Basic Real Analysis
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Lecture 56
Metric Spaces - Part I

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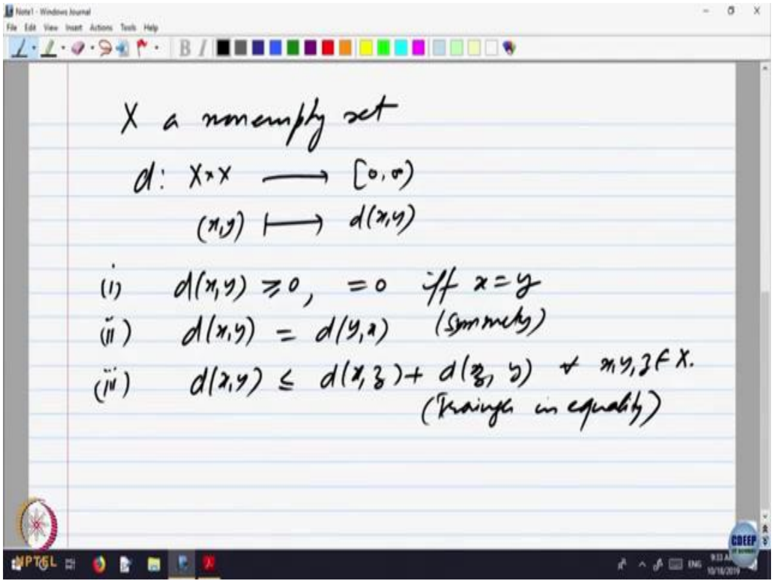

Okay. So, let us start looking at the notion of a metric space. So, x, a non-empty set and d is a function on x cross x to non-negative real numbers. So, x y goes to dxy, with the following properties. One, the distance we will call it as, D as the distance is always bigger than or equal to 0. Equal to 0 if and only if x is equal to y. Distance between x and y is same as the distance between y and x.

And the third property the distance between x y is less than the distance between x z plus distance between z y for all x y and z belonging to x. So, this is symmetric and this is triangle inequality.

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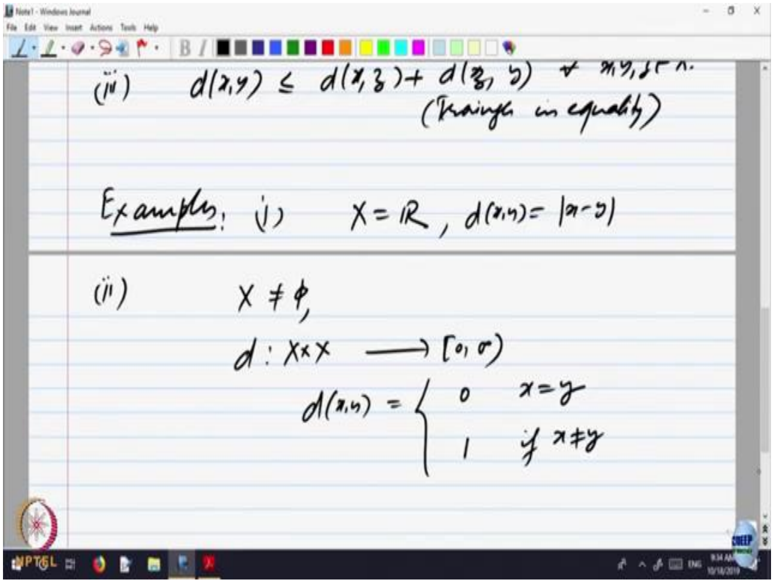

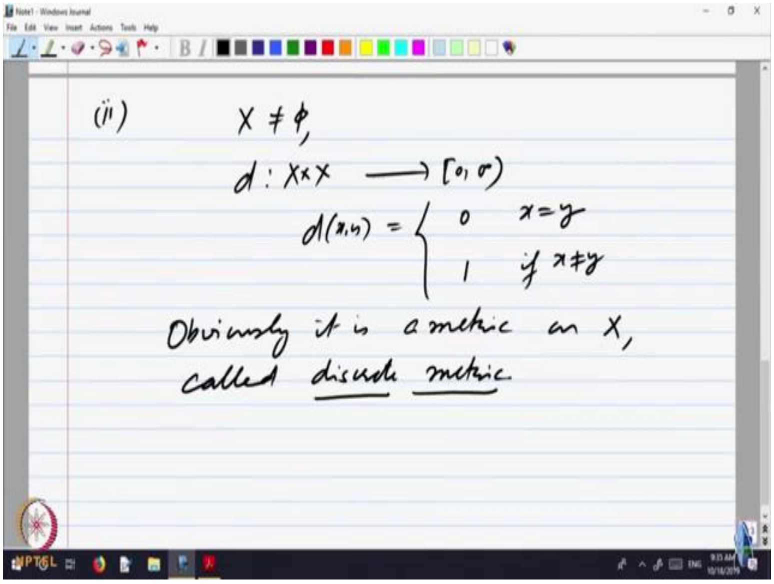

So, essentially this is modelled on. So, let us look at examples. The motivating example was that x is the real line and dxy is mod of x minus y. So, the idea was that this notion of distance gives you the idea of saying something is close to, two points or close to each other and gave us many concepts on real life. So, let us look at some more, for examples, let x be any non empty set and define the d on x cross x.

