Basic Real Analysis
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Lecture - 54
Change of variables - Part III

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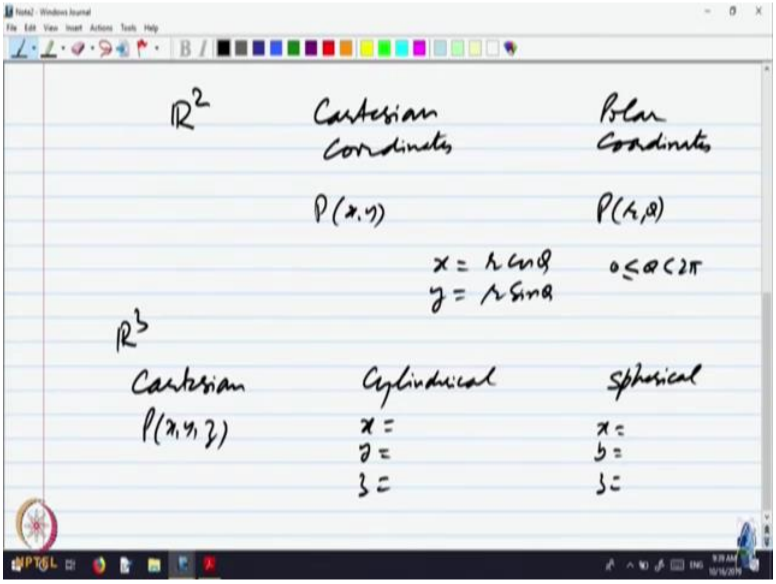

So, let us start looking at the change of variables. So, we saw yesterday in the previous lecture that in R2, there are Cartesian coordinates and there are polar coordinates. So, for a point P with Cartesian coordinate x y, the polar coordinates are r and theta, where the relation is x is equal to r cos theta and y is equal to r sin theta, theta goes from 0 to 2pi. In R3, we had 3 coordinate systems, namely Cartesian and then we had cylindrical and then we had what is spherical.

So, we will revise them again today, what are these coordinates. So, this is P x y z and what is x, what is y and what is z and similarly what is x, what is y and what is z. So, we will describe them again. But before that, let us start looking at what is called the change of variables.

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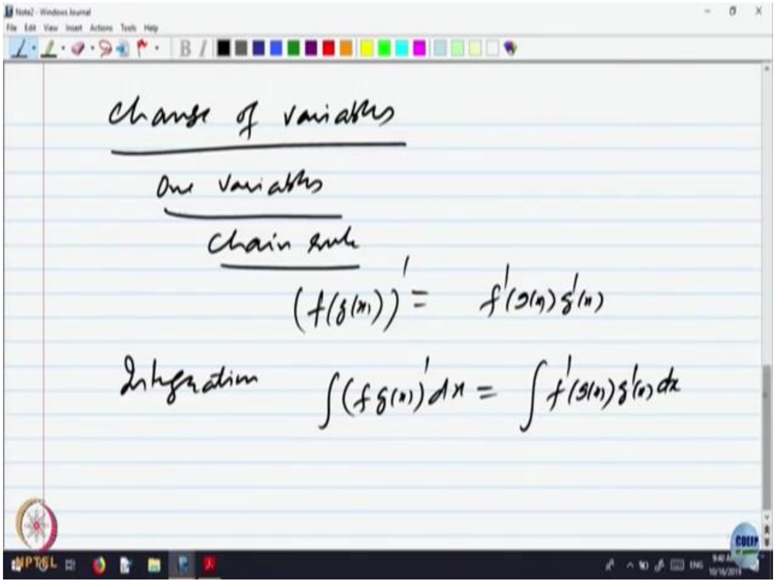

