Basic Real Analysis
Professor Inder. K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture - 55
Change of variables - Part IV

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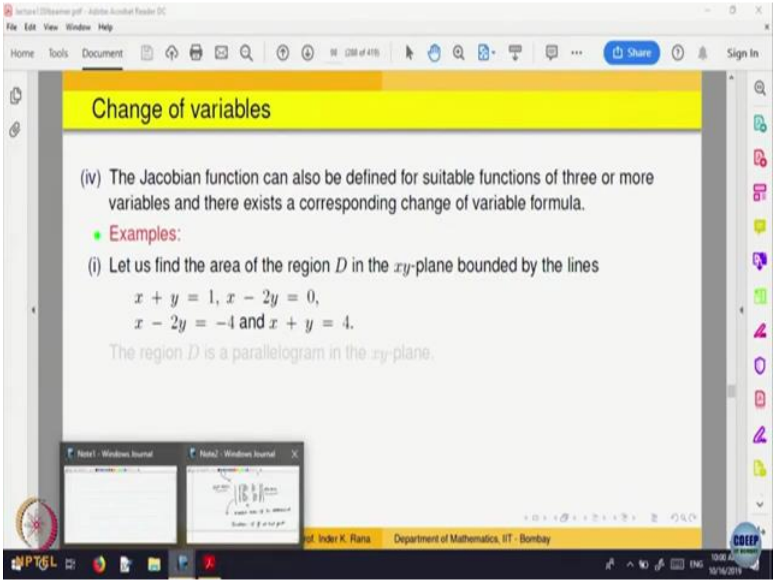

So, let us look at how does the things change, let us find the area of the region D in the x y plane. Okay, I am going to give a very simple illustration of the change of variable in the x y plane bounded by these lines. x plus y equals 1, x minus 2 y equal to 0, x minus 2 y equal to minus 4 and x plus y equal to 4. These are so we are looking at a region in the x y plane whose area we want to find out, okay.

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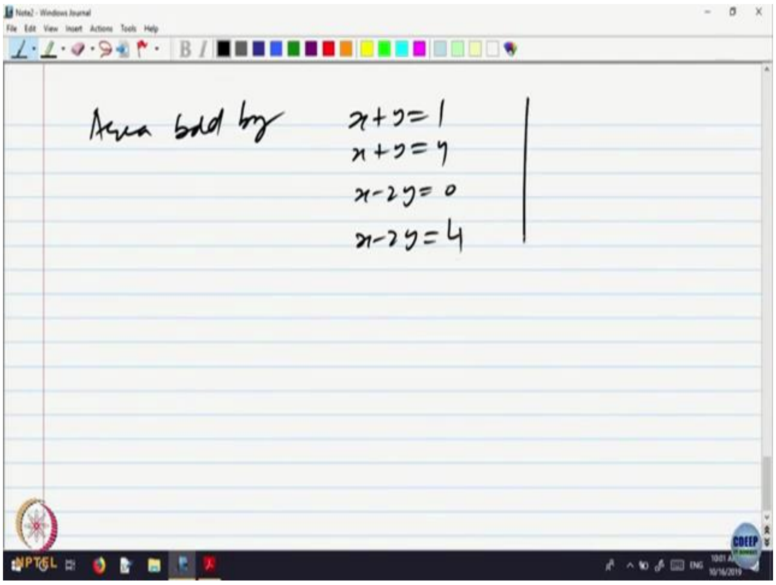

So, how does the, what is this object? So, what is the area is, area is bounded by x plus y equal to 1 and x plus y equal to 4 and the other 2 lines are x minus 2 y equal to 0, x minus equal to 4. So, this is in the x y plane.

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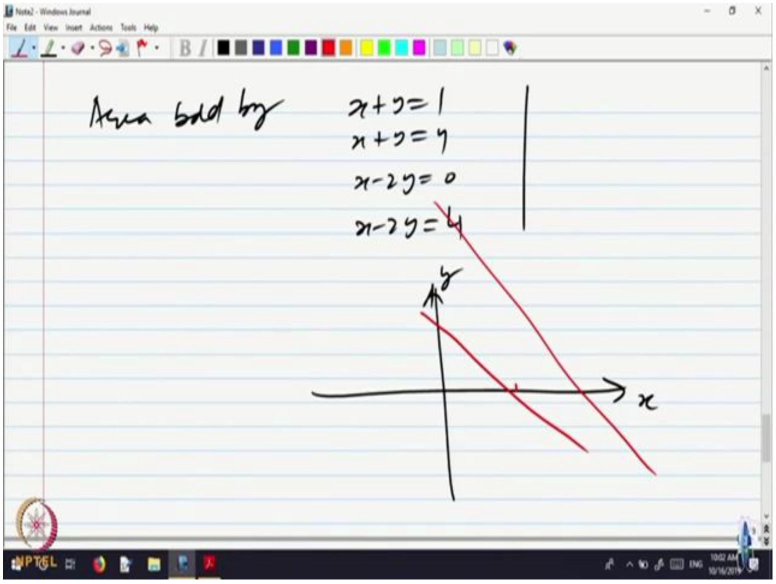

So, let us see what does it look like? So, what is this x plus y equal to 1, what is that? That is a line passing through the points on the x axis passing through the point when. So, let us draw that line x plus y equal to 4 is the same line basically. But it has been shifted, where it goes now? 4. So, when for example, when x is equal to 0, y is equal to 4, so it goes to this line somewhere here, okay, they do not look parallel, but assume they are parallel, okay. Figure is badly drawn, it is okay.

And then x minus 2 y equal to 0. So, what is that line? So, it is a line passing through the origin. So, x is equal 2 y, so it is a line passing through the origin. So, where is this line? x is equal to 2 of y. So, when this is equal to 1 this is 1 that is equal to 2, so, it goes like this, and x minus 2 y equal to 4. So, where is that? Shifted. So, somewhere is it okay? No, it should be shifted other side, so this side.

So, this will be a parallelogram. Of course, ordinary geometry you can use it to give a parallelogram what is the area and all that, but we want to transform this parallelogram into a object whose area is easy to find. So, what is the transformation we should be doing? So, that is what we are looking at. So, the region is a parallelogram in the x y plane, okay.

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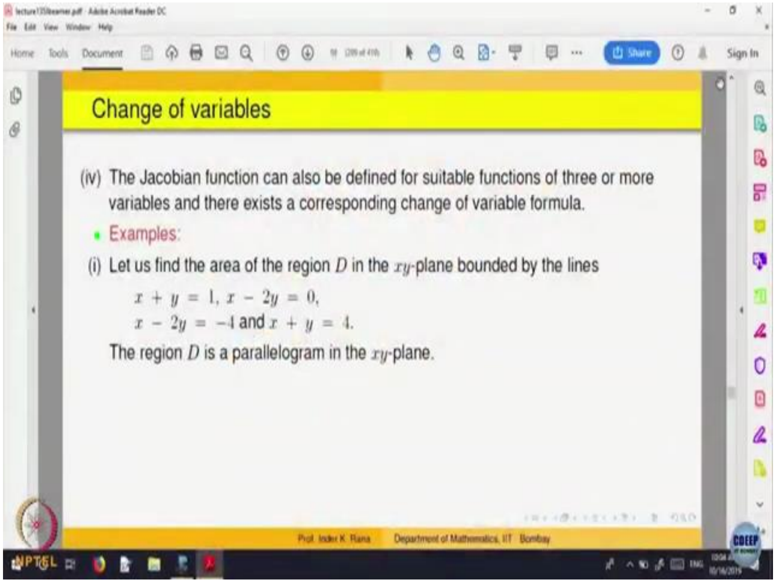

So, okay so what do you think I should be change of variable I should be making? x plus y, I should call it as some variable u and x minus 2 y as a variable v.

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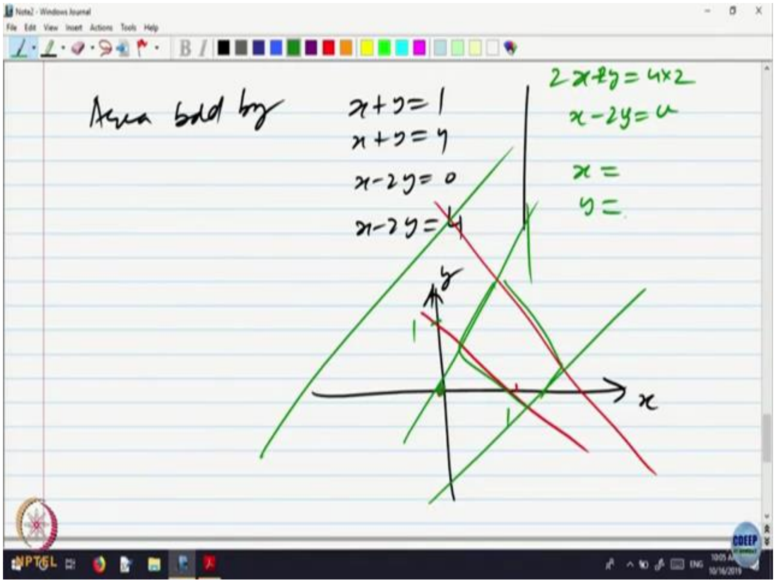

So, let us make that change and see what happens. So, x plus y equal to u and x minus 2 y equal to v. So, what is my function g 1 here? g 1 is obtained by writing x as a function of u and v. So, I have to write, what is x is equal to, what is y equal to from these two? So, what is x is equal to? From these 2 equations, if you multiply by 2, and add, so, what do you get? You can write x and y variables.

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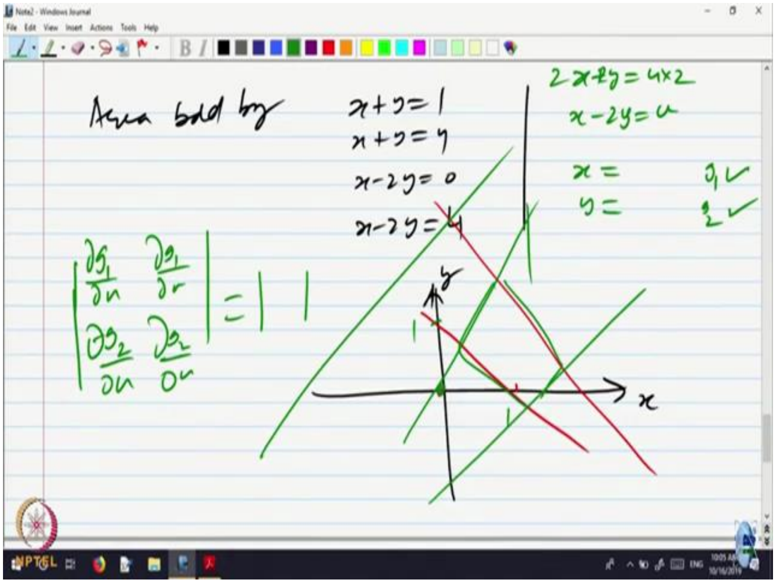

Once you got an x this is your g 1 and this is your g 2, find out the partial derivative of g 1 and partial derivative of g 2 with respect to x and y compute the Jacobian? Okay, once you do that, so, Jacobian of g 1 partial derivative with respect to u, partial derivative of g 1 with respect to v. So, from these 2 equations, you will get partial derivative of g 2 with respect to u, partial derivative of g 2 with respect to v, compute that and take the absolute value of that.

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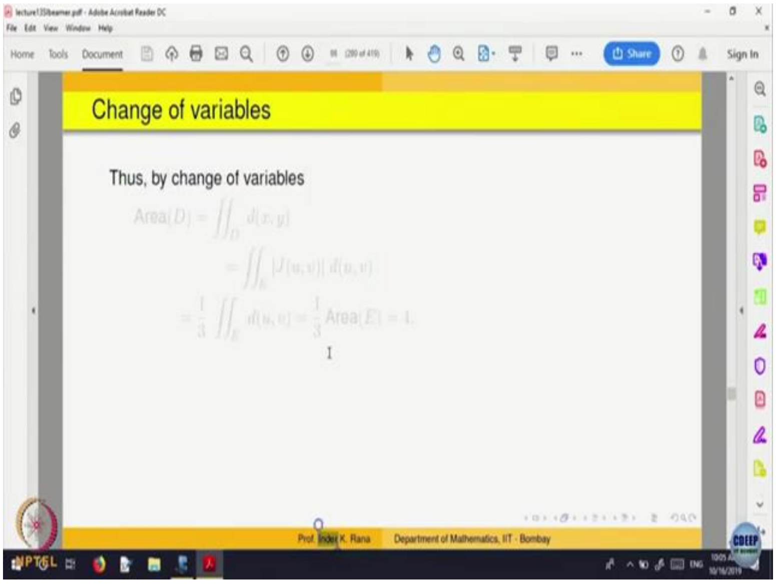

So, once you do that, you will find that in this particular case it comes out a nice thing, it is just 1 by 3, that Jacobian absolute value is 1 by 3, I want you to compute yourself later on.

