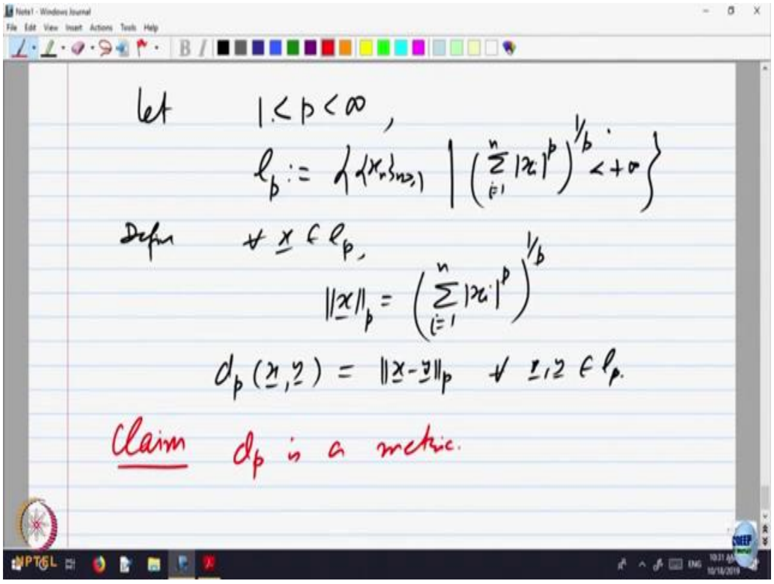

So, let 1 bigger than p less than infinity, look at okay. So, look at LP, now you realize why I was using 1 and 2 and so on. It is all sequences such that okay, sigma mod Xi. In L1 we just took the sum, in L2 we squared them and now we take the Pth power and take the Pth root is finite. Okay. So, and define for every x belonging to LP, p to be equal to sigma i equal to 1 to n mod x i to the power p raised to power 1 by p.

And then you can define dp x, the distance all right to be equal to x y belonging to Lp, here is something which we would like to prove. Claim, dp is a metric, this is a metric, okay. We have time, so let us try to prove it is a metric.

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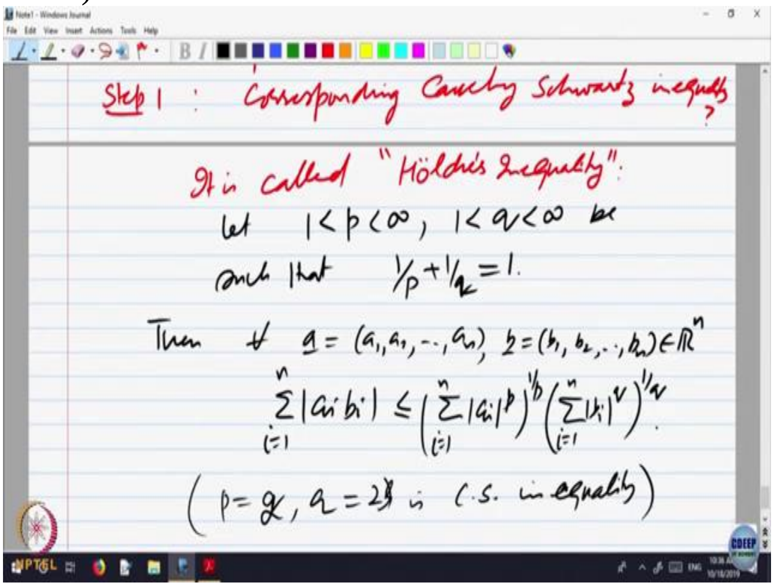

So, to prove this the first step will be trying to prove something Cauchy-Schwarz inequality and then using that prove, triangle inequality. The main problem is the triangle inequality, right? All other two properties are okay, they do not cause any problem. So, only thing is the triangle inequality. Even for 2, p equal to 2 we use Cauchy-Schwarz inequality and then use that to prove that it is a norm, it is a metric only, right.

So, the same route we will follow okay. So, right. So, let us okay. So, step 1, corresponding what should be Cauchy-Schwarz inequality? Right. So, that is called so let me write, that is called, it is called Holders inequality. It is called Holders inequality. Okay. And what does it say? What does it say? So, holders inequality says the following namely, okay. So, let us first state Rn and then do for general thing, okay.

So, let 1 less than p less than infinity. 1 less than q less than infinity be such that 1 over P plus 1 over q is equal to 1. Then for every a equal to say a1, a2, an belong an, let us write another part, a vector b is b1, b2, bn belonging to Rn. If I look at product ai bi sigma i equal to 1 to n, that is less than or equal to okay summation mod ai raised to power p raised to power 1 by P into I equal to 1 to n mod bi raised to power q raised to power 1 over q. That is what the inequality says okay.

Now, let us see why it is a generalization of Cauchy-Schwarz inequality if you take p equal to 2, if p is equal to 2, 1 over P plus 1 over q equal to 1 should give you the sorry p equal to 2, what is q? That also should be 2. So, what is this? 1 to n, 2 raised to power 1 over 2, that is precisely Cauchy-Schwarz inequality. So, then for 2 is Cauchy-Schwarz inequality, it is precisely Cauchy-Schwarz inequality.

And we were able to prove Cauchy-Schwarz inequality very easily by looking at the linear combination of the vector and norm. A more general thing for this, another route of proving this is via looking at arithmetic geometric mean, the relation between them. What is the relation between the two numbers a and b? What is the relation between the arithmetic mean and the geometric mean?