So, let us define dxy is equal to, it is to be 0 anyway when x is equal to y and let us put it equal to 1 if x is not equal to y. So, distance between any two points is defined to be 0 if of course x is equal to y and two distinct points, the distance is defined to be equal to 1, okay. So, easy to check that obviously, it is symmetric, non-negative, 0 if and only if x is equal to y and dxy is less than or equal to the triangle inequality is again obvious, because the right hand side could be either 0 or bigger than 1.

So, obviously, it is a metric. So, any set has got this kind of metric called discrete metric. So, this is called discrete metric on any set. Not very interesting, but in some way okay. Let us look at some more.

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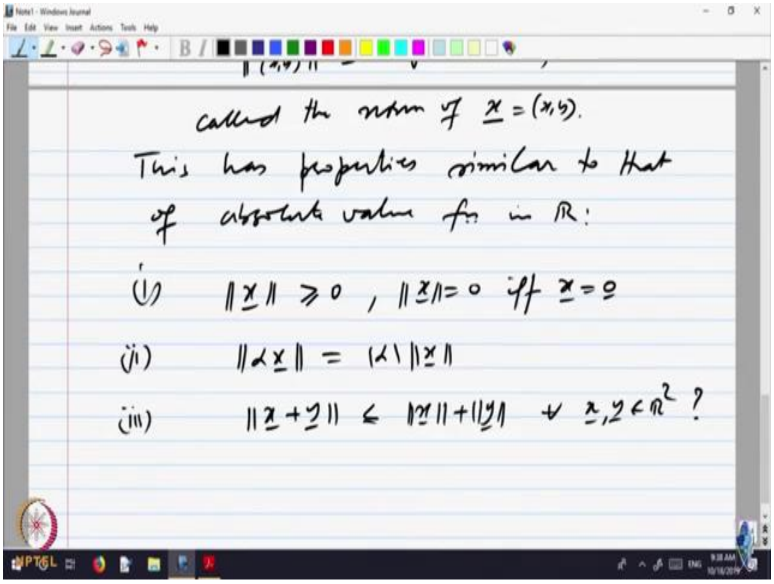

We had, let us take x is equal to R2 and let us take a point x y belonging to R2. In the real line we had the notion of absolute value of a number, right and that was the distance of for the point from 0. So, in R2 let us introduce that, so the distance of the point xy from 0 so let us, so that is x square plus y square called the norm of the vector with components xy. I would like to check that this, this has properties similar to that of absolute value function in R.

So, meaning what? One, the norm of, we will be using underline x for the vector with components x and y is bigger than or equal to 0 and it is equal to 0 if and only if the vector x is the 0 vector, right. Let us look at the second one that says that if I multiply a vector by alpha, then this is mod alpha times absolute value, absolute value of ab is equal to absolute value a into absolute value b.

And the third property namely, x plus y is less than, less the triangle inequality property for all xy belonging to R2. One way of looking at, that would be that actually if we look at a triangle in R2 with sides X and Y then the hypotenuse, right, the length is less than or equal to or triangle inequality, any two sides, the sum of any two sides is bigger than, let us try to prove it analytically. I would like to prove this that for any two vectors this is so.

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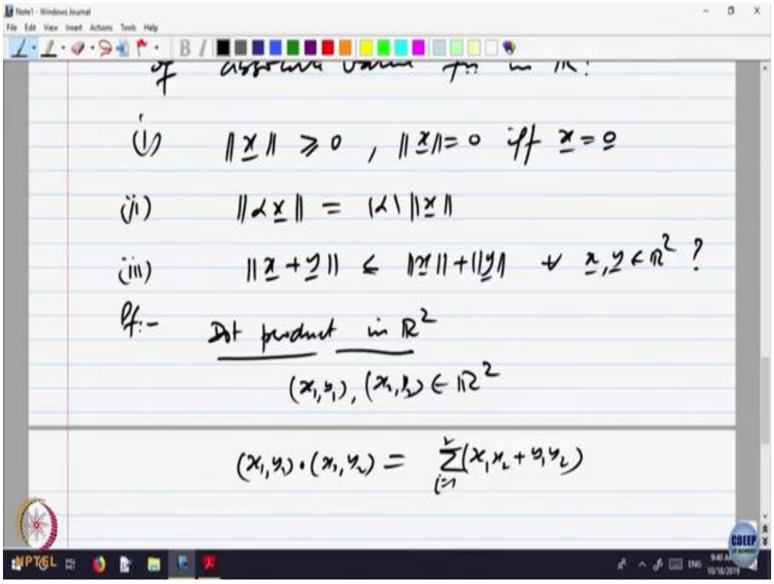

Let us try to introduce something which is interesting in its own respect. So, what is called the dot product in R2? So, given X1Y1 and X2Y2 belonging to R2. One defines the dot product. So, this X1Y1 product X2Y2 to be equal to sigma X1X2 plus Y1Y2. In fact, it will be good idea to prove it for all, not only R2, let us prove it for Rn, okay, but anyway we will do it a bit later probably.