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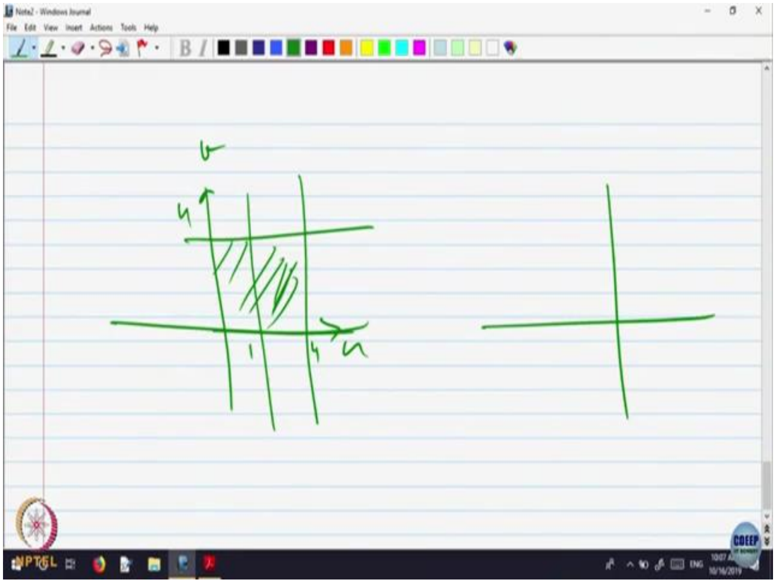

Okay now, what happens to these lines? More important is what happens to these lines x plus y? In u, v plane, what does it look like? That is u equal to 1 and u equal to 4. So, in the u, v plane, this is v and this is u, here was x y, the domain was a parallelogram. So, here it is u equal to 1 and u equal to 4. So, here it is 1 and here it is 4, and v equal to 0 and equal to 4, okay. So, v equal to 0 and v equal to 4.

So, it is this in the u v plane, it is there a rectangle, whose area you can easily write without doing anything, the length into. So, that is the basic idea that transformation of the coordinates change of variables can help you to write the domain in a simpler form, where it may be easier to compute.

For example, this is both of type 1 and type 2. Rectangle, whereas a parallelogram as such is neither type 1 or type 2, you might have cut it into many parts to it write if you want to compute via integration, so, that is the advantage of this kind of change of variable.

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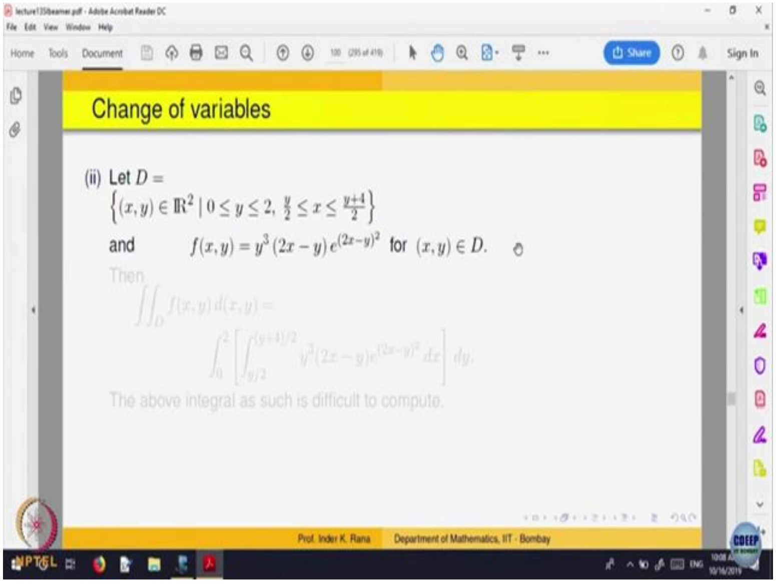

So, let us look at some more examples, so, let us look at this. The domain is y between 0 and 2 and x so, this is at what type of domain it is? Is type 1 or type 2? Type 2, y is between something 0 and C and D and x is lie between function of y, okay. And we want to integrate this function f x y is equal to y cube, 2 x minus y, e raised to power something and something, okay.

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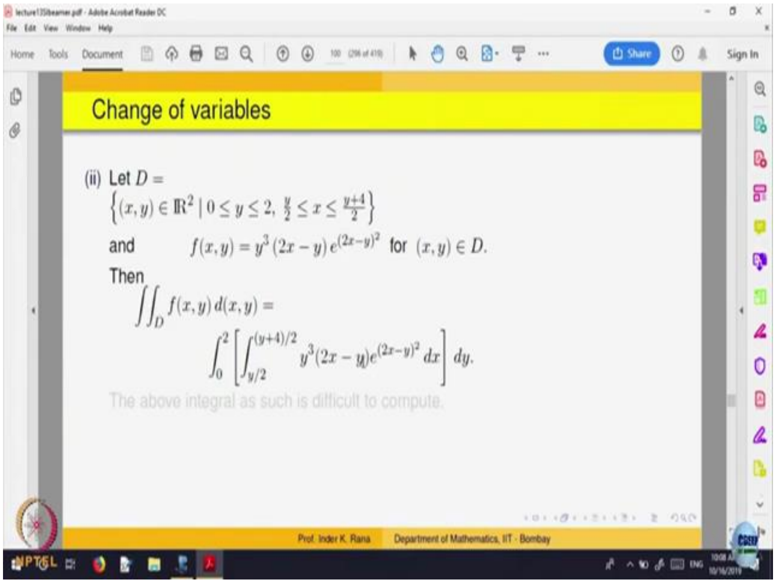

So, what will the double integral look like? Domain is very nice and easy, so, I can compute it. So, 0 to 2 it is at its integral Fubini's theorem, integral from y by 2 to y plus 4 by 2 of this function. Now, this says to if I want to compute this, I have to make a change a variable that will be easier for me to because it is 2 x minus y and 2 x minus y that power is coming. So, it says that I should make a change of variable, I should call 2 x minus y equal to another variable.

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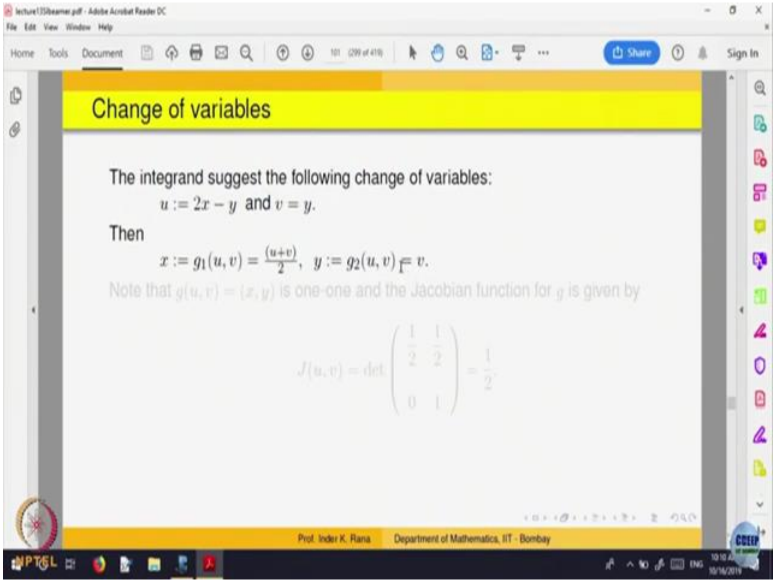

So, let us make that change and see what it comes out. So, u equal to 2 x minus y and y variable we do not have to change, so v equal to y. So, once again when you solve these 2 equations, you have to write x in terms of u and v, y in terms of u and v. So, solve these 2 equations, you get x is equal to that is a function g 1 that is u plus v by 2 solving this, and y is equal to v that is as it is, so, that is a function g 2, okay. So, you get the function g 1, g 2, how do you get g 1 and g 2? By solving the substitution and in solving it for x and y.

So, now, let us find out the Jacobian partial derivative of g 1 with respect to u. So, what will be that? Partial derivative of this function with respect to u, so, that is 1 by, 1 by 2, partial derivative of g 1 with respect to v that also is 1 by 2. So, the first row is going to be 1 by 2, 1 by 2, partial derivative of g 2 with respect to u that is 0, because is only a purely a function of v and partial derivative of g 2 respect to v that is 1. So, you get the Jacobian 1 by 2, 1 by 2, 0 and 1.

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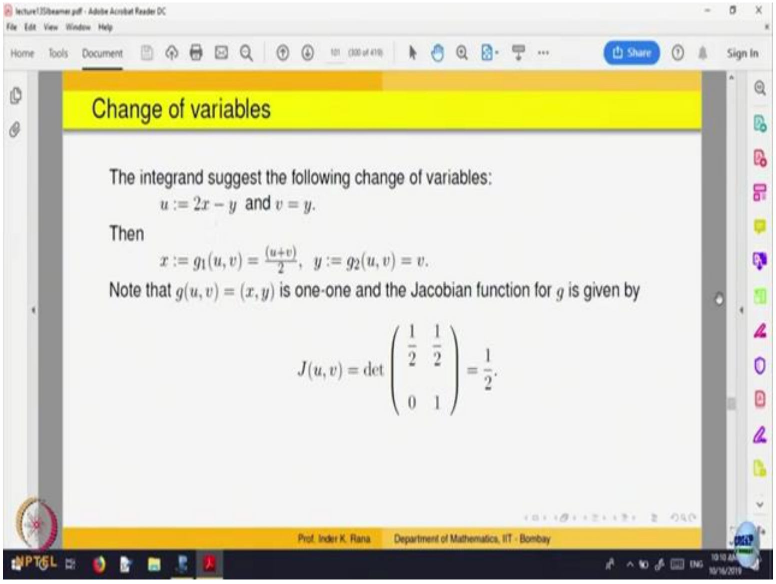

So, the Jacobian, okay, and you solve it, you get 1 by 2. So, how does a transformed integral look like? We will had to write this also. So, the previous one, we have not yet computed the domain, keep in mind this is a domain D, we have to find how does it look like in the u v plane also.

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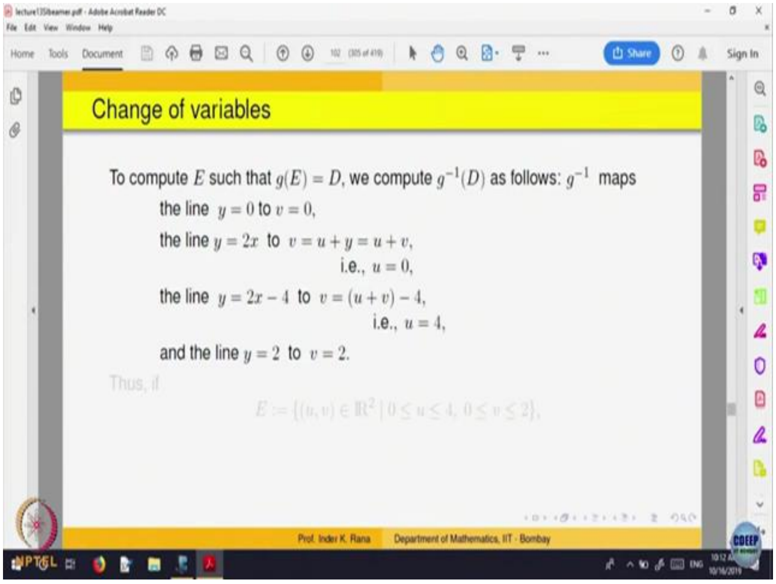

So, let us find that. So, we have to solve our equations again, so, we have to see where does we are putting y equal to v, so, y equal to 0, where does that go? y was equal to v. So, that goes to v equal to 0, the domain bounded by those lines we have to transform them how they look like in the u v plane. So, each boundary line will have to be transformed. So, y equals 0, y equal to 2 x, so put the value v was equal to u plus y, so that gives u equal to 0.

So, v equal to 0, 2 x is equal to this, whatever 2 x plus 2 x minus 4 that gives you u equal to 4. Solving those same equations the domain is given by those lines, so how does these lines look like in u v plane? That is what is to be solved. Using those same equations substitutions that 2 x minus y was equal to u and y was equal to v, so using that you get these are the lines, okay.