Arithmetic mean is always bigger than the geometric mean, right? But if you think of in the arithmetic mean, in the geometric mean you take the square root kind of right. So, instead of square root if you think of P, 1 over P, you want to generalize that inequality arithmetic mean, geometric mean. So, let me state that and then prove that and then come to this thing, okay.

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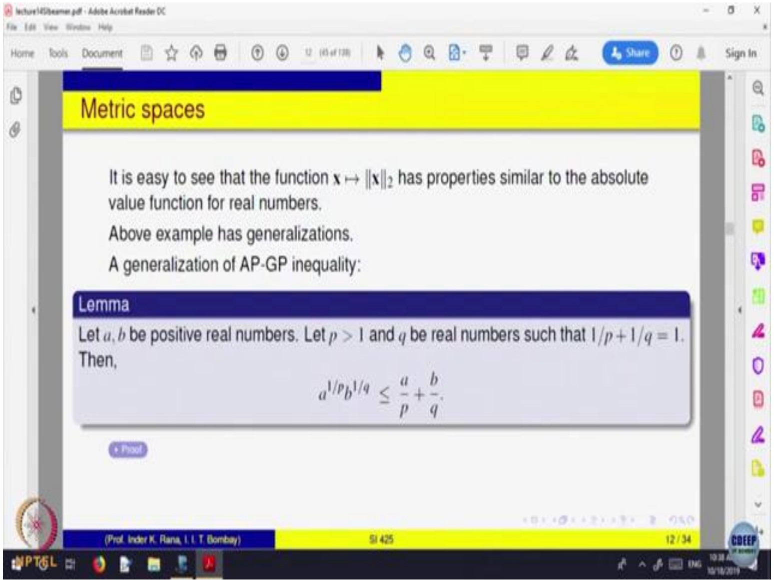

Okay. So, here is the, so to here is the general case, if a and b are positive real numbers and p is bigger than 1, and q is a number such that 1 over p plus 1 over q is equal to 1, then a raised to power 1 over P into B raised to power 1 over q, that is the geometric mean. If p is equal to 2 is equal to 2 is less than a by p plus b by q. Okay, that is the, right hand side is the arithmetic mean.

So, it is a generalisation of that once you want to go beyond powers of 2, okay. So, we will be, one can prove it very nicely, easily. So, probably I think I will go through the proof of this and then using this, one proves holders inequality. That is the generalization of Cauchy-Schwarz inequality and then we will prove what is called the triangle inequality. So, let us see what is the proof of this. So, here things involved are p, q, a and b. So, the idea of the proof is first step, try to bring it to a equation in one variable kind of a thing.

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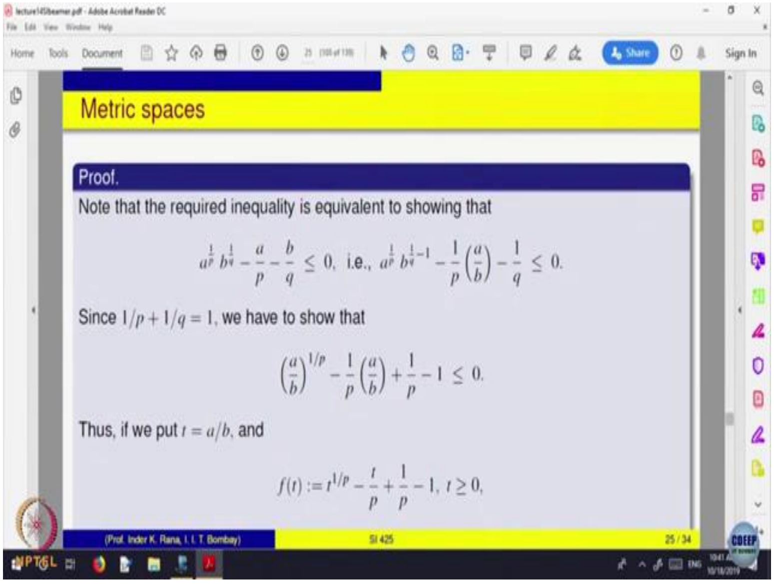

So, here is the, how do you do it? So, here is the idea of the proof. So, this is what you want to prove. I bring everything on one side. You want to show it is less than or equal to 0, right and now write everything in terms of instead of ab, write in terms of a by b. a and b are two numbers given positive, right. If I write everything in terms of a by b then one variable only comes, a and b together do not come.

So, that is the idea. So, let us write. So, this a raised to power p, b raised power 1 over q. What is 1 over q? Keep in mind, 1 over p plus 1 over q is equal to 1. So, 1 over q is 1 minus, so write that okay, is less than or equal to 1. And pq is equal to 1. So, this is same as this equation. The given equation, okay is same as this equation in a by b. And now, q is also gone, only p is there.

So, only two variables, p is fixed for us, a by b, you can call it as a variable t. So, what is this? If I call a by b as t, are you following what I am saying? The given thing less than or equal to 0, I divide by b. Right, so that everything becomes divided by b and 1 over q, write in terms of 1 minus 1 by p. So, that gives me a by b if we call it as t. t raised to power 1 over p minus 1 over p into t plus 1 over p minus 1 is less than or equal to 0.

For every t bigger than a by b you have called as t. So, if I can show that for every t bigger than 0, this holds, then we are through. What we are saying is for every value of t, the function f of t. So, this is a, what is this? This is a value of the function f of t, is less than or equal to 0. That means what? If I want to show f of t is less than or equal to 0, that means what?

The minimum value of the function should be 0. So, let us and that calculus now comes to calculus, you have got a function which is differentiable, derivative equals to 0, that gives you, what is the point of minimum maximum analyze. So, is the calculus proof.