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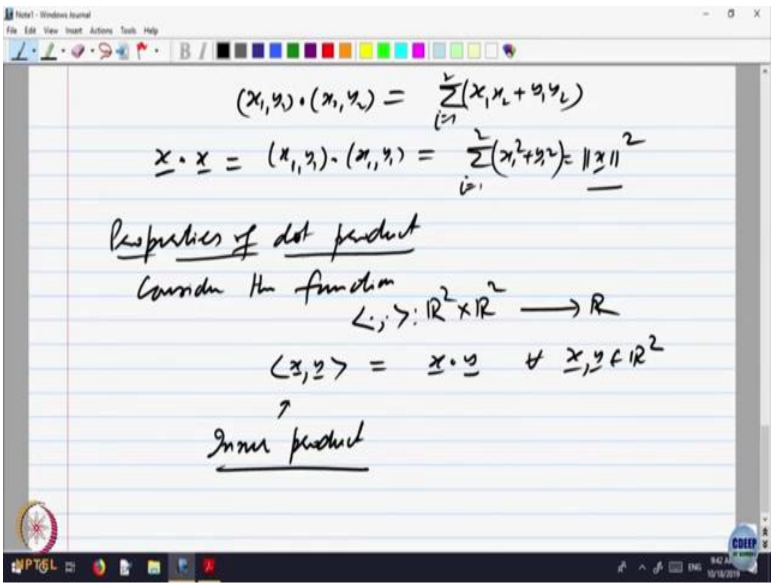

So, this is called the dot product in R2 right. What is a, why we want to define that? So, dot product, okay, so if I look at dot product of x dot x itself, there is X1Y1 dot product with X1Y1, that is sigma x i square equal to 1 to 2 and that is norm of x square. Right. So, that is the most interesting thing that happens. It relates the dot product with the notion of magnitude or absolute value of a vector.

Oh, I have written xi, okay. So, let me, because I have used a different notation here, so let me write that. Because components are, so it is x1 square plus y1 square. So, that is equal to norm x square, right where X has components X1 and Y1. Okay. So, this dot product is related with the distance in this following way, right. It also has interesting properties. So, let me write some properties of dot product.

Let me write the dot product as a function. So, consider the function R2 cross R2 to R. So, how was it defined, it is a dot product. I am just using a different notation so that things become very clear. So, for a vector X and a vector Y, okay, that is the dot product of x with y for every x, y belonging to R2. For any two vectors x and y the dot product is this notation I am using.

So, with this notation it is called the inner product. Dot product is also called inner product, it is a product of a vector with itself, right. So, why we are doing this?

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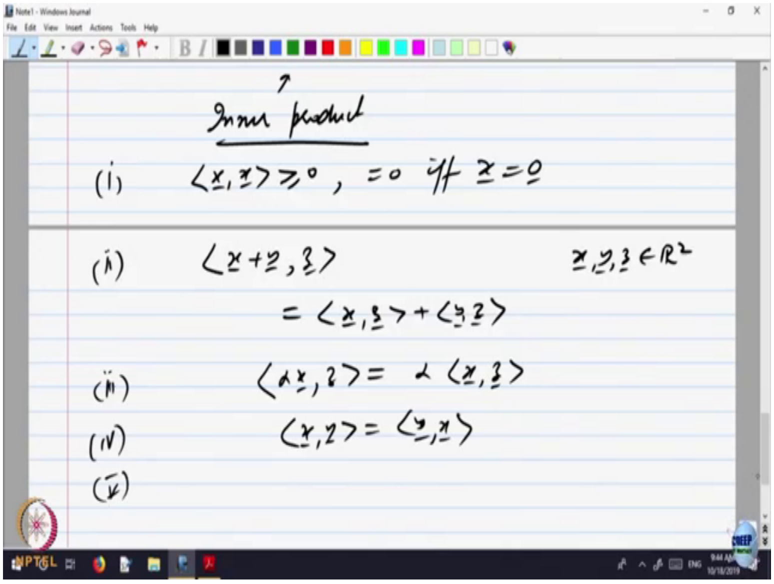