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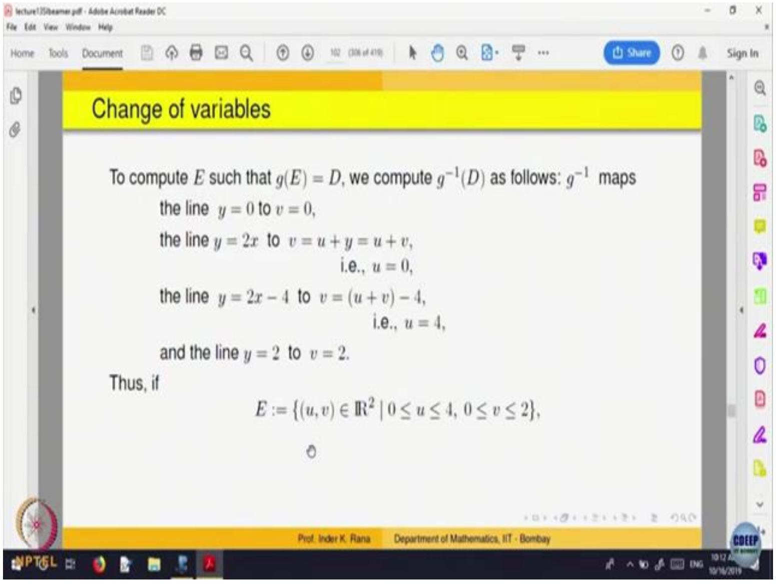

So, what is a domain in E in the u v plane that gets transformed into the domain D that is what we want will be, v equal to 0, v equal to 4, so, u will lies between 0 and 4, v lies between 0 and 2, from these equations, from the these equations, okay, these give me that E must be this domain. So, what is the transformed integral? Integral of f x y d x, d y, will be equal to integral of f in terms of u and v, x is a function of u and v, f of g 1 u v comma g 2 u v into Jacobian of the transformation in D u D v over this domain.

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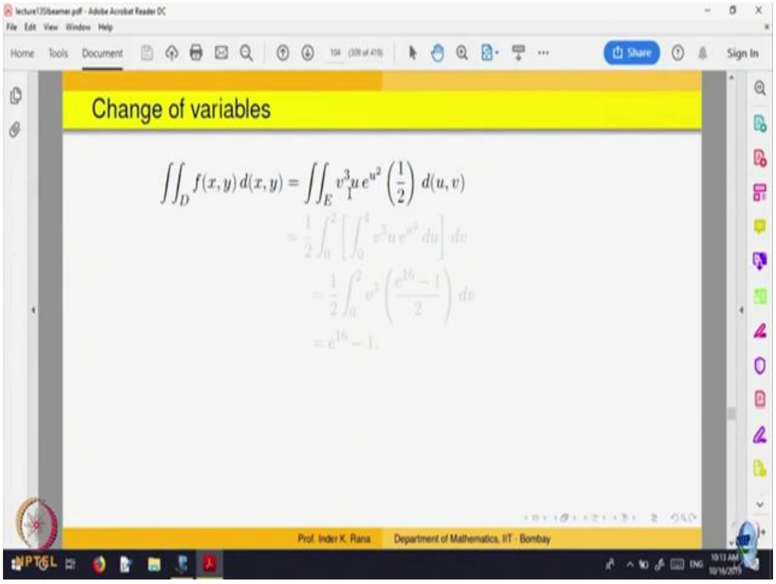

So, integral over this domain which is a rectangle, okay, so, integral over E, that function. So, remember that function?

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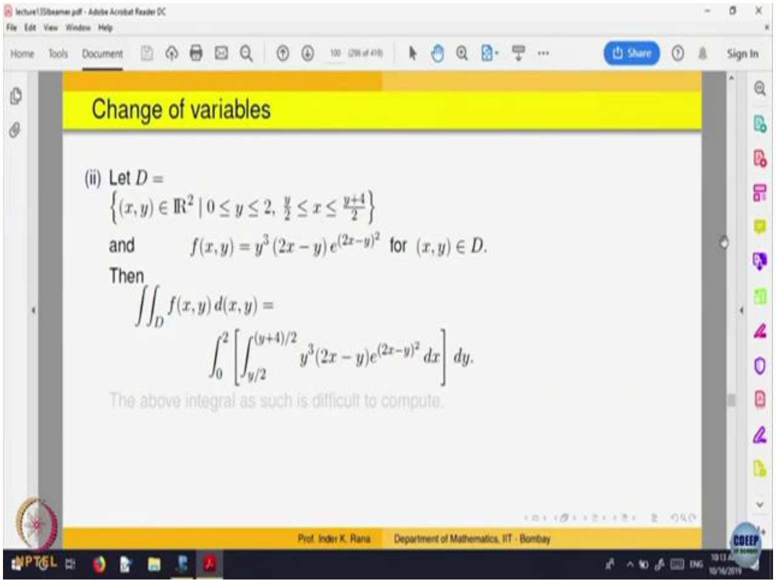

Let us just write, go back and see what was that function, this was the function. What is y? y is v, so v cube, 2 x minus y is u, so v cube u e raise to power u square, Jacobian D u, D v.

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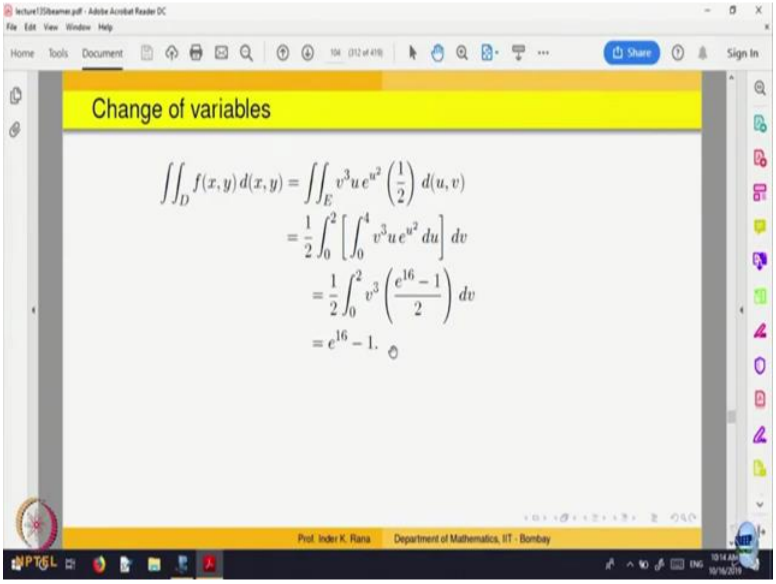

So, write that, so, v cube u e raise to power u square Jacobian is half D u D v and this e is nice now, it is both of type 1 and type 2. So, what has happened is? Not only our integrant has become nice by the change of variable, our domain has also become nice. So, that integral becomes very simple you can split 0 to 2, with respect to v with respect to u, when you integrate u e raise power u square.

So, that we can know we know with one variable, how to integrate? Derivative is sitting next, so integral you get the value. So, change of variable we are using only when either our domain becomes nice or the integrant becomes nice.

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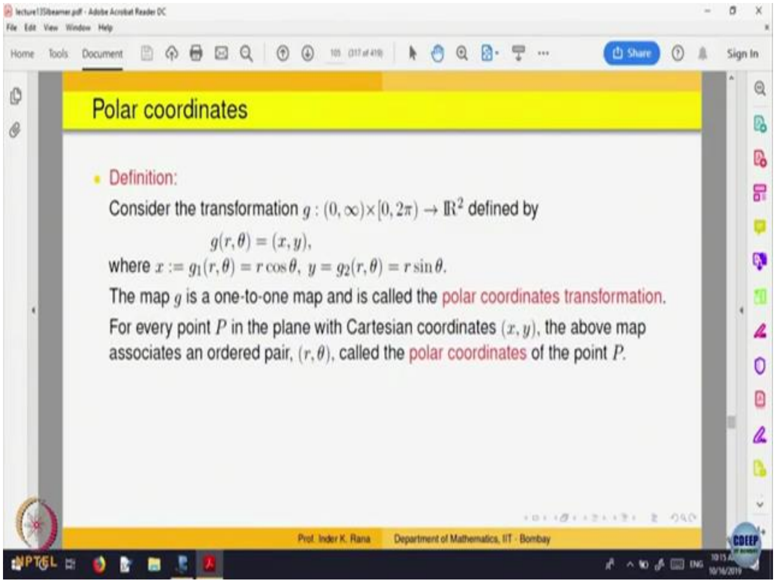

So, let us see how do we apply this to our, so, for the polar coordinates theta r, theta comma r sorry r comma theta, r is between 0 to infinity, theta is between 0 and 2 Pi, r theta goes to x y and what is x? r cos theta, r sin theta, so, g 1 is r cos theta, g 2 is. So, x is written in terms of r and theta u and v, here u and v are replaced by r and theta and second variable g 2 is y and that is r sin theta. So, what is the Jacobian of this transformation?

First row g 1 with respect to r, g 1 with respect to theta, second with respect to r second. So, first with respect to r, so cos theta with respect to second variable, g 1 with respect to theta. So, that will be minus r sin theta, similarly the other 1 so when you do that transformation, calculate the Jacobian, what does it come out?

Partial derivative of r cos theta with respect to r that is cos theta, partial derivative of r cos theta with respect to theta. So, that gives you sin theta, okay, there is a second one with respect to r and with respect to theta, so, that gives you Jacobian is r.

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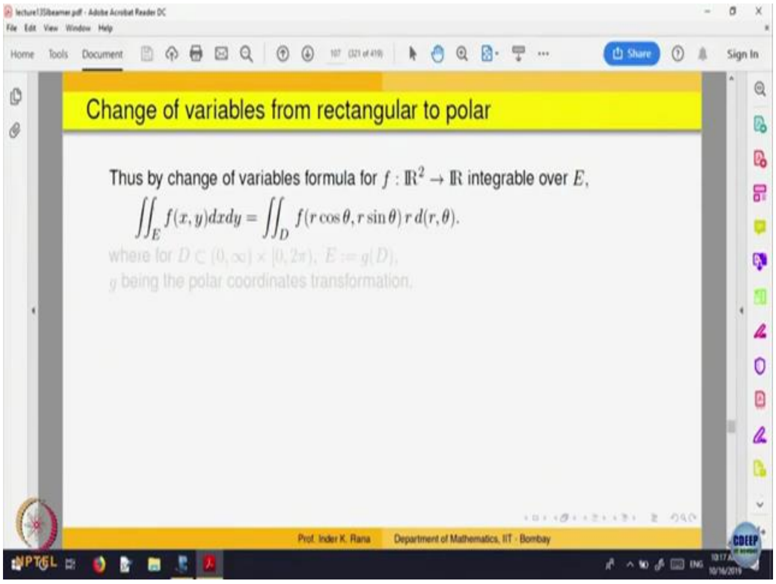

So, it says when you transform f r cos theta, r sin theta and d x d y changes to r d r d theta, that normally we start using in our school without any hesitation, but this is a special case of change of variable. And what is this domain D? You had to write domain D in terms of polar coordinates, E is in Cartesian coordinates, but that has to be interpreted in terms of the polar coordinate. So, D gets transformed into E by change of variable, so, that has to be done.

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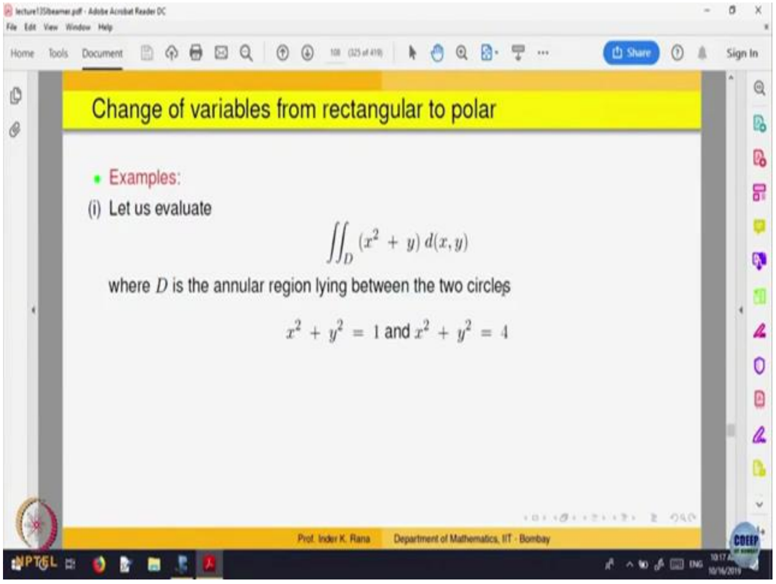

So, let us look at probably some examples of that. So, let us look at this thing. You want to integrate x square plus y, d x y, where d is the domain lying between 2 circles, so, that is the annular's portion, domain is the annular's portion, okay. So, if you want, you can try to do it by cutting the annular's region into domains of type 1 and type 2, and then adding up all those integrals.

But we understand that if we look at the domain in 2 circles, it is very easy to explain that domain in terms of polar coordinates. So, that domain what it will look like? It is a region between 2 circles, so, our goal is from the inner limit to the outer limit, smaller radius to the bigger radius, and theta goes from 0 to 2 Pi, so, very easily described, so, we make a change of variable and use the transformation.