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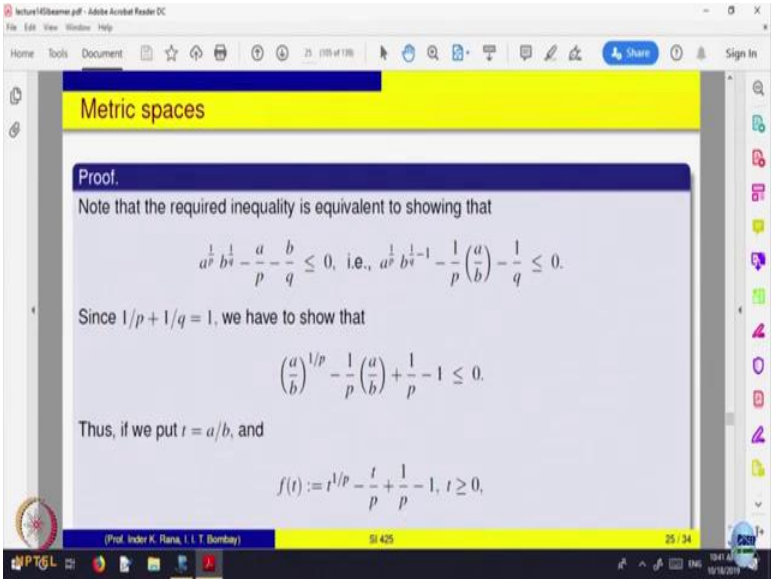

So, let us look at the proof. So, we want to show f of t is less than or equal to f of 1 which is equal to 0. At 1, the value is 0. Is it okay? At 1, what is the value at the point 1? That is 0.

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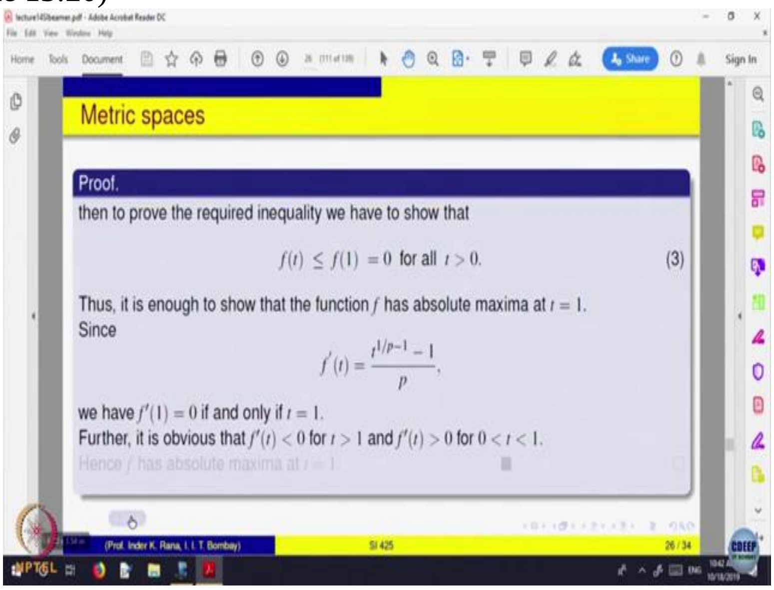

So, we want to show f is a function defined by this quantity and this function has the minimum value 0 at the point x t equal to 1. So, calculus problem. Now, so how do you solve that problem in calculus? Take the derivative, derivative equal to 0 and analyze, either you can analyse the second derivative at that point or just look at the left and right. So, you do that okay. So, one variable calculus. So, let us go back.

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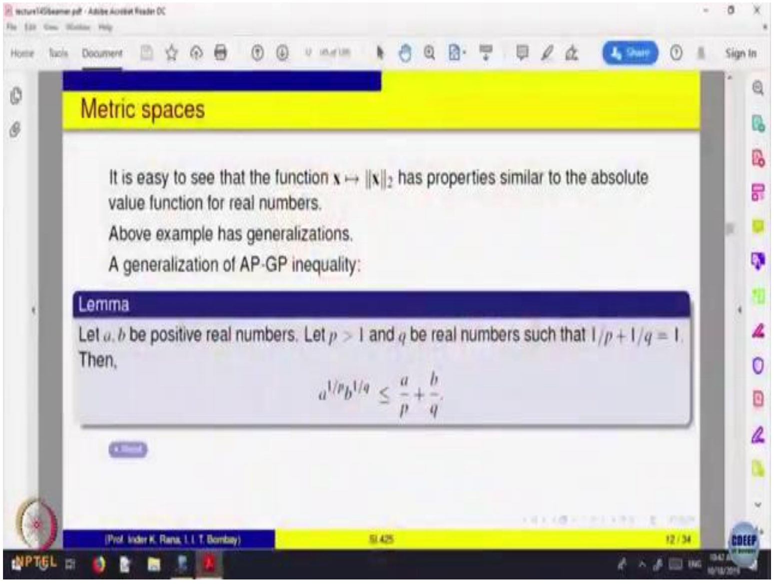

So, this is right, one variable you can improve calculus only, no problem. Using this, we will prove our Holders inequality. So, what is Holders inequality? So, let us go to Holders inequality.