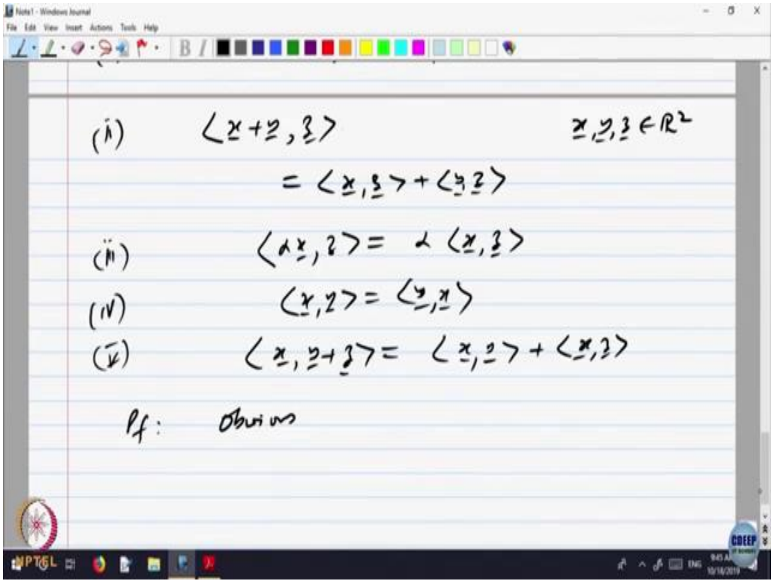

The reason is the following. So, you can treat it as a function of two variables okay x and y, x belongs to R2, y belonging to R2. So, the properties are, one, is always bigger than or equal to 0, equal to 0 if and only if x is equal to 0, right that is obvious that we already seen that. But anyway because of the sigma xi square, so that is 0 means each component must be equal to 0.

Second, let us take two vectors x and y. So, let us take vectors x, y and z belonging to R2 and let us look at the vectors x plus y comma z. So, what is that equal to? So, that will be, components of this will be sigma xi plus yi and comma that will be zi, so product, product distributes over a addition, so this is same as x comma z plus y comma z, right. So, saying in the first component if I add and take the dot product is same as the sum of the dot products.

And third property, in fact I can write here alpha XZ is same as alpha times XZ, multiply scalar comes out in the dot product because sigma xi, so alpha will come out from the sum. Fourth property, x y is same as y x. It is symmetric. Right? Dot product of x with y is the same as dot product of y with x and four and fifth. Similarly, this combined with the, second combined with this one will give you that x comma y plus z is same as x plus y, sorry, x comma y plus x comma z.

So, essentially what we are saying is positive definite the first property, it is linear in the first variable as well as in the second variable. And it is symmetric. Linearity because of alpha also comes out. So, it is a linear thing. Okay, we will come to the generalizations a bit later. So, let us say, it has obviously these properties, so proofs are obvious, right easy to verify, just ordinary algebra, right.

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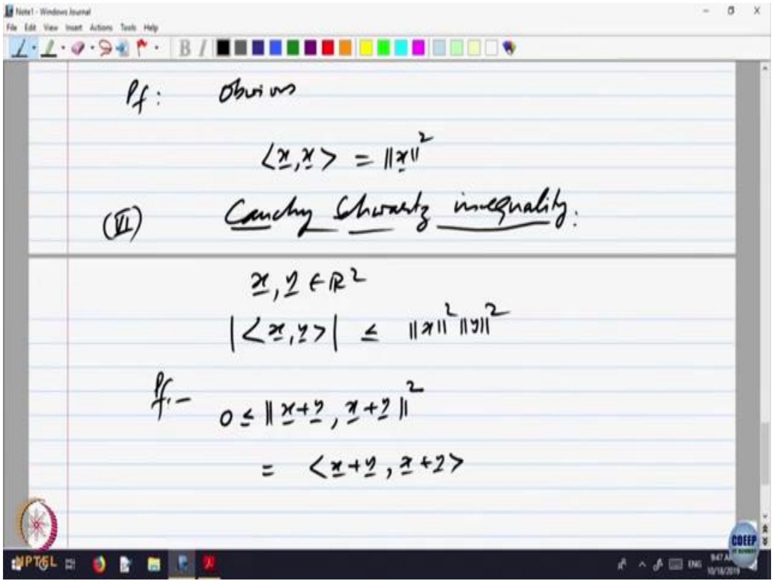

And of course, we had that fact, namely x dot x is equal to norm of x square. Right. What we want to do is prove a property of this, which is very important called Cauchy-Schwarz inequality. So, what does it say? It says if I take vectors x and y belonging to R2 and I look at the dot product of xy, that is a real number, take its absolute value that is always less than or equal to norm of X square into norm of Y square. Right?

And there are many ways of proving it. Let us look at a way which we generalize it to everything. So, let us proof of this. Look at x plus y, x plus y, the vector x plus y, x plus y and look at the norm square of that. That is always bigger than or equal to 0. Let us write this in terms of the dot product. So, this is x plus y, x plus y, norm square is equal to the dot product of the vector with itself.

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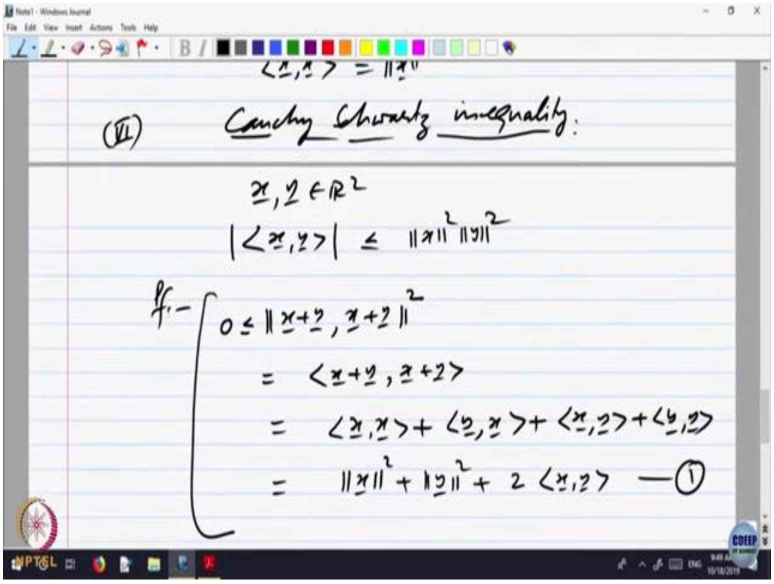