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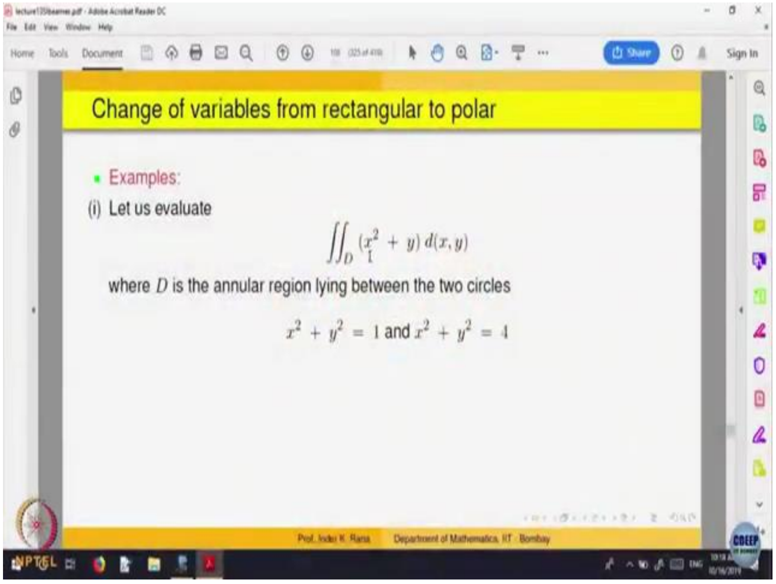

So, what will x square look like? r square cos square theta, y r sin theta. So, r square cos theta plus r sin theta integrated over, r goes from 1 to 2, inner radius is 1, outer radius is 2, theta goes from 0 to 2 Pi, r d r d theta.

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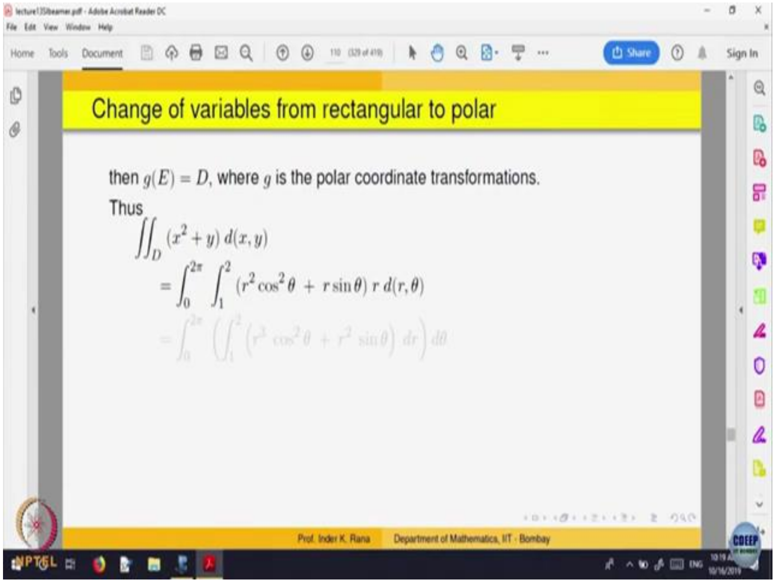

So, let us put those values and see what it comes out. So, this is what it comes out. So, r square cos square theta, r sin theta, 0 to 2 Pi, 1 to 2 r d r d theta and now you integrate out this quantity, as a function of r, as a function of theta, 1 variable at a time, okay. So, because the domain was easier to explain, we did that.

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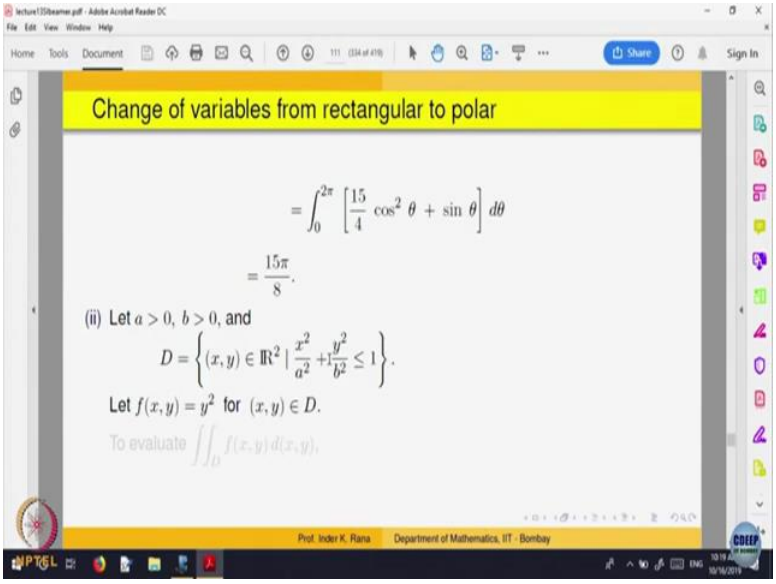

So, there are many examples I think let us not do all, some of them you should read yourself. f x y equal to y square over the domain x square over a square plus. So, what will the transformation you will think of doing there? Is, x square over a square plus y square or b square. So, instead of putting r it should be in terms of a and b, polar coordinate slightly. So, once you want to represent a ellipse in terms of polar coordinates, that will come here, okay.

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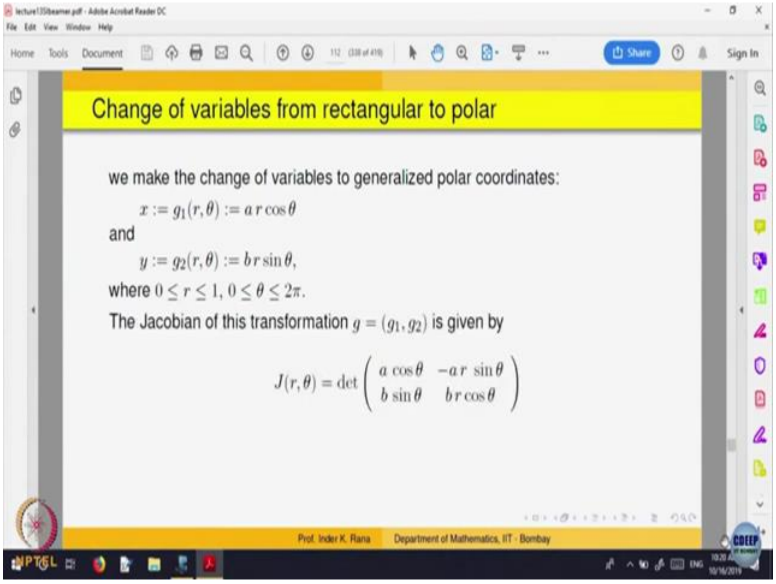

So, a r cos theta a r sin theta, so, you make that a r cos theta, b r sin theta and make the transformation and do it, so, read that okay, calculate the Jacobian.

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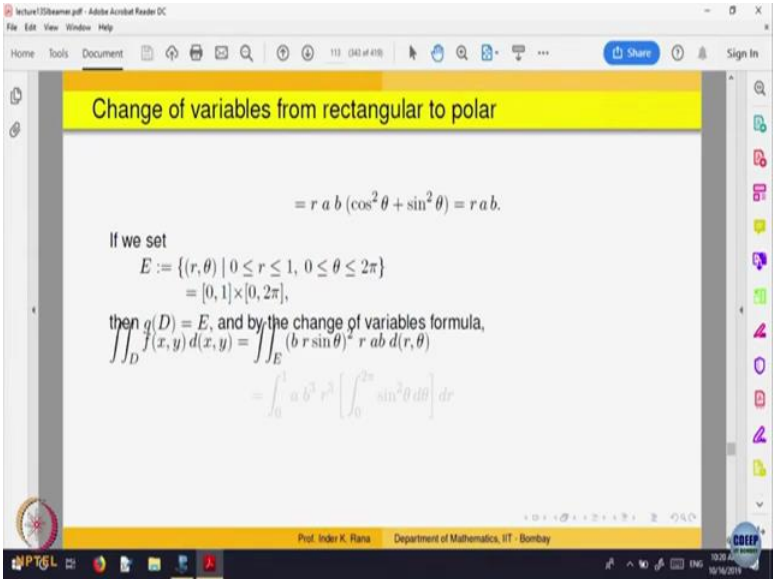

So, finally it comes out r a b and 0, 1 cross 0 to 2 Pi, because we have already put a r, okay, so, it becomes very nice to compute, so, I will leave it as it is, try to do it yourself these problems.

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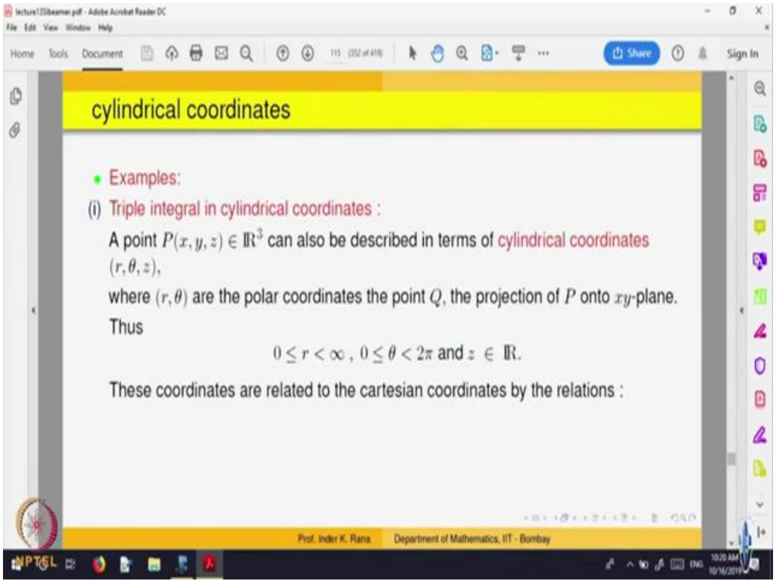

Now, let us come to cylindrical coordinates. So, I am going to revise again what we described last time. So, given a point x, y and z in r 3, you can describe the point also in terms of what are called cylindrical coordinates. You can imagine the point lying on a cylinder and the tip of a cylinder, and cylinders are expanding, and how much ware on the cylinder you should be going.

So, it is described by r, r is the distance or the point how much is you are away from the origin and then you are doing the cylinder. So, theta is 0 to 2 Pi is that is as it is, okay. So, these are the cylindrical coordinate.

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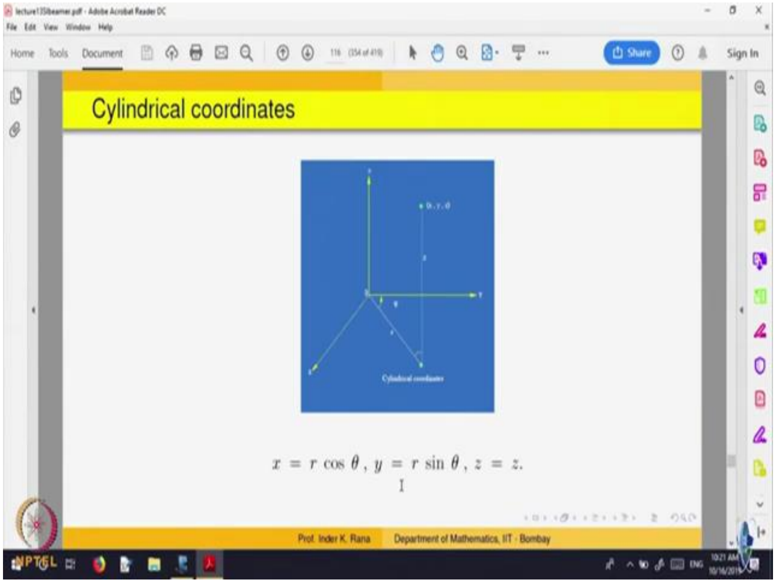

What is the relation between them? x is equal to r cos theta, y is equal to r sin theta, z remains as it is, z we are not changing cylinder, okay. So, with that let us compute Jacobian of this. So, Jacobian will be a 3 by 3 now, because there are 3 variables coming into picture. So, this is g 1 r cos theta, g 2 r sin theta, g 3, okay, equal to z.

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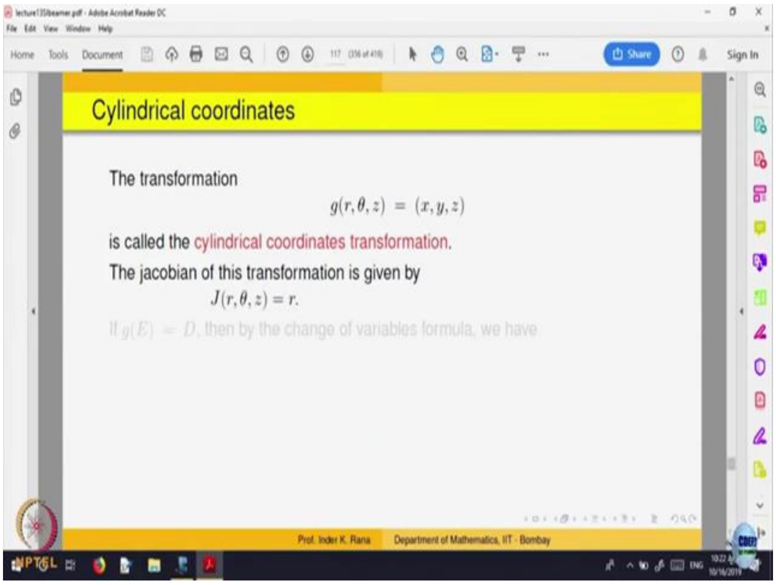

So, calculate the Jacobian for this and you simplify it comes out just r, because z does not change, only the polar coordinates are making a change, so, x and y should give you r, okay. So, it is Jacobian is just r as in polar coordinates.