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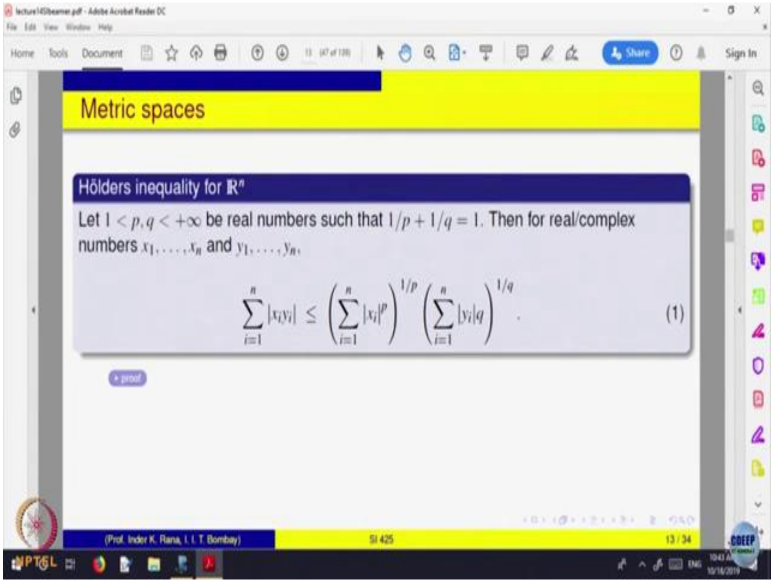

So, that says that if p and q are between 1and infinity then the dot product right, is less than or equal to this p raised to power 1 over p into you raised to power 1 over q, right, generalization of the usual Cauchy-Schwarz inequality. For p equal to 2, q equal to 2, it is a usual Cauchy-Schwarz inequality. And the proof is very simple if you look at the idea, the same idea that if all xis and all yis are 0 right?

Here there is a typo here, this q is the power okay. This q should be the power, I will correct it. If all xi or and all yi are 0 then both sides are equal, nothing to prove, right. So, assume that the vector x with component x1, x2, xn and the vector y with components is not 0. Divide by that right if required, then you get vectors with components x is equal to 1, y equal to 1 and you have to show that this inequality holds.

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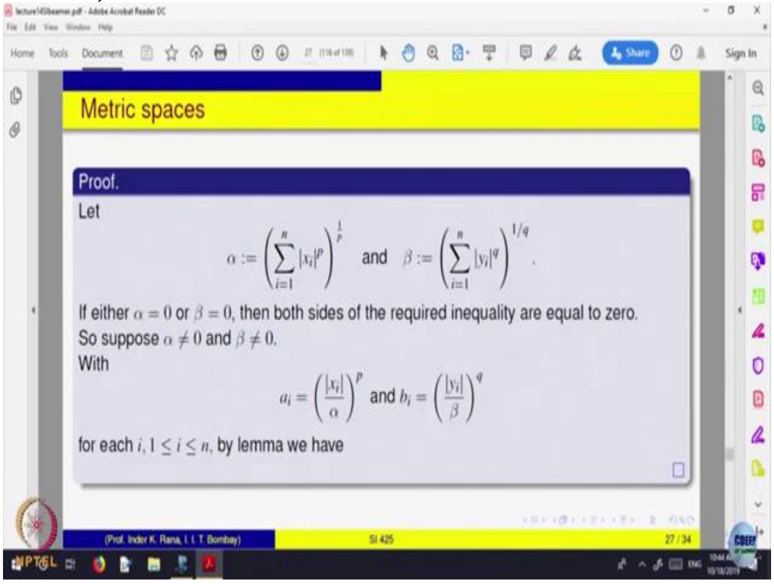

So, let us look at a proof of that. So, that is the basic idea. So, divide alpha is norm right? Here the norm is p raised to power 1 over p. So, divide the vector by each component by alpha, each component by beta right? Call this as ai, call this is bi. Apply that lemma, a raised to power p into b raised to power q is less than or equal to right?

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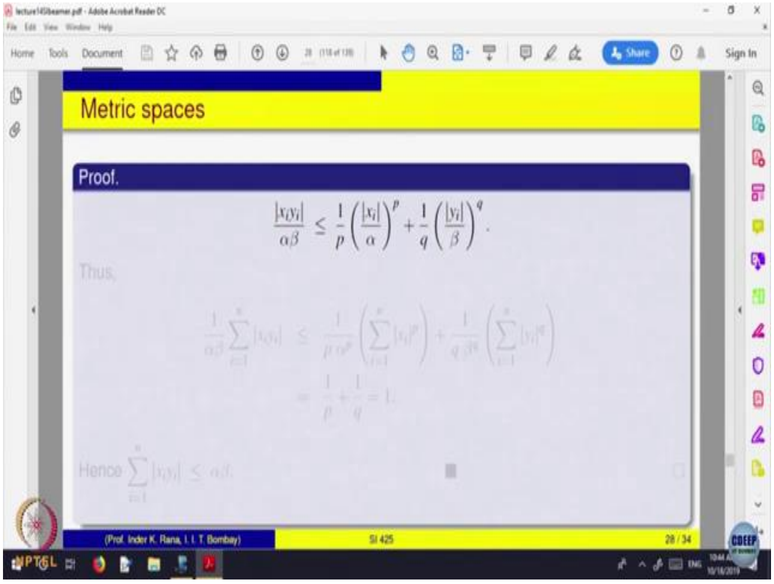

So, apply that with this each alpha I, so you get right, apply for each alpha I, you get this component, okay. a raised to power p plus b raised to power q. So, ai raised to power okay xi to the power p.