So, let us expand it using linearity okay. So, that is x comma x plus y comma x plus x comma y plus y comma y. So, that is equal to norm of X square plus norm of Y square plus two times norm x norm Y, two times xy, right. Okay, that will not lead it to is bigger than or equal to 0. I think this let us just keep it as 1, this will not give us the required thing.

Let me just look at Okay. All right. So, one way of this would lead to the proof but just keep it as it is. There is one obvious way of looking at it, but that requires a knowledge of saying that the dot product a and b is norm a norm b into cos theta but how does that formula come? Okay, so we are just debating about that, whether I should.

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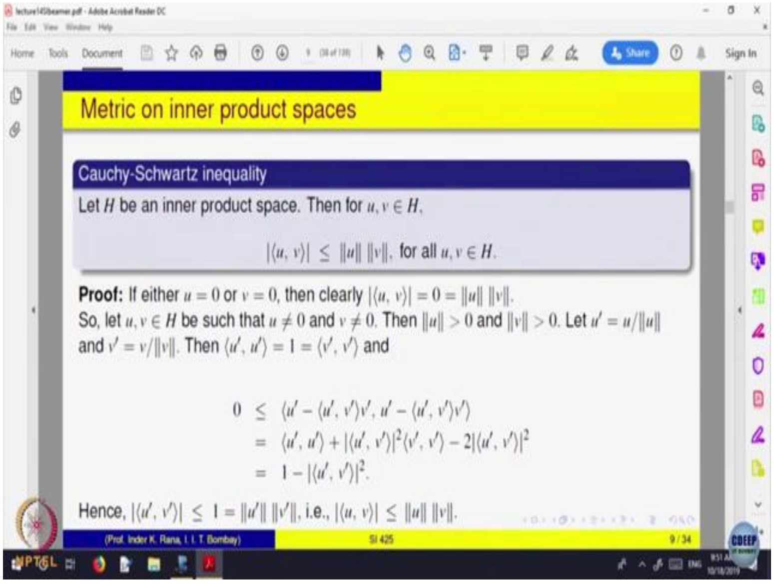

Yes, this is another way of, S just look at this proof. One can use calculus also, we will do the calculus proof later. So, we want to prove this inequality that the dot product less than equal to norm u norm v, obviously if either u or v is 0, then both sides are 0, right. So, no problem. So, let us assume both of them are not 0 and divide u by norm of u. Right.

So, define a new vector called u dash and V dash. u dash is u divided by norm u. Because it is not 0 I can divide by it. So, what will be the norm of u dash? That is 1 now. Norm of V dash also is equal to 1 okay. So, let us look at this term u dash minus the dot product V dash with itself okay. So, that is a vector we should look at. So, let me, are you able to follow from here or shall write it up? So, let me write properly.

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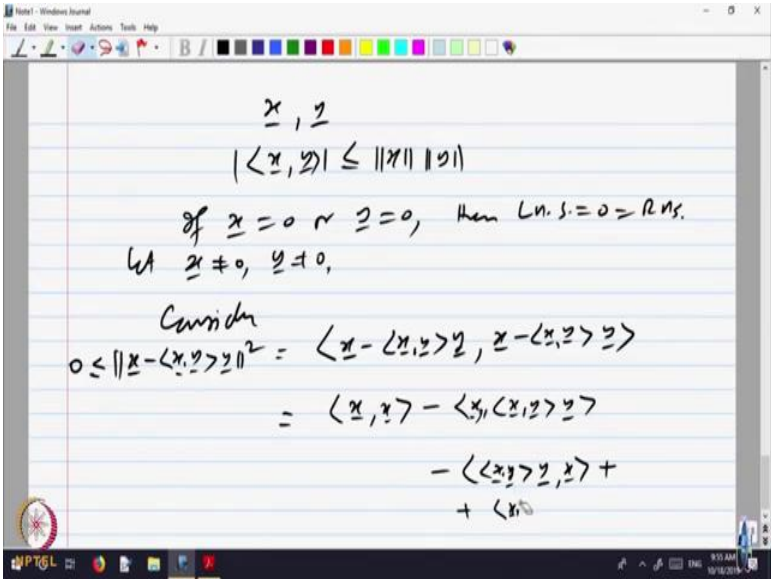

So, we had what we have written x and y. So, let us keep it x and y. x and y are vectors, so we want to prove norm of is less than or equal to norm x norm y. Right. So, let us, so if x is equal to 0 or y equal to 0, then left hand side is equal to 0 equal to right side. So, no problem. So, let x not equal to 0, y not equal to 0. Consider so this is what, this is a key thing, okay.