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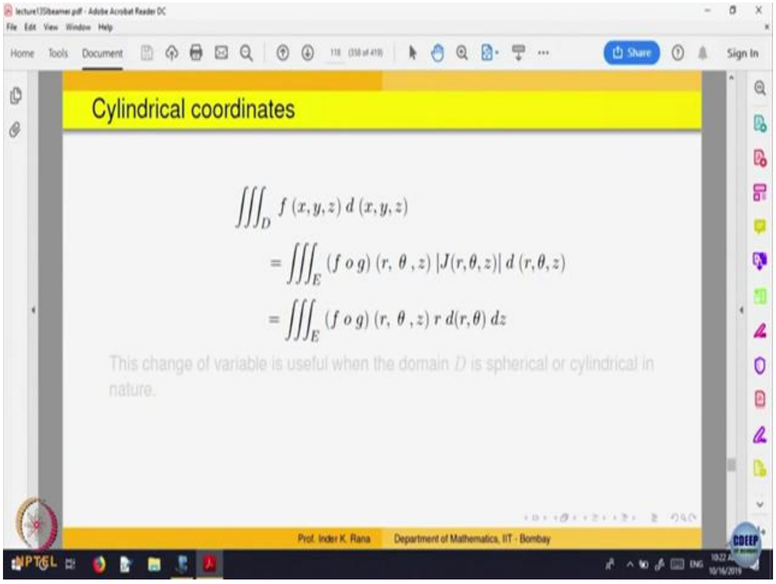

So, the change of variable formula will give you f in terms of r theta and z, of course E has to be written in terms of cylindrical coordinates and d r, r d r d theta, d z as it is. So, d x, d y, d z goes to r d r d theta d z, z coordinate we are not changing, z is z only, x is r cos theta, y is r sin theta, so, that change comes. Once again why you we should be doing that? Depends upon what kind of integral you want to.

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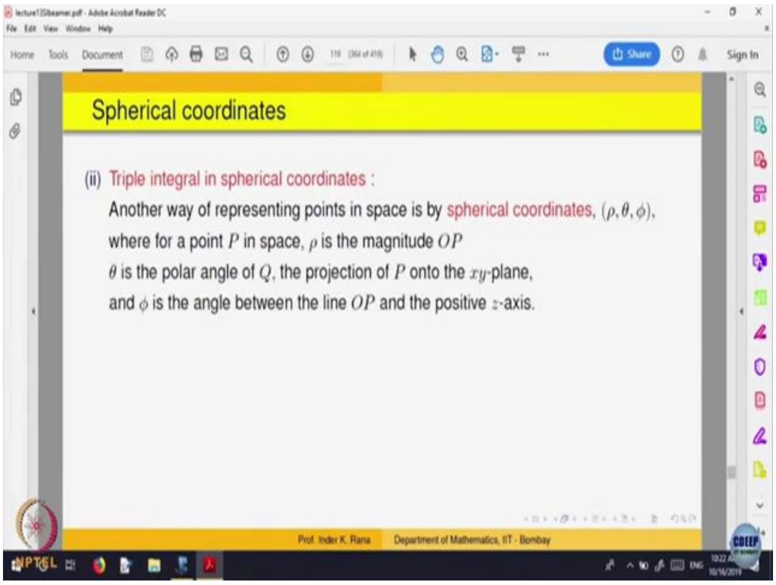

Let us do the spherical also and then we do the. You remember spherical coordinates? How would we describe these spherical coordinates? We will imagine all of r 3 as concentric spheres, okay. So, every point in the space will rely on some sphere of some radius. So, let us say the radius is rho, if the radius is rho, let us see that position vector o p, how much is the angle that is making with the z axis. If we know that angle, so, imagine a rod which is rotating with that angle, so that will give me a circle on the sphere of radius rho.

So, all the points on the circle of radius rho can be described, by looking at the polar coordinates for that points on the circle and looking at this rho. So, as this angle changes, you will get different circles, so, on the sphere of radius rho, you get different circles. So, these are what are longitudes and latitudes in describing the position of any particle on Earth, but call them as spherical coordinates, because we imagine the whole of r 3 as concentric spheres.

So, what was the relation with the (respect) so what is it required? You need how much is the angle with the z axis, that is Phi, how much is the distance? That is rho, and on the thing how much you are rotating, point on the circle? So, you need how much angle you are going to make theta, so, rho angle Phi and angle theta, so, rho, theta and Phi are the 3 variables that will determine a point on in space by looking at the spheres.

And what is the relation between that? So, that rho if you recall, we said, when you drop a perpendicular that is same as z and the angle being phi, so, that gives you, the radius giving rho sin phi and then polar coordinates, that is the radius of that circle okay. So, once you know the radius, you know the polar coordinates.

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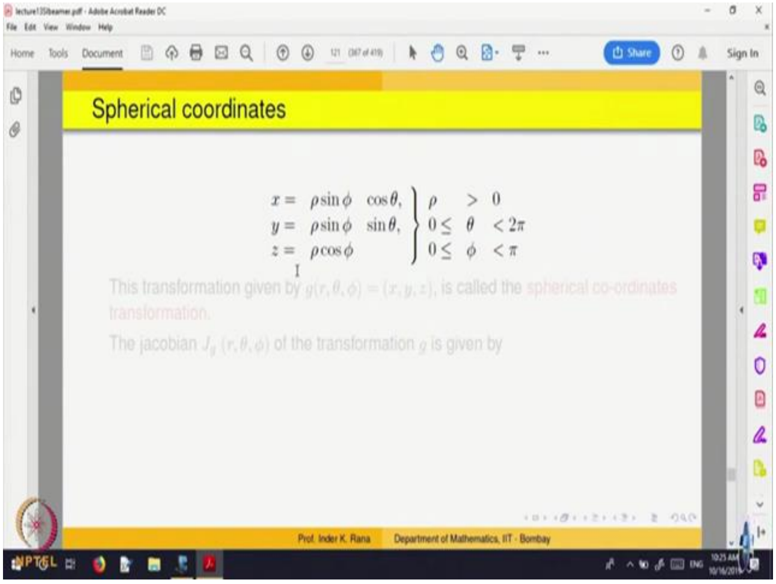

So, this is a relation. Phi rho is a distance of the point from origin that is rho bigger than 0. Theta is a angle on that circle, so, that is 0 and 2 Pi and Phi is the angle with respect to z axis. So, that is theta is 0 to Pi, okay, from top to the bottom. So, relation is that z coordinate is rho cos phi and the remaining things instead of rho cos phi, sin phi, rho cos phi y, sin phi, so, these are the relations.

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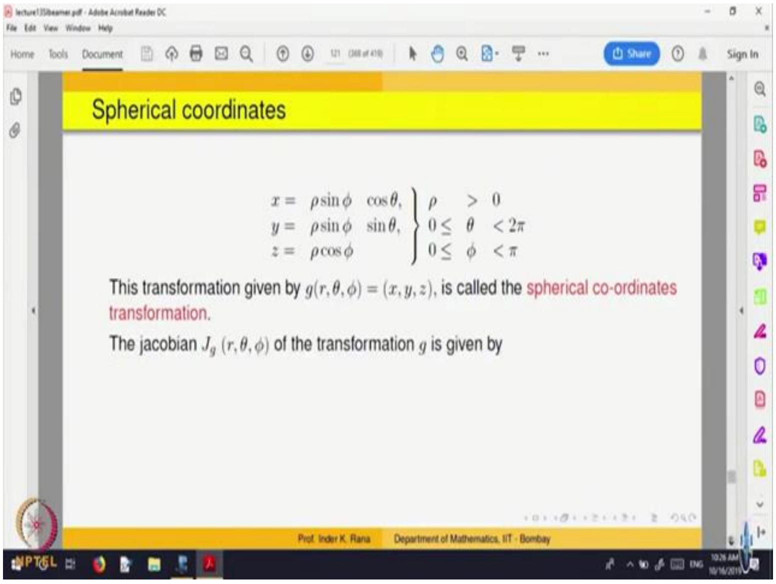

And now, to find the, so, change of variable from spherical coordinates you are going to Cartesian coordinates, rho, theta and phi goes to x y and z. So, g 1, g 2, so, this is your g 1 the first one, y that is a g 2 function and z that is a g 3 function. So, 3 by 3, determinant you have to compute, okay. So, one can do that, partial derivative of x with respect to rho, partial derivative of x with respect to theta, partial derivative of x with respect to phi, that is a first row second, row third row.

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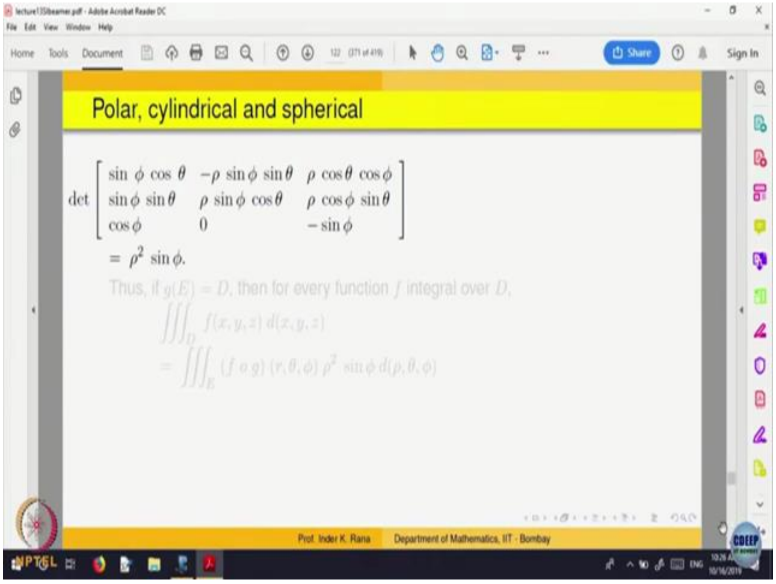

So, if you compute that, that is what it looks like. And you simplify and use your trigonometric relations, that sin square phi plus cost square phi is equal to 1, sin square theta, cos theta is equal to 1, and so on. So, you get the Jacobian is simply rho square sin phi, it does not depend upon theta, the change of variable, only the phi angle, that you make with the z axis.

Here is a minor thing which probably, see we are taking angle phi, the position vector makes with the z axis. So, phi goes from 0 to Pi, top to bottom, but as far as position of the Earth concerned one likes to measure it from a great circle which we call it as meridian, and then the angle is measured how much up you should go and how much down you should go.

So, that phi is taken as from minus Pi by 2 to Pi by 2. So, in all GPS calculations, they take it phi as going from minus Pi to 2 to Pi by 2. So, there are 2 ways of writing only, the only change comes it changing phi to phi minus Pi minus theta, so, accordingly change can come, okay, but that does not matter.

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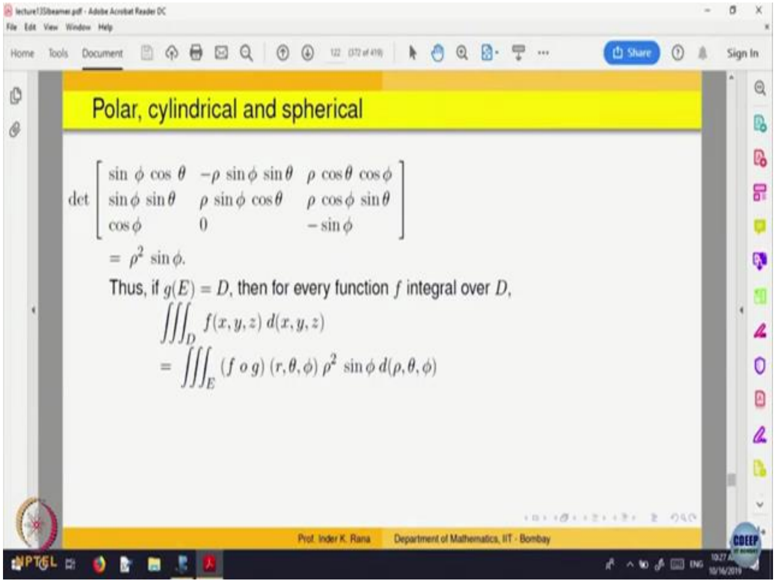

Spherical coordinates will be using that. So, that gives me the relation, change of variable.