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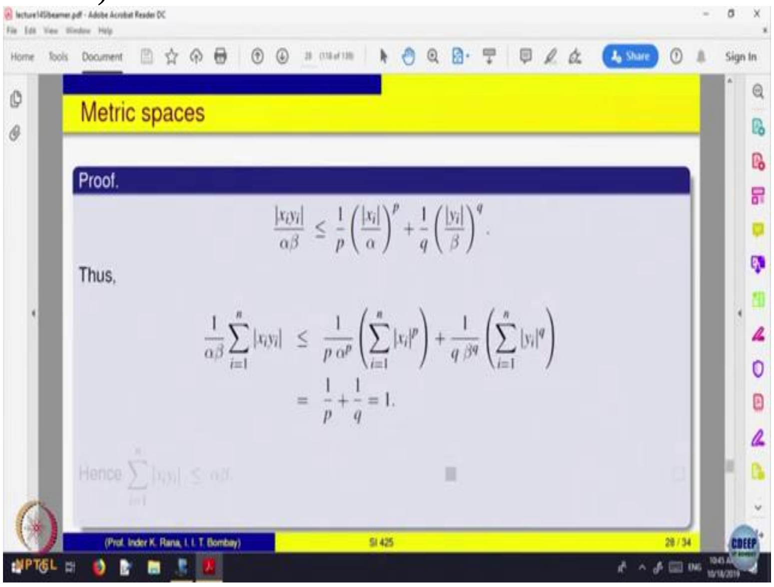

So, if you apply to each you get this. Summation now this is for every i. So, for each component I am applying that previous inequality right. Now, sum it both sides over i. So, when you sum it up i equal to 1 to n, i equal to 1 to n plus i equal to 1 to n, what is sigma x i to the power p? What is this quantity? What is this quantity? That is precisely alpha to the power p right, alpha was the norm and this is beta to the power p.

So, they cancel out so it is 1 over p plus 1 over q, that is equal to 1. So, it says this is less than or equal to 1. So, that means sigma mod alpha i y i is less than, almost the same proof. Only instead of using that arithmetic mean is less than, is bigger than the geometric mean, we are using generalized inequality. a raised to power p into p raised power q is less than or equal to ap divided by. So that, so that is so now I think let me do some writing also.

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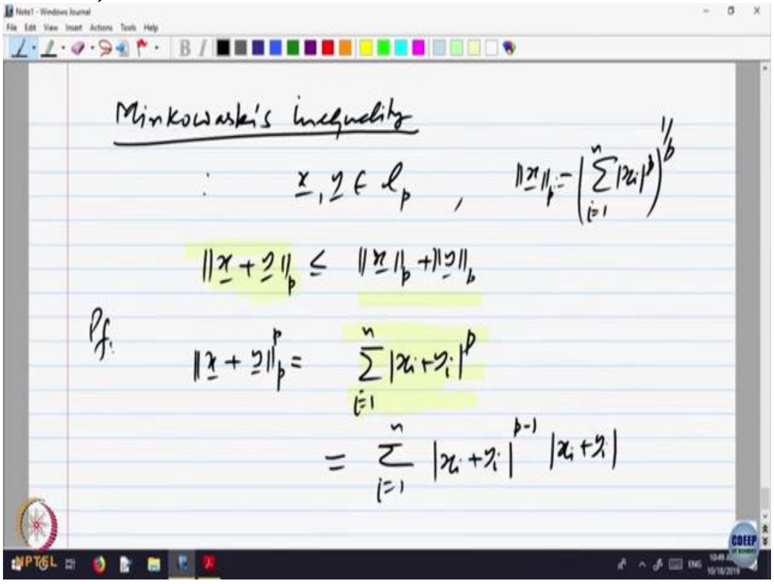

So, we got a Holders inequality using this we want to prove that what is called Minkowski's inequality. Do not go by, get scared by the name. It precisely says for X and Y belonging to LP norm of x plus norm of y is less than or equal to the pth the triangle inequality for this. So, what was norm? So, that is defined as sigma mod x i to the power p i equal to 1 to n raised to power 1 by p right, that is the definition that this is equal to this.

That goes by the name Minkowski's inequality. Okay. So, let us give a proof of that. So, proof is reasonably simple. So, let us look at the left hand side. So, that is norm of x plus norm of y p. What is that quantity? So, that is equal to sigma mod x i plus yi raised to power p 1 to n whole thing raised power 1 over p. So, let me raise the power p so that I do not have to write every time.

So, left hand I raise the power. So, that is equal to this, right? Now, you see, here is how one thinks of a proof from this product, from this product, this is a product. Which is the left hand side? I want to split it into two parts, sum. Right-hand side is a sum, left-hand side is a product. So, I have to split this left hand side into a sum somehow, two terms, otherwise I cannot reach the target.

So, how I can split it? The idea is very simple. So, let us look at that. So, write this equal to summation i equal to 1 to n mod xi plus yi raised to power p minus 1 into, right, is it okay? Why I am doing that?

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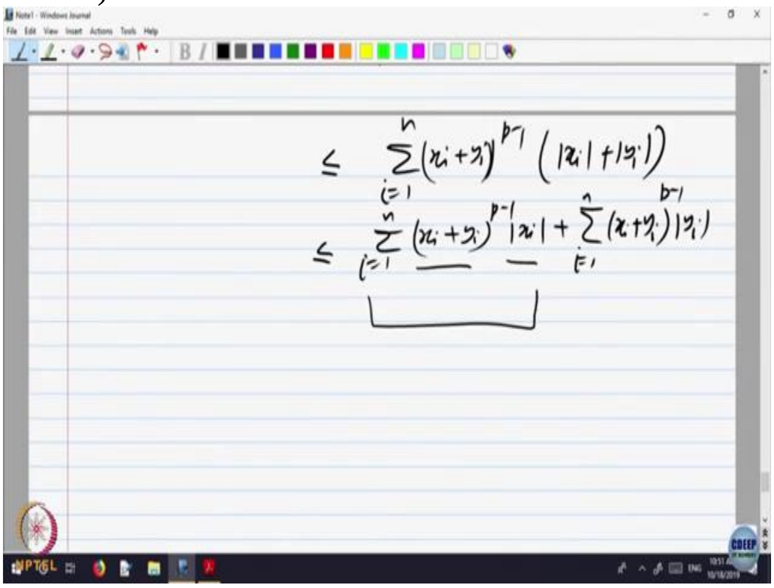