So, let us right, x minus x y y and product with itself. So, x minus x y with y. So, what is this quantity? On one hand this is equal to norm of x minus x y times y square, right, norm of that square. Same vector with not product with itself, which is bigger than or equal to 0. On the other hand, let us expand this, what is it equal to?

So, this is x comma x first term minus x comma x y y, right, then so it will be 2 times, right? x x first term and then x with this one, x with itself and then x comma minus yes. So, that we have taken, so this one. So, that minus sign, so minus x y y x plus algebra. So, last term will be minus minus plus. So, x.

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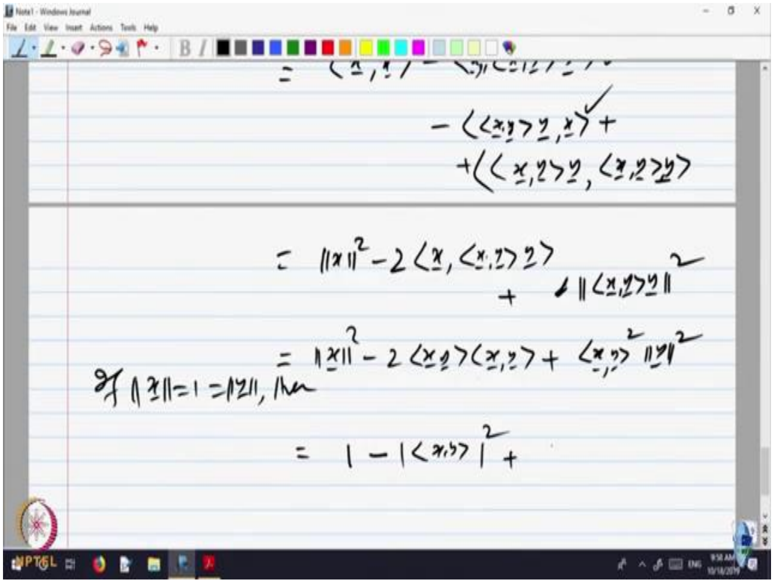

So, fourth term will be minus minus plus, so x y y, comma x y y right. We will have four times, first term x with itself, that is x comma x, dot product, second term will be taken x with the second term here, so that will be x comma this with a negative sign, negative alpha comes out and next will be minus of that thing with respect to, is that okay? So, minus x y y, that is x okay and this thing. So, that is equal to norm of x square minus two times, so this is two times x.

So, xy okay so let me write x y and y. So, these two, this term and this term together, so minus 2 plus, so this is norm of xy y square right. Yes, okay. So, this is, so scalar comes out, norm of x square minus 2 times x y okay. And this is another scalar, so that is into x y. One scalar comes out and the same thing, plus XY square norm of y square, right.

The scalar comes out, XY square and norm of Y Square, fine. This is also same quantity, so let us write it x square minus the price of that same coin. If, so let us write, if norm x is equal to one is equal to norm y, then what do we get? In case so that is what we were looking at, 1 minus, so this will be x y. Right? So xy xy Square, so you can write plus yes, that is all.

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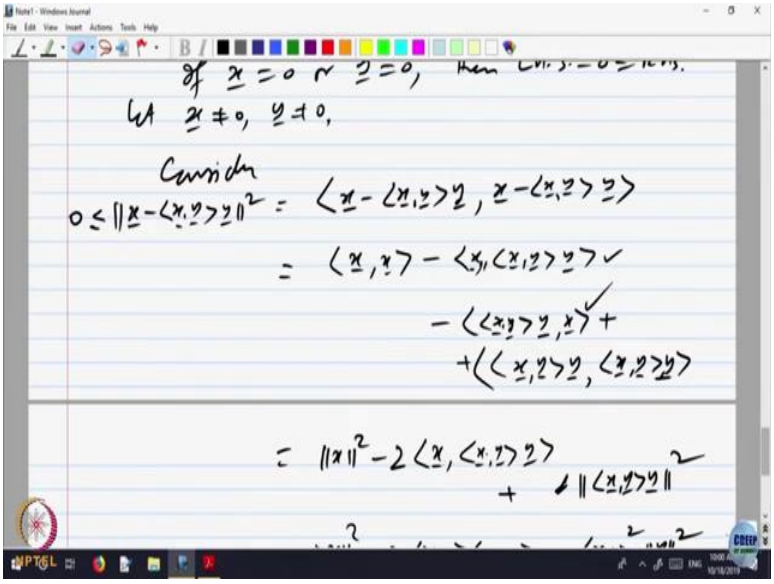

Now, okay let us go ahead and see what we are writing is. So, I am looking at this vector, x minus dot product of x with y times y. This is a scalar, right? This dot, this is a scalar, scalar times y. So, I am looking at a x minus a scalar times y, norm of that. That is bigger than or equal to 0. So, that will be equal to dot product of this vector with itself, okay? Now, open this, this is this minus this plus not product this minus this, so linearity will give you four terms.