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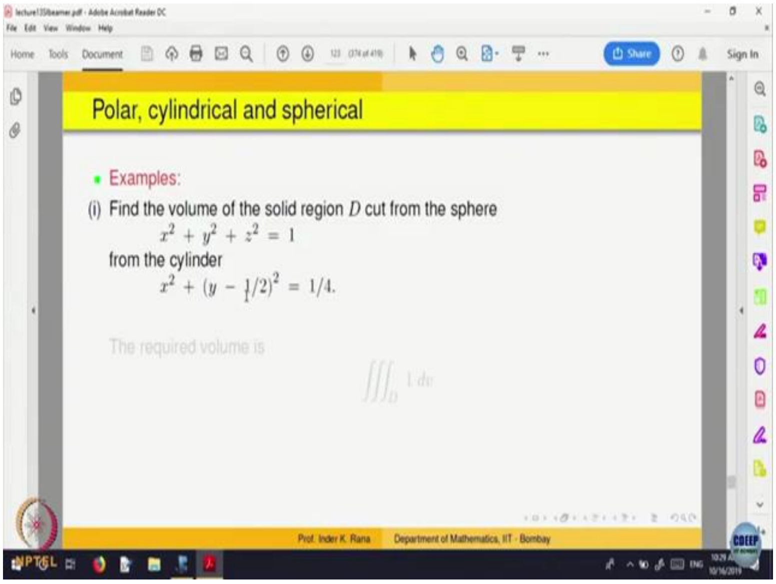

So, let us look at probably some examples. Find the volume of the solid cut from the sphere, x square plus y square plus z square equal to 1, from the cylinder x square equal to y minus half square equal to 1 by 4. So, can you try to imagine that what it, find the volume of the solid region cut from the sphere, so, it is a part of the sphere. And there is a cylinder, so, what is that cylinder? x square plus y minus half square equal to 1 by 4. What is that cylinder? There is no z. So, it is moving vertically.

But, the axis is not passing through the origin, it is a circular cylinder, when z is equal to 0, what is the circle you get? That is x square plus y minus half whole square equal to. So, it is off origin on the y axis, so, the center of that circle will be 0 comma 1 by 2, so it is off. So, when you go off and you vertically, you will get a part of the sphere inside the cylinder.

So, how do you find out what is this looks like a complicated problem. Find out the volume of this solid, okay. So, because there is a cylinder involved and there is a sphere involved, so, probably we can think of cylindrical coordinates or spherical coordinates, change of variable, okay.

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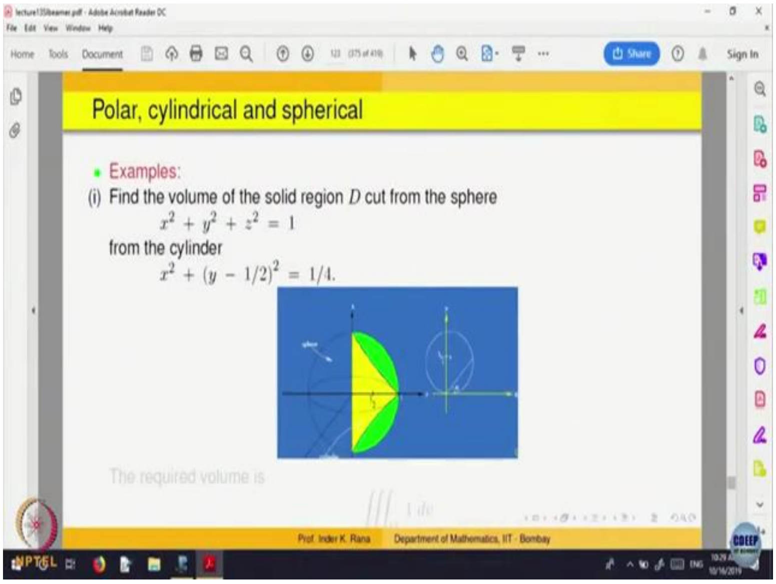

So, let us make a change of variable. So, this is what the picture looks like. So, this is a sphere and the here is a cylinder, so, cylinder is only one side, y equal to 1 by 2, x is equal to 0, so, that is a centre, so, that is a cylinder going, okay. So, you want to describe this in terms of say cylindrical coordinates or spherical coordinates, which one will be easier?

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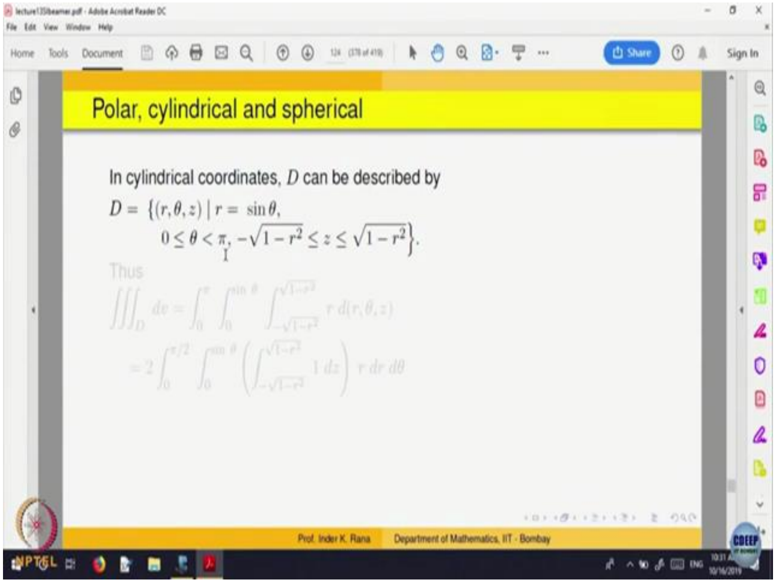

So, let us look at cylindrical coordinates, because it is cylinder involved in it, okay. So, the domain D can be written as r theta and z that is a cylindrical coordinates, where does r vary and where does z remains as it is, what is it z? What is the top of the surface and what is the bottom of the surface for that? It is a part of this sphere. So, top sphere and the bottom was a sphere that is a top part of the solid and bottom part of the solid.

So, z is not going to be in problem, we can write from the equation of the sphere, what is a top, what is the bottom, z goes from minus, part of the sphere to the top of the sphere the positive part. Only thing is what are our r and theta?

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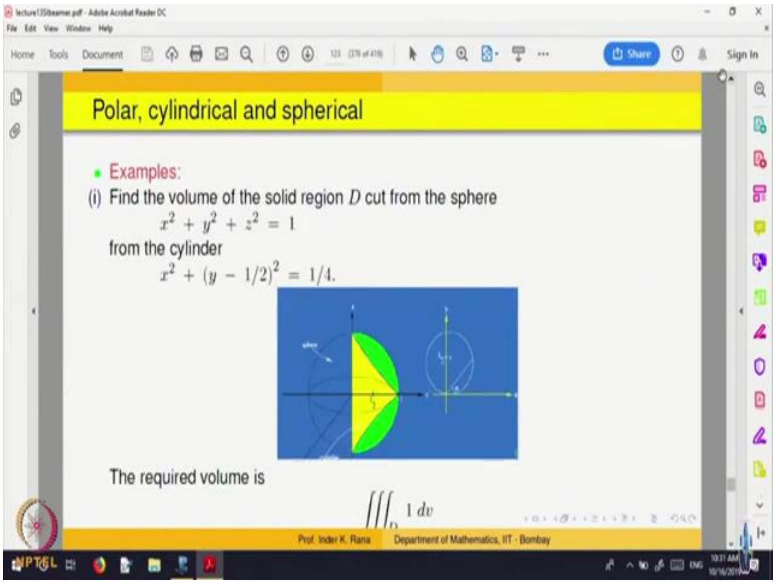

So, if we look at the circle, this is what the circle looks like, angle is theta here, any position vector how much is the angle that it makes? Okay. So, what is your this distance r, this is your r, angle is theta polar coordinates, you have to calculate r, okay. So, if this angle is theta, what is this and you can draw a perpendicular here if you like so right angle triangle you can calculate in terms of theta, so, that gives you the r. See r will start when your bottom will start at 0, and then it will go up to the vertical one.

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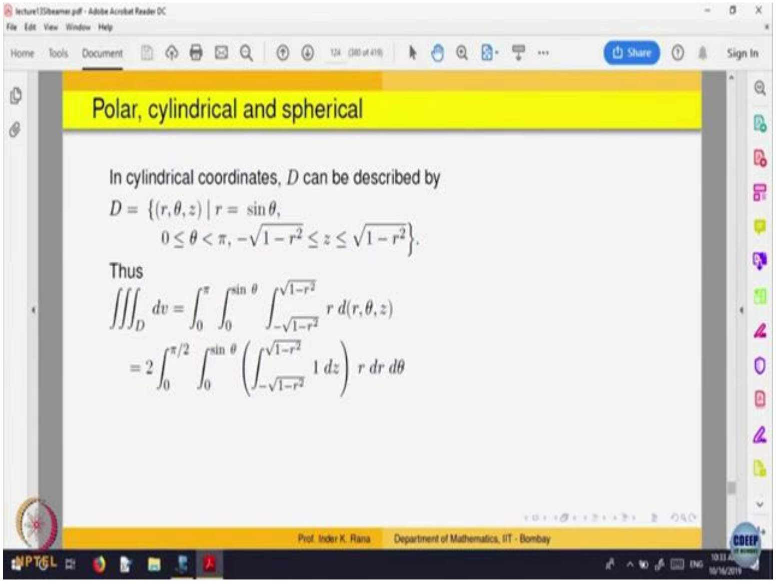

So, r is equal to sin theta when theta is equal to Pi by 2, it will be the full of, it starts with 0 the top, is it okay the bottom circle!? See this it is not circle centred at 0, it is off centre. So, I have to describe that in terms of r and theta the off centres, circle. So, that is given by r is equal to sin theta, theta between 0 and Pi. Theta will go from 0 to Pi only and z goes from bottom to the top, so, one.

So, what is it z? Sphere, x square plus y square plus z square equal to 1. So, what is z? 1 minus x square plus y square plus but x square plus y square that is equal to, x plus y that will equals r, so, 1 minus r square 1 plus r square, okay, x and y polar coordinates. So, that is, so it becomes r goes from, so, r goes from 0 to Pi, theta goes from 0 to theta goes from 0 to Pi and z goes from this region, r is equal to sin theta, so, that gives you.

So, that domain splits nicely, okay, in polar coordinates, are you following what I am saying? How would the domain looks like in the cylindrical coordinates? In the cylindrical coordinates, the cylinder top and bottom are the sphere, part of the sphere, that is no problem. We have to describe what is r and theta, but that is a circle centered at 0 and 1 by 2, from the origin what is that angle? Angle is theta. So, this is a circle, centred at 1 by 2. So, what is this r in terms of theta? Right, r and theta. So, r is equal to sin theta, because the radius is equal to 1, radius is equal to half, so, the diameter is equal to 1, are you following or not?

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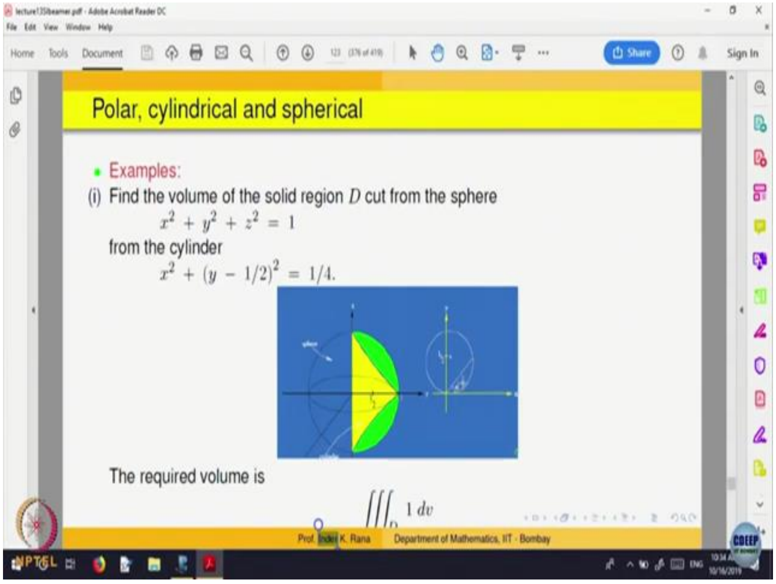

Okay, let me just go back to the picture, here is a picture. So, in this, this is a origin, any position on the circle, if this is angle theta, okay, then this is a this vertical is the y axis, this is a centre is here somewhere in between okay. So, if you draw a perpendicular here, you get a right angle triangle. So, you can calculate, what is this distance o p, that is your r and that is sin theta. So, that is how do you calculate, okay.

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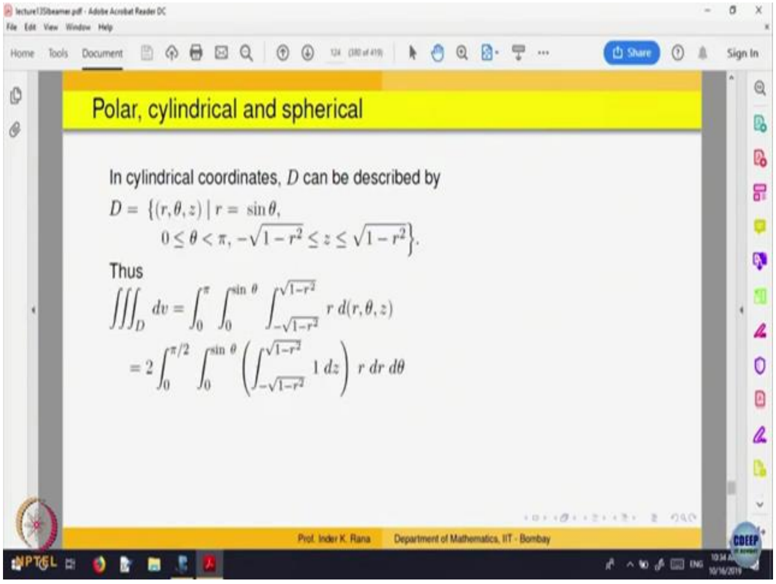

So, basically the idea is your domain looks very nice to be explained in terms of cylindrical coordinates, because there was a cylinder involved, z was not causing any problem it was only x and y. Let us probably look so, you can compute all this integral, okay. So, let us not, let us just vary this problem a bit. Find the volume of the solid cut from the sphere by the cone, okay.