Because this gives me immediately that this is less than or equal to if I look at this quantity that is less than or equal to mod xi plus mod yi, so let us write that. Summation i equal to 1 to n xi plus yi raised to power p minus 1 into mod xi plus mod yi right, using triangle inequality. So, that is less, now let us split the two terms okay. So, less than equal to summation i equal to 1 to n xi plus yi raised to power p minus 1 into mod x i plus sigma of x i plus yi raised to power p minus 1 into mod of yi.

Now, I have got two terms, at least I am nearing the target right. Now, idea is somehow make these to each sum right? What do we want? We want this quantity to be less than or equal to norm of xi plus, right? So how do we do that? So now, this product okay now, let us use our Holders inequality in this. You remember in proving Triangle inequality for two we use Cauchy-Schwarz inequality.

Same thing we are doing here. Now, at this step we will use holders inequality, this is one term, this is another term, product of these two sigma mod should be less than or equal to, what is Holders inequality?

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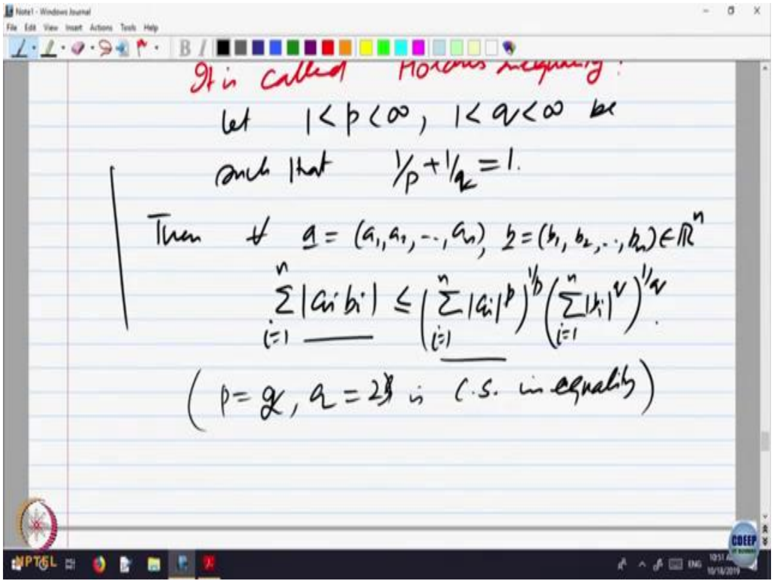

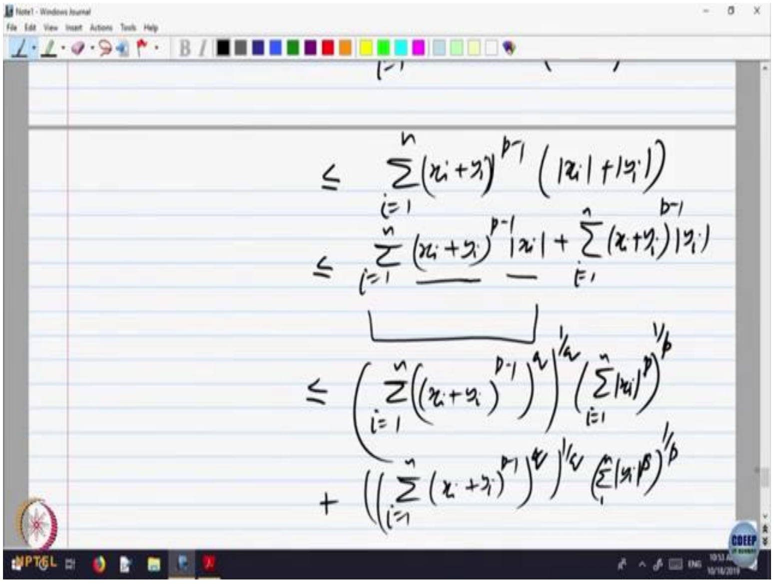

Here is Holder's inequality. Product of ai bi is less than ai to the power p raised to power 1 over p into bi raised to power you 1 over q right. So, here let us use holders inequality, so less than or equal to, so sigma i equal to 1 to n xi minus yi raised to power p minus 1. So, this one divided by raised to power P right sorry. So, this is my ai, that is bi. So, p minus 1 raised to power p raised to power 1 by P, let me write because xi, okay i will write q here and instead of p let me use q for this and p for because xi is belong to 1 power p.

So, q to raised to power, 1 over q sigma mod xi to the power p i equal to 1 to n raised to power 1 by p. Okay. Is it okay? I have called the ai bi, so ai is xis, bis are the other ones. Similarly plus the second term, so second term will be sigma i equal to 1 to n xi, sorry it is plus or minus. It was plus, plus yi raised to power p minus 1 raised to power q raised to power 1 over q into mod of summation mod yi raised to power p 1 to n raised to power 1 by p.