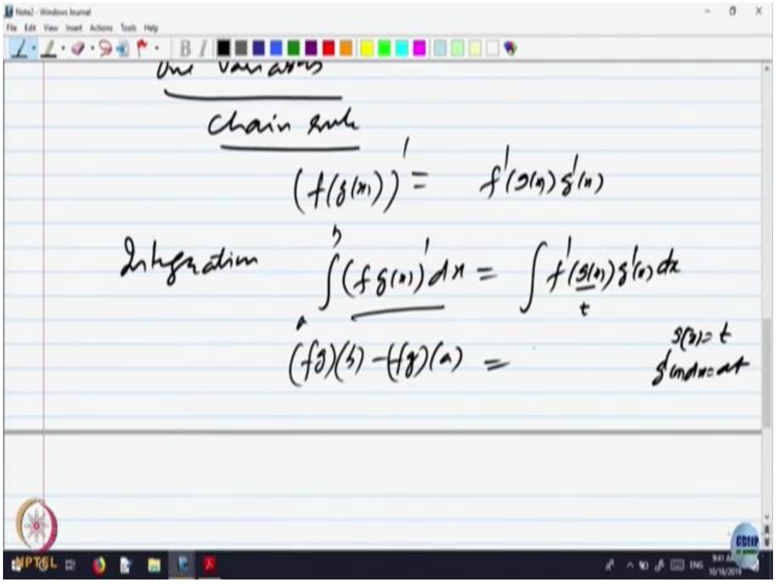

How these are related with change of variables, in one variable there is a chain rule for differentiation, which says that if we have a composite of functions f of g x, then its derivative is f dash at g x into g dash of x. So, that gives you a change of variable formula for integration. So, it says that if you want to integrate, if you integrate both sides, you will get integral of f g x derivative dx is equal to f dash of g x, g dash of x dx, with appropriate limits of integration.

So, one way of viewing this is if you call this g x as t, then the derivative f dash t is equal to f, then what is g of x is equal to t, then g dash of x dx is equal to dt, that is what is normally the change of variable formula in one variable it is called. So, and this side is nothing but integrating the derivatives, so f composite g if the limits or a to b, so b minus f g at a.

So, that is equal to this, so change of variable formula helps you to compute the integral by making a change in the variable. Instead of looking at the variable x, you look at the variable g x, so that is a new variable you introduce. A similar thing is possible in say in 2 dimensions.

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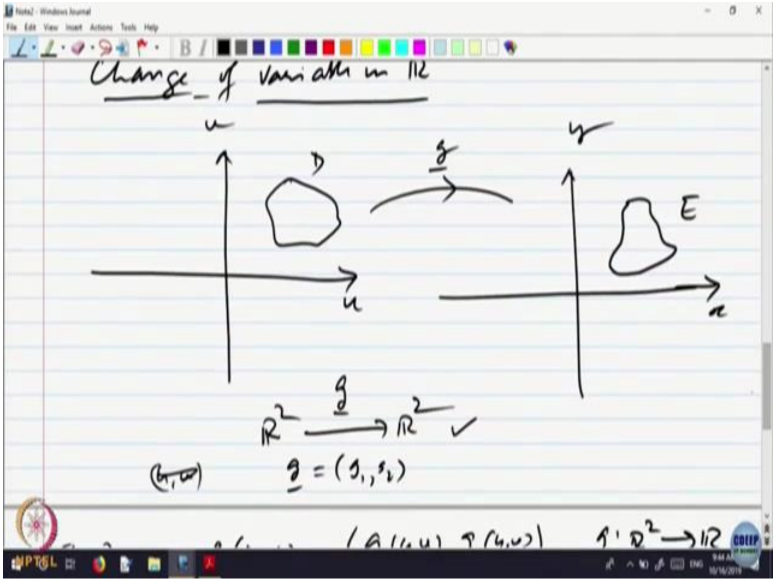

So, change of variable let us look at in R2. So, to describe this let us have a variable u and v and we are making a transformation of this to x and y. So, there is a domain here and there is a transformation which changes it to some other domain E, so call it as D, call it as E. So, what do I mean by this transformation?