So, here is a cone, z is equal to x square minus y square, sphere is x square plus y square plus z square equal to nine, and this is a cone, cut from the sphere by the cone. So, what does the object look like? Is a part of the cone inside the sphere basically. So, a very familiar figure would be, when you go for an ice cream, something a cone and top is a ice cream there, is a ice cream cone kind of a thing, okay.

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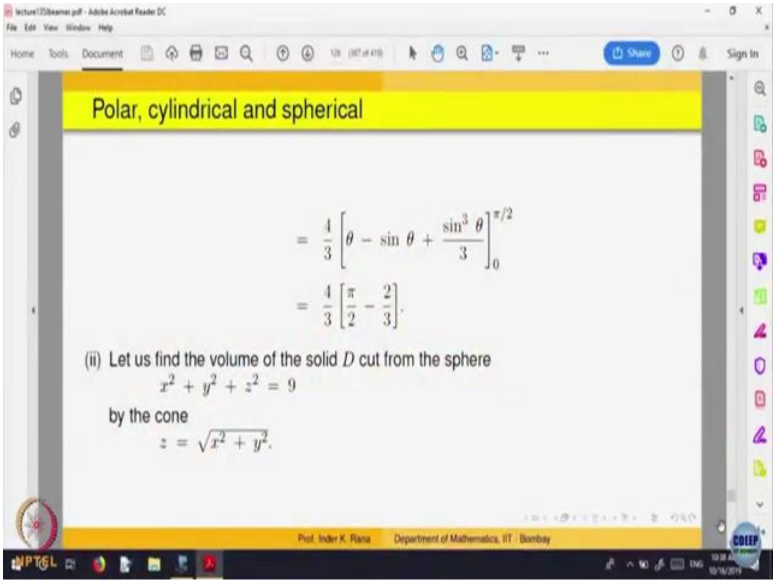

So, now, here sphere is involved and a cone is involved. So, what will be the bottom of the in that cone, z goes from where? It will go from 0, to the top of this sphere, wherever the cone is cutting, so, part of will be the, that ice cream will be the part of the sphere. Is it okay? Bottom is 0, z is z is 0, bottom to.

So, now, we want to know whether how best we can describe the cone. Whether Cartesian coordinates, this is given by the Cartesian coordinates, okay, equation, z is equal to x square plus y square or you can do it in cylindrical coordinates, what does it look like in cylindrical coordinates? It does not help, z also is varying, cone is also varying in cylindrical coordinates, so, cylindrical coordinates are not of use here.

What about spherical coordinates? Can you describe a cone in terms of spherical coordinates? That is a basic question. If we can do then probably the life will be easy, because we are inside a cone, we are inside a cone. So, imagine a cone, z is equal to x square plus y square, in spherical coordinates inside of a cone. So, what does it look like?

Actually, when I waved my hand, in the spherical coordinates you have to go away from z axis, and then you have to rotate. So, when you rotate, what do you get? When you rotate you are precisely getting a cone. So, this line when you rotate at some angle, it gives you a cone with that angle, and top is a sphere. So, the best thing is to use spherical coordinates.

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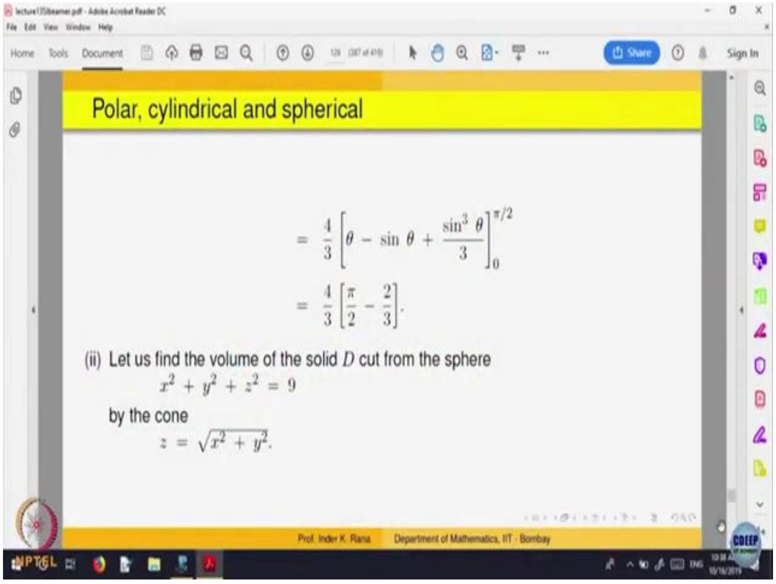

So, for this, z is equal to x square plus y square, what is that angle? At what angle, if I go, very thin angle, thin cone, bigger cone, bigger cone, and so on. So, what is that cone? What is the angle of the cone I have to find out? If you can find out that angle, then your problem is solved. So, try to find out yourself I think, let me but when I send you the slides do not look at the solution try to do it yourself and then try to look at the solution, what that angle should be? Is very nice angle, okay.

And once you know that angle, that is Phi is fixed, Phi is between 0 and that angle. Phi is fixed. And what is the projection of that? That is a circle, for every point on the cone is a circle. So, you theta and r, r theta and Phi. So, that describes fully what is your cone?

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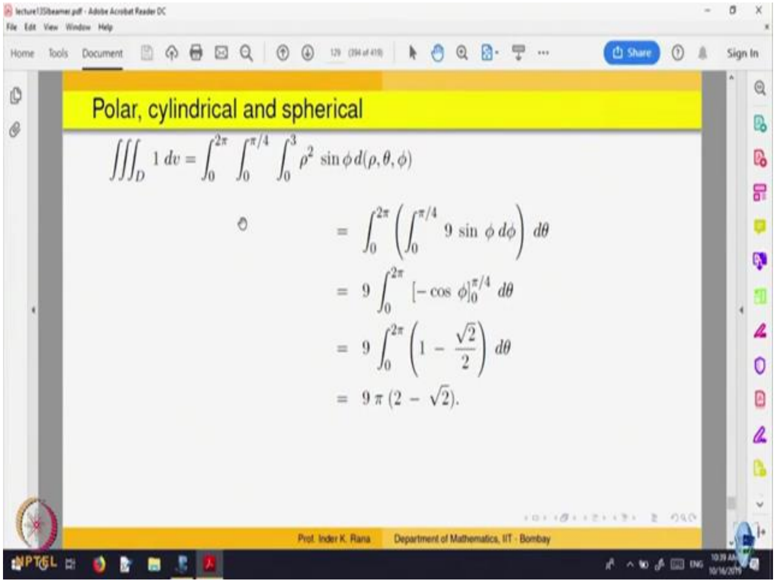

So, let me, so, that is what it looks like, okay. So, it comes out very nicely, theta you have to go on the cone, so 0 to 2 Pi that is your phi and you have to find how much is the r? The distance of a point from the cone, okay. So, you find out, so try to read if you cannot find out, so, there are many examples of this type. So, let me not I think go into all of them.

So, let me just summarize what we have done so that we can go ahead a bit. There are a lot of examples you can do a lot of examples, I would say if we want to do this thing, then you can pick up any book on calculus actually and look at multiple integrals there are very I think our library has lot of books on calculus, you can pick up anyone of them, to practice visualizing these domains and how you change them to various coordinates in the plane to polar coordinates in the space, cylindrical coordinates and spherical coordinates.

The basic is how much is the change coming in the plane d x d y, goes to r d r d theta appropriately the limits for theta. For cylindrical coordinates, it is r d r d theta and d z, x y z, z does not change and in spherical coordinate it is r sin phi, d r, d theta, d phi. Okay. I think we should not be using r in the r 3 we should be using D rho, because r and theta, r and rho can confused. So, distance should be taken as rho, rho, theta and phi instead of saying r, theta and phi, sometime it can be confusing, okay.

So, basically what we are try to do is? In integration of 2 and 3 variables, we have tried to define what is the integral of a function of 2 or 3 variables over a domain D. And we looked at intuitively as for the plane because the graph is a surface we looked at intuitively it is the volume below the surface, okay.

And for functions of r 3 you cannot visualize what it, what the triple integral looks like. Why cannot you visualize because the graph of a function of 3 variables is an object in r 4, and we cannot visualize r 4, we live only in r 3, okay. But triple integral, when you integrate the function 1 that gives you the volume, so, that is the advantage of it, triple integral gives you the volume of the object. You will looking at triple integral of D 1, that gives you the volume, okay.

And if I looking at double integral, looking at between the surfaces that volume between surfaces or the volume below a surface, okay. So, those are the and as far as computation is concerned, of 2 variable or 3 variable is done by Fubini's theorem. You can integrate a function of 2 variables or a function of 3 variables by integrating 1 variable at a time. For 2 variables your domain should be either of type 1 or type 2, then only Fubini's theorem can be applied.

So, what was domain of type 1? x y, where x lies between some limits, and y lies between 2 curves, so, that is type 1 and type 2 is other way round when your y lies between c and d. And when you move along x lies between some curve phi y and some psi y, then only Fubini's theorem is useful.

So, it says if your function is integrable over the domain then you can integrate 1 variable at a time, okay. Some domains may be interpretable as type 1, some may be interpretable as type 2, some maybe of both, some may be of none, you may have to cut up into parts and join them by known overlapping and then compute, so, that was of 2 variables.

And 3 variable, again the same thing, but in the 3 variable your domain D should be either projectable on to x y plane or y z plane or z x plane, if it is projectable your domain is predictable on to x y plane, then what does it mean? Projectable means there is a projection of the domain on the x y plane. So, whatever is the domain in that projection if it stand then you can reach the domain by going up or down from that point in the domain. So, you have to see how much is the z varying to be inside the domain.

So, your point x y will lie in the region r, which is a projection and z will be from surface to another surface, so, that is projectable on to x y plane, similarly, other. And if in particular your region r which is a projection onto either of the planes is of type 1 or type 2 in that plane, then you can further decompose your 2 variable integral over the region into 1 variable at a time.

So, the triple integral can be written as either integral with respect to x, then with respect to y and then with respect to z or depending on the type 2 with respect to y and then with respect to x and the inner most is with respect to z that is projectable onto x y plane. So, 2 possibilities, two iterative integrals. And similarly, for each plane projectable, so, there are six possible, it will be a good idea you take some region and try to write all possible six iterative integrals for that, that will clear all possible doubts in your mind what it is possible.

So, that is computation part, sometimes computation becomes easier by making a change of variable. And for the change of variable, you have to compute what is the Jacobian of the transformation, find out the absolute value of the Jacobian that is a factor by which you have to multiply d x d y. So, d x d y will be equal to Jacobian, as a function of u and v into D u D v. Of course, do not forget to write that domain in terms of u and v, and the function in terms of u and v, then only you can integrate, because the integration variable is the D u D v, so, everything has to be in terms of u and v, so, that is multiple integrals.

So, let me just summarize what we have tried to do till now is, we started looking at a function of 1 variable, we looked at limit continuity, the sequences limit continuity, differentiability integration of 1 variable. Same thing we have tried to do it for functions of several variables, we looked at notion of sequences, we looked at limit, we looked at partial derivatives and differentiability, and we looked at integration.

And all this was possible because there was a notion of distance available on real line or r 2 or r 3, the notion of limit, everything limit continuity, differentiability, integration, everything depends on limit. You define the concept of limit that is by using closeness, 2 points are closed, images are closed, and the domain there is a notion of closeness, then the range there is a notion of closeness.

Whenever points come close here, points come close there, and this closeness allowed us to define the notion of differentiability of sequences ohh sorry, convergence of sequences, this allowed us to say, when is a function continuous? When the limit is equal to the value of the function, again the concept of limit, closeness, differentiability, again differentiability of function of 1 variable was, when the slope of the secant, approaches a value limit, take a point nearby P and Q, look at the slope of the secant, f of x, f of x plus h divided by h limit, again the concept of limit is coming there into picture.

Whether you want to do, then derivative, applications of derivative, second derivative, all are limits only. Once you do that, curve sketching becomes possible, maxima, minima, and then (())(48:51) points, asymptotes, all these things become possible because of the concept of limit, limit of something happening. And all this we did it in r rho, r, r 2 or 3 and it all dependent on the notion of distance.