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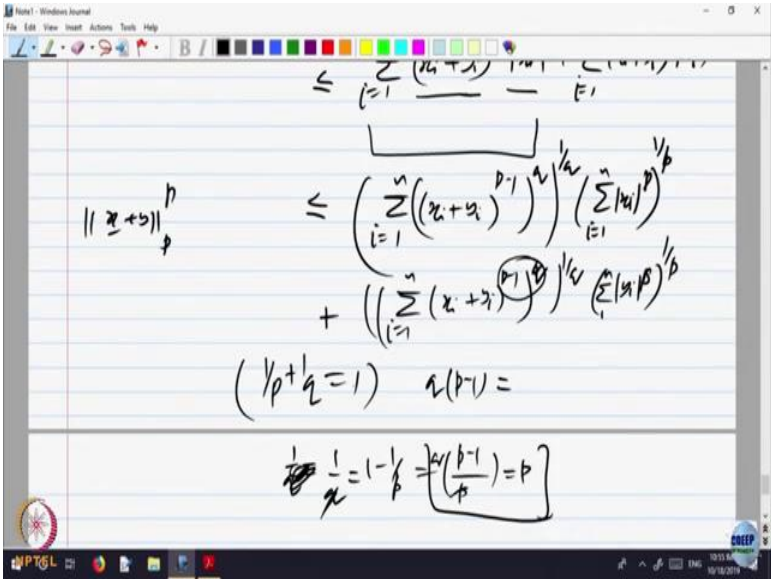

And what was on the left hand side? Keep in mind, on the left hand side was mod x plus y p to power p. So, that is less than or equal to this right? Now, one uses the fact that 1 over p plus 1 over q equal to 1. So, use that, okay. So, what is this? So, this is p minus 1, I am just working out roughly here. p minus 1 times q. So, what is that equal to using this fact? So, if you want q time, so 1 over q is equal to 1 minus 1 by p.

So, that is equal to p minus 1 divided by p, right. So, we have put q here. Is it okay? 1 over p plus 1 over q is 1. So, what is 1 over q? It is 1 minus 1 over P and that is p minus 1 divided by p, q I take it there. So, this quantity is equal to 1. So, this quantity is 1. So, this quantity is q times, so what is it ? Divided by p. So, this is equal to times p right.

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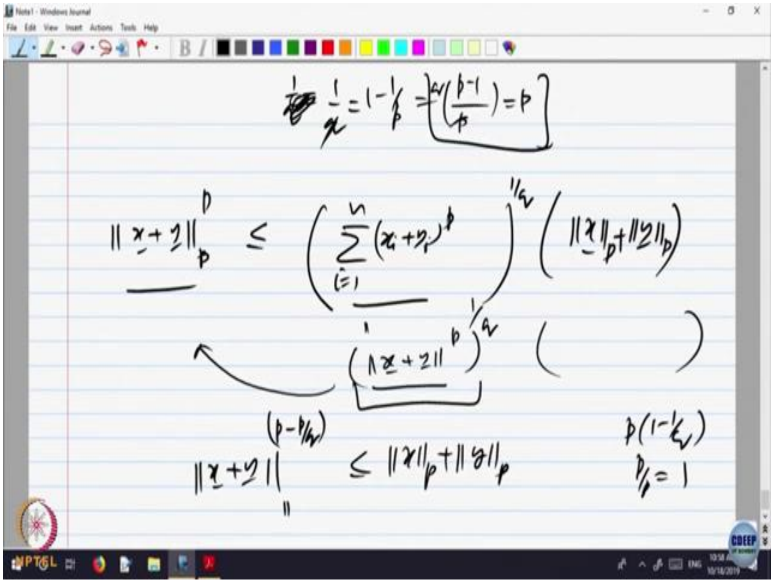

So, this quantity comes out to be, so the right hand side So, let me write. It is just simply arithmetical simplification of this powers, nothing more than that. So, mod x plus mod y p to the only 1, not 2 p is less than or equal to this quantity. That is same in both right. So, let me write it outside that quantity. So, that is summation xi plus yi summation raised to power 1 over p. summation of xi plus yi raised to power p equal to 1 to n.

So, that is equal to p raised to power 1 over q into norm of x and that second term is norm of, right. Can you see a relationship between this and this? This is norm of, so this quantity is norm of x plus y right, p raised to power 1 over p is the norm, so it is just norm of p raised to power q 1 over q. Okay. You can transport it on the left hand side. So, this shifted on this side. So, what do you get?

So, norm of x plus y raised to power p minus 1 over and on this side divided by both sides by this term, so when you divide what you will get? It is less than or equal to, is that okay? This is to the power p, yes? This is to the power p and this is to the power p, okay raised to power p by q, sorry, not the same okay. So, this is okay, this is okay, so p minus sorry anyway I will be getting problem if I do not write it correctly. p minus p by q right because this thing is p by q.

Mod norm of x plus y raised to power p by q and that is power p, so shift it to other side and what is this quantity left hand side? What is p minus p by q? 1 over P plus 1 over q is equal to

1. So, that is the quantity. p into 1 minus 1 over q if you like. So, that is 1 over p, p by p, that is equal to 1. So, this quantity is just 1.