So, what will be the 1st term? Dot product of x with itself. So, that is the first term here. Second term will be dot product of x with the negative of this thing. So, that negative sign I have taken it out, so dot product of x with that vector, other will be the switched over negative sign. So, two cross terms and then the last one will be the dot product of this vector with itself.

So, this vector with itself, four terms right. Now, let us simplify. The first one is norm of x square, okay. This term is same as this because of symmetry, okay. And this scalar comes out. So, it is two times norm x, dot product of x y with y and y. So, that is these two terms are combined together to give you this, two times that. And the last one is norm of dot product x y y, okay.

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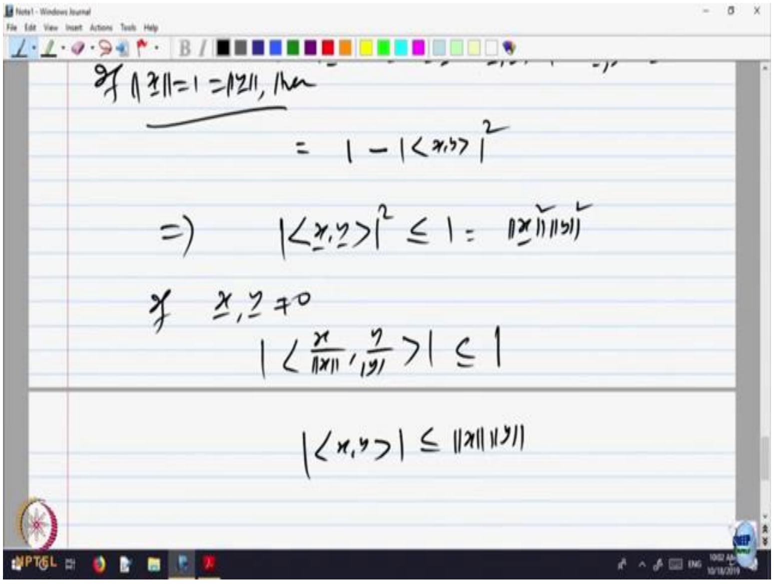

Let us simplify that a bit further. So, this scalar comes out x square norm of y square. So, this is this term. And now let us assume norm x is equal to 1, norm y equal to 1. So, what do you get? So this quantity is 1, this is 2 times and this is 1 here. So, it is only minus 1 times this is ok. So, what does and this is bigger than or equal to 0. So, that is what we said, this is bigger than or equal to 0.

So, implies the norm of x y, sorry x y squared is less than or equal to 1 okay. And what is 1? You can write it as norm of x, norm of y because norm x we assumed to be 1. So, what we are saying is this proof works very well when norm x is equal to 1, norm y equal to 1. If not then what you will do? We already seen if not, you can divide x by norm x, y by norm y.

So, if x y not equal to 0, the same thing will give norm x over norm x, y over norm y. Now, both are off norm 1. So, if I take dot product, this will be less than norm x over is less than or equal to 1, right? And that means dot product of x y is less than the norm of x into norm of y. Okay? That is all.

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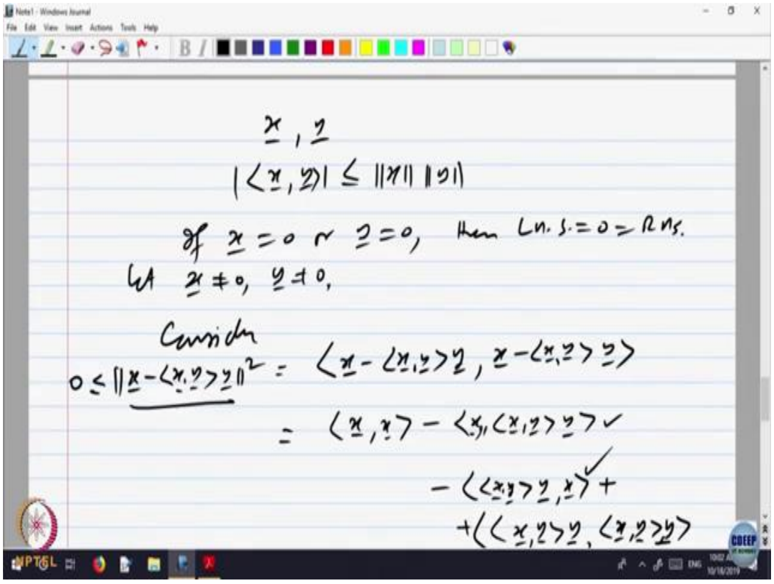

So, basically the idea is take that linear combination namely and expand it. One side it is bigger than or equal to 0, other side by using the property of the dot product, okay. So, this, this proves Cauchy-Schwarz inequality. So, what I wanted to do was using that, so this is not required in Cauchy-Schwarz inequality.