So, now, the question comes, can we define a object which may not be real line, which may not be plain, but it just the some set x, on that can we define the notion of a distance, closeness? Can we define the notion of closeness on some set, any set? If we want to define the notion of distance on any set, what should we the properties of that function?

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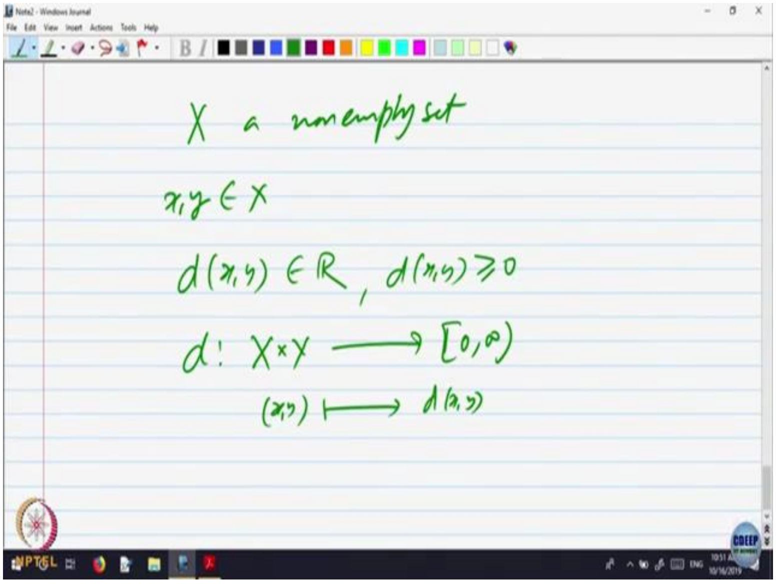

So, let us just look at what we want to do, x is a set, a non-empty set, we are given 2 points x and y belonging to x, I want to say that x and y are close to each other that means what? That should be some way of measuring closeness. So, that is going to be, when d x y, that is the notion of a distance between them, what is how much is the distance between them?

So, this is going to be a number, it is going to be a number, distance is a number and distance is always a number which is non-negative. And it depends on x and y, it changes as x or y chain.

So, this is a function defined on x cross x, taking values in 0 to infinity. So, a distance function, if we want to define? It should be a function defined on the cross product x cross x x comma y goes to d of x y, which is bigger than or equal to 0.

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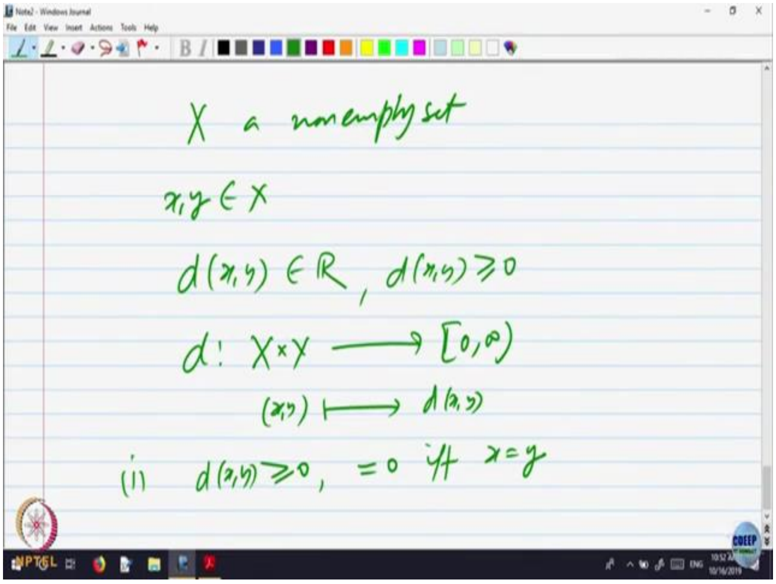

We want such a function but any, will any function do a job like this? Physical notion of distance, what are the properties of physical notion of distance? Of course 1, we have said d x y is bigger than or equal to 0. When is it equal to 0? If x is same as y, can it be 0 when x is not equal to y?

No. So, it is equal to 0 if and only if, x is equal to y, the distance between the two point is 0, if and only if distance is 0, x should be equal to y and if x is equal to y then that distance is equal to 0, this is a second property, one.

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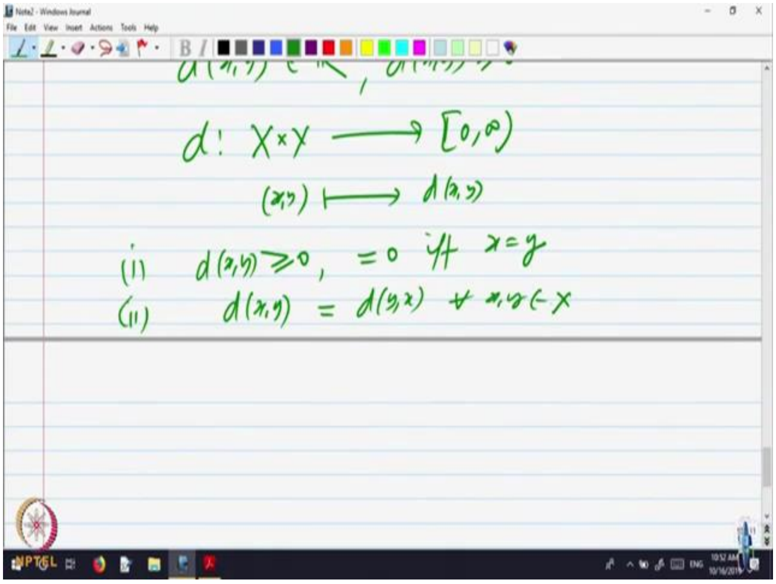

What is the second property? Let us look at the distance between x and y and distance between y and x. Logic says, physical say that does not matter whether you go from x to y or y to x, so, it should be equal for every x y.

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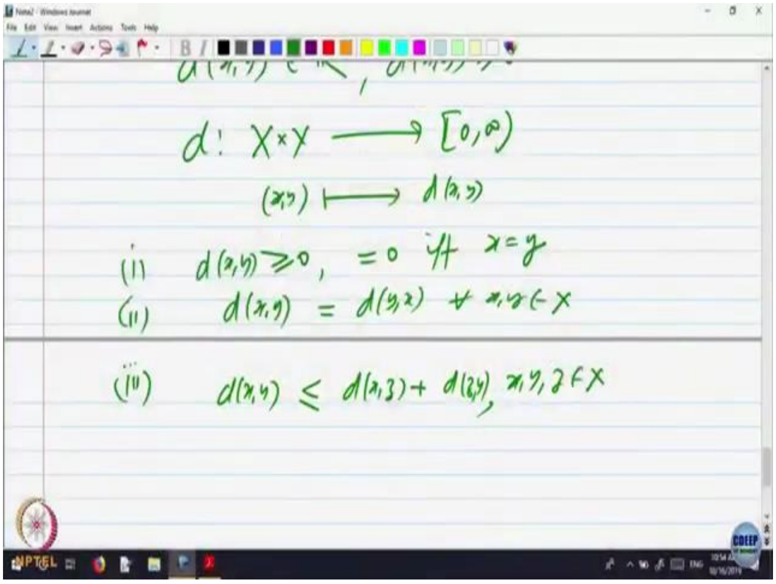

And then comes the third point, let us take three points x, y and z belonging to x, look at the distance between x y, distance between x z, and distance between z and y. What is the relationship between these three? Can I say, distance between x and y if z is a point in between say, this equals this plus this or what about if it is outside? So, let us go back to real line, let us look at our previous knowledge on the real line, absolute value gives the notion of distance, distance between two point x and y was absolute value of x minus y.

So, what is the property of the absolute value function? If x y and z are given distance between x and y, absolute value of x minus y is not equal to always x minus z plus z minus x, it is not always equal, it is only less than or equal to absolute value of x minus y is less than or equal to absolute value of x minus z plus absolute value of z minus y. So, in general this may be the best possible thing we can expect, because, one notion of distance which we have used and has given us nice results has this property.

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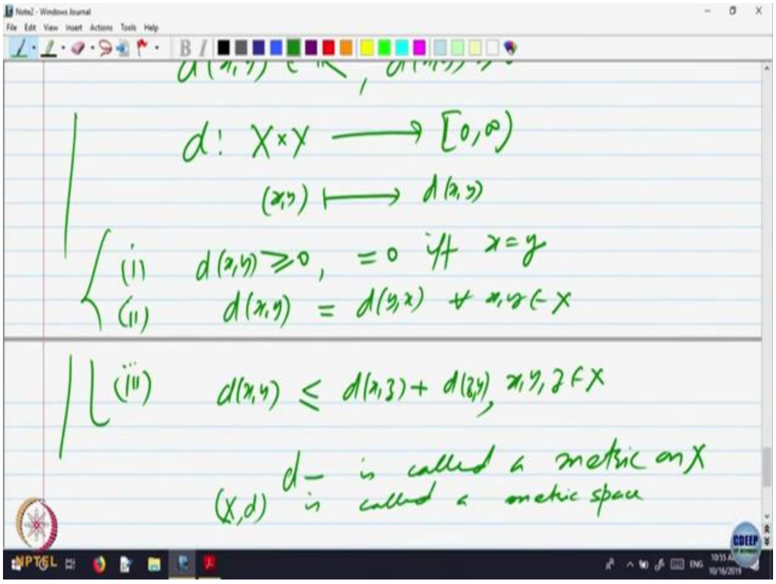

So, this is these properties if we want to define notion of distance on any set that function should have these properties and wherever the function has this properties this is called d is called a metric, it is called a metric, metric is the distance, it is called a metric on x. And x, this pair is called a metric space, it is called a metric space. So, it is a pair, x is a set and d is a function defined on x cross x with these properties, on the same set you can have more than one notion of distances, that will mean what?

If, in this pair if either x or d changes, then the metric space changes, you can have real line with absolute value one metric space, you can have the non-negative real numbers with the same notion of absolute value that is a different metric space, because a set same is not same. So, when you say two pair, ordered pair x comma d is same as y comma some something that means x is same as y and other object is same, if either of it changes your example your metric spaces has changed.

So, what we are going to study is, look at sets and different examples of notion of distances, on the same or different sets, okay. And see, what are the things that we have done? I will not be proving everything because this goes into a topic called metric spaces which is a different not a different which is in itself a topic for a study, one semester course can be run on metric spaces, but we will try to at least expose you what are the concepts that carry over in metric spaces, what are the concepts which does not carry over?

But remember, notion of distance allowed you to define what is called an open set, what is a closed set, and then in terms of open sets, closed sets, you defined what is compactness, connectedness and so on. So, all this concept you can study on metric spaces, you can study continuous functions on metric spaces and so on.

So, what are the things possible, in what form they generalize? In what form they do not generalize? Will not prove, we will only give you exposition that there is a concept of metric space which looks like a general set a notion of distance, you can do some things which you have done on real line r 1, r 2 and so on, there are some things you cannot do.

For example, in r 2 itself you cannot compare points, in the real line you can compare, one is bigger than the other, in the plane you cannot say a point x 1, y 1, is bigger than x 2, y 2, order is gone. So, whatever you do with respect to order on real line you cannot do with respect to order on, there is no order on r 2 or r 3 and so on, so, we will see.

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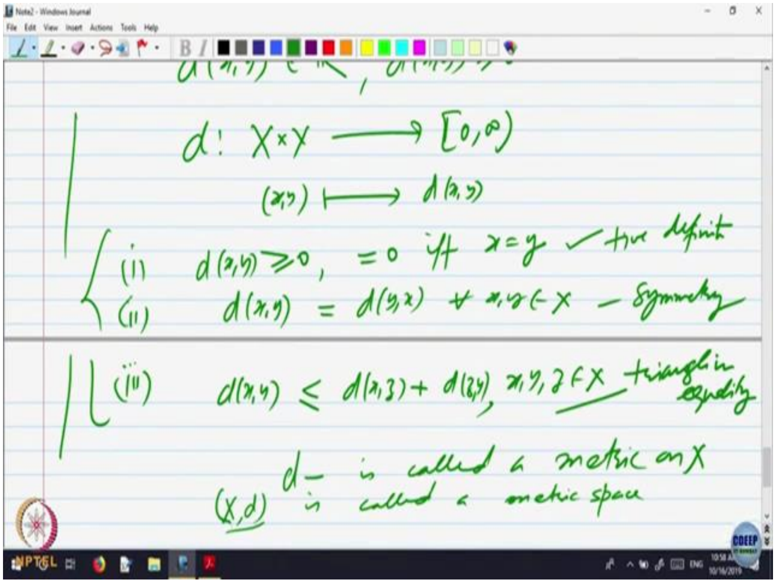

By the way, this property is called positive definiteness. Distance is positive definite, is non-negative, and 0 definitely when x is equal to y, this is called symmetry, the distance is symmetric with respect to x and y and this property goes by the name of triangle inequality. It is something saying that the sum of two sides of a triangle is always bigger than the third sight, so, it says that thing. So, whenever there is a notion of a set a distance that is called a metric space. So, obvious our example is real line with absolute value, that is a metric space. We will see others, okay.