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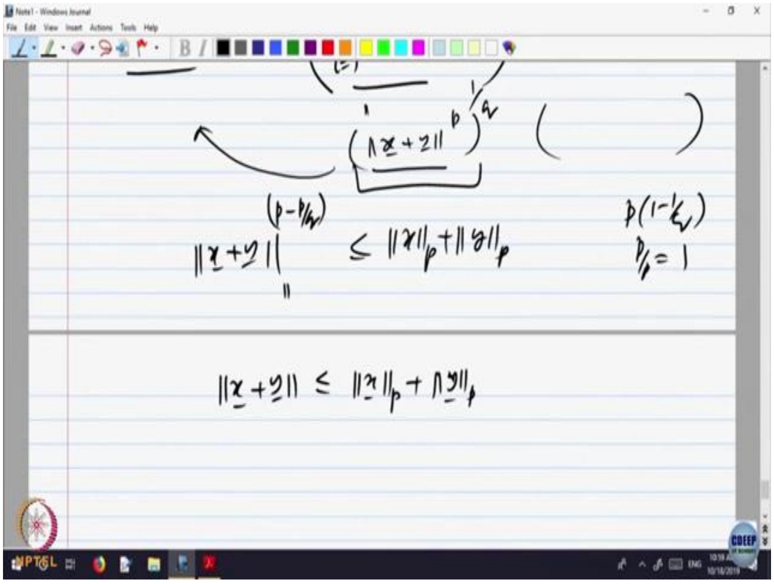

So, you get the required thing that this is less than or equal to p plus norm y. The basic idea is of the proof is going parallel to L2 case, right. You first generalize Cauchy-Schwarz inequality to instead of p equal to q equal to 2 right, you generalize it to Cauchy-Schwarz inequality called Holders inequality where a product, right, is less than or equal to a sum.

Basically that is what the AMGM is. A product is less than equal to sum, arithmetic mean is bigger than the geometric mean. So, generalise that quantity, okay and that gives you the Holders inequality and the proof from how do you use holders inequality and Minkowski's inequality is precisely from the product you want to go to the sum.

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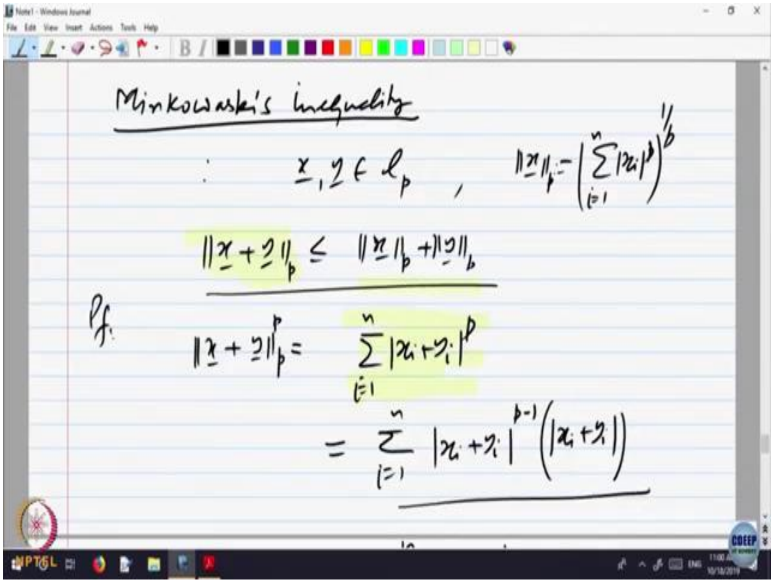

So, this is a crucial step. Split the power into two parts, that into minus 1 right, absolute value this minus 1 into this and this gives you less than or equal to norm. So, splits into two terms right and then there is only the arithmetic jargon, you simplify the powers and all that thing. So, that is okay. So, that is Minkowski's inequality.

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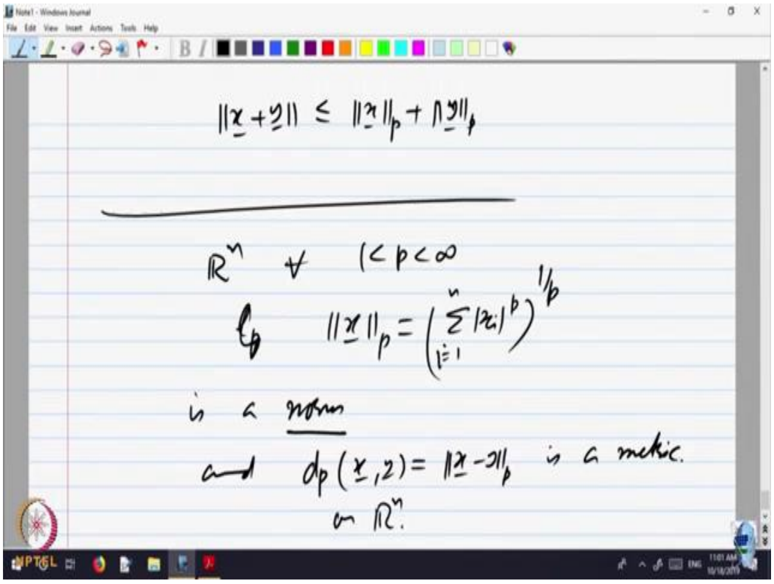

So, that says okay, on RN for every 1 less than p, actually we have got 1 also. You can define on Rn, you look at what is called, because it is finite, so no problem. So, define norm of x to p to be to be equal to sigma mod xi i equal to 1 to n to the power p raised to power 1 over p is a norm. So, what does norm mean? It is something like absolute value. Okay. We give it a well name as norm okay.

And dpxy equal to dp of xy is equal to x minus y p is a metric. This is a metric on Rn. So, on the same set Rn you have got infinite number of metrics. For every p, even p equal to 1 also we have done it. p equal to infinity also we have done it. Every power in between p bigger than 1 and less than infinity, also were getting a metric. Okay.

And the next thing what we will show is not only a metric on Rn, it is a metric on R infinity also. It is a metric on. Oh, that is a minor step because these are partial sums already, 1 to n kind of thing we have it, okay. And from there we will go to what should be the next step? R infinity what should be the next thing? R1, Rn, R infinity. What should be the next thing? Anybody has a guess? Is there anything beyond R infinity? R to the power infinity? So, think about it till the next lecture. Okay.