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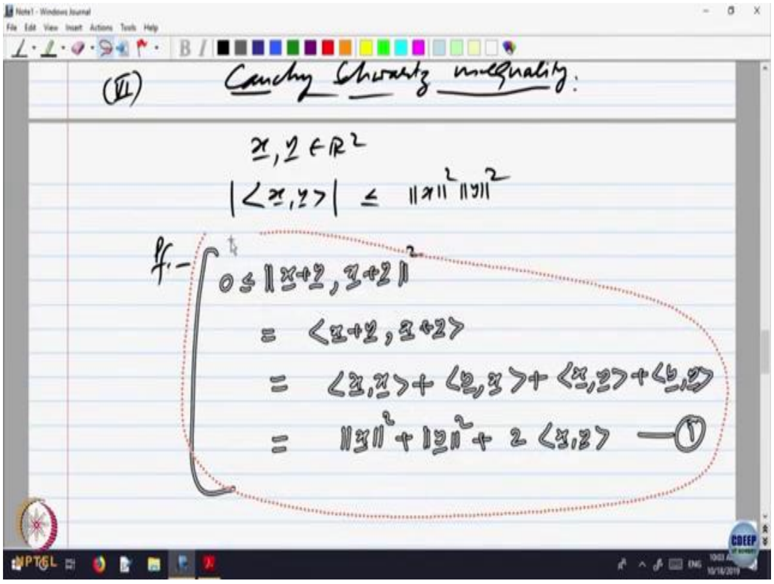

So, let me remove from here. Okay, so let us remove it from here. Cauchy-Schwarz we have proved. So, let us prove now what is the property.

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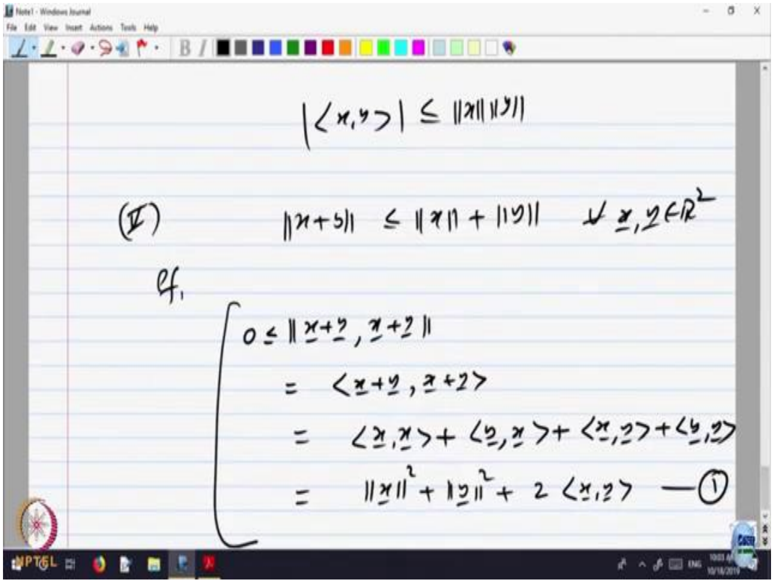

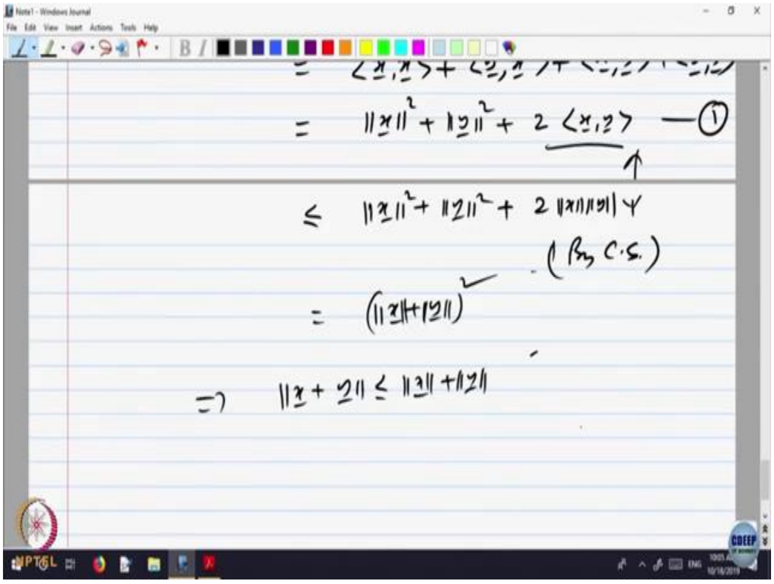

That norm of x plus y is less than or equal to norm x plus norm y for every x y belonging to R2. So, proof, that is what the other thing we should be looking at. Okay. So, we are trying to prove that the norm of x plus y is less than or equal to norm x plus norm y. So, let us compute it. Right. Norm x is the dot product with itself and that is equal to this, right. We are computing the norm of x plus y, right. So, write it as a dot product, expand. So, you get this.

And now, the idea is this is less than or equal to norm of x square plus norm of y square mod of 2 times in a product. So, that is less than or equal to 2 times norm x norm y by Cauchy-Schwarz inequality. So, that is a crucial thing here that Cauchy-Schwarz inequality gives you and this is the thing what norm of x plus norm of y whole square. So, that implies that norm of x plus norm of y is less than norm x plus norm y.

So, finally, what we are saying is the norm or the magnitude in R2 also has the property similar to that of absolute value function.