It is a map from R2 to R2. So, g is a map from R2 to R2. For example, u and v okay. So, g is a map. So, it is a vector valued map because it is in R2, so it will have 2 components g will have components let us say g 1 and g 2 Okay.

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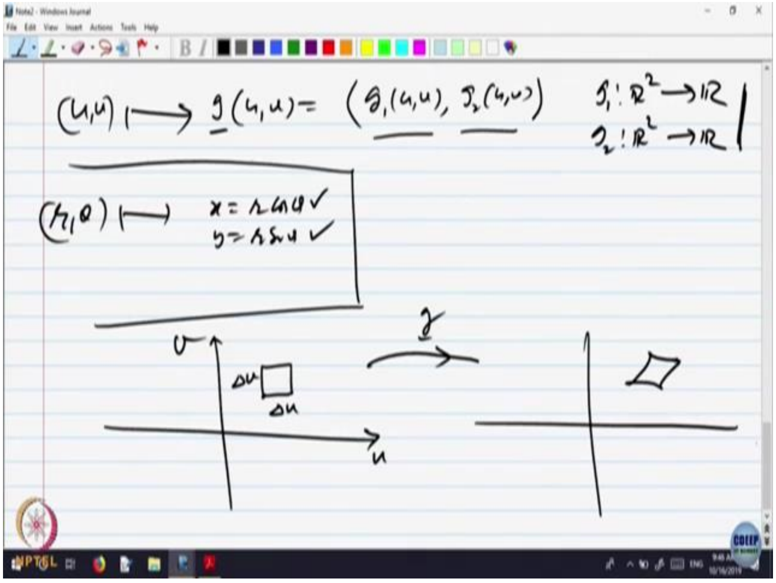

That means what that means, g of u, v is equal to g 1 of u, v, g2 of u, v, where g 1 itself is a function from R2 to R and g 2 also is a function from R2 to R. See, for every point u, comma v, okay. So, let us write u, comma v that goes to g of u, comma v, that is a point in R2. So, it will have 2 components, this is a first component, that is a second component. So, each component is a function of the point u and v. So, they are functions of 2 variables.

So, now, for example, you can have if you like in the mind you can keep an example, r, theta goes to. So, what is g1, that is x is equal to r cos theta and y is equal to r sin theta. So, this is g 1, this is g2. So, polar coordinates representing something in polar and Cartesian that relation can be thought of as a transformation from polar to Cartesian or Cartesian to polar, either way it is okay because is the one-one onto map.

So, in general we will have a transformation g from R2 to R2. So, in the domain the variables will be denoted by uv and in the range we are denoting it by x y. Once would like to know, if I make this transformation, domain D changes to a domain E, what is the relationship between the area of g and, area of D and area of E, what is the relation between the 2 areas when you make a transformation.

So, to do that, let us imagine, let us imagine a small portion of inside that D. So, this is a point and let say it is delta u and it is how does a small area element change, we want to look at that and this is, this length is delta v.

So, this is a rectangle of length delta u delta v and there is a transformation g, we want to know how does it change. So, it will change to some kind of area like this okay. So, what is the relation between the 2 areas?

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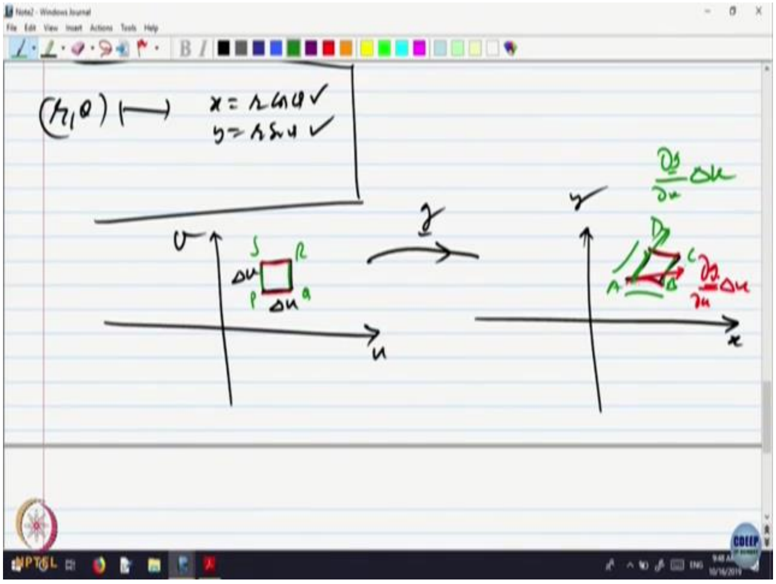

So, this is x and that is y. So, let us think it of as a movement, at this point you are traveling in the direction of u by distance delta u. So, when you transform it, this is what you will get and this upper one will be transferred here and similarly this side will be transferred, that is image of the small portion. We would like to know what is that area. So, we can think of this small portion, this small portion on the right hand side in the xy plane to be a parallelogram.

And to find the area of a parallelogram if you know what is, what is this vector and what is this vector, then the cross product will give me the area of the domain generated by the 2 vectors. So, what is this area? Now, this is not going to be a straight line, but for a small portion we can assume it is a line. And how will you move, as you move in du, let us call it P, Q, R, S. If you as you move on PQ, you are going to move here along AB. And the distance you are going to move in the domain is delta u.

So, we want to know what is the rate at which you are moving along in the direction of A to B. So, what is that rate, rate of change is given by the derivative. So, this AB you can think it off approximately. So, this can be thought of as a tangent vector. So, what is that tangent vector? So, that vector has got. So, partial derivative of g, this is a partial derivative of g with respect to u direction so u. So, this is a vector function, is a vector.

And distance is Delta u so this is going to be the vector and similarly the other one if we look at this one, this is going to be partial derivative of g with respect to v delta v. That is approximately A B and this is approximately AD, we are approximating the curve by the tangent vector. So, what will be the area, the small area on this side?

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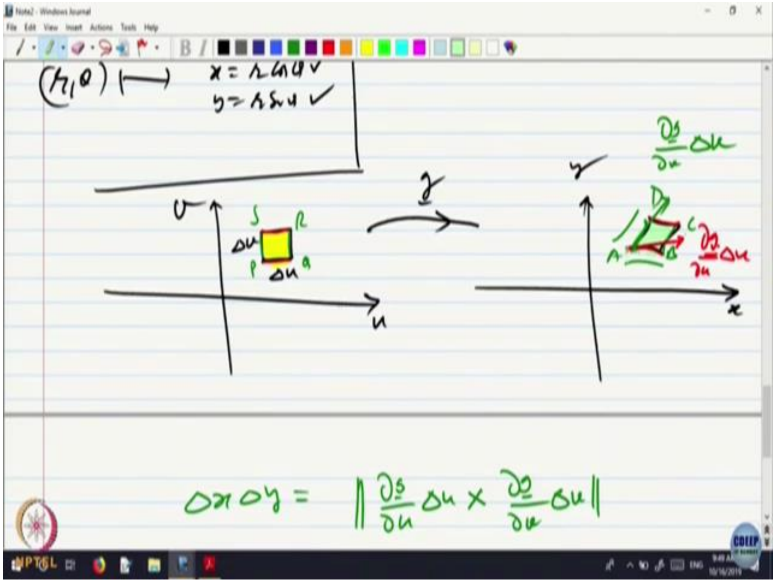

So, delta x delta y will be given by. That is a cross product of delta g Delta u, cross product with the I am using the vector algebra that you are supposed to be familiar with. That you are given 2 vectors a and b, then the cross product, absolute value, magnitude of the cross product gives you the area generated by the 2 vectors, the parallelogram. So, that is a definition of cross product actually, you can also think of, so v, delta v.

So, that is this, area of this small rectangle or small parallelogram. So, basically what we are saying is this is going to be change to and if we call it delta x cross delta y, delta x delta y that is going to be, what is this equal to, let us just compute. So, how do you compute that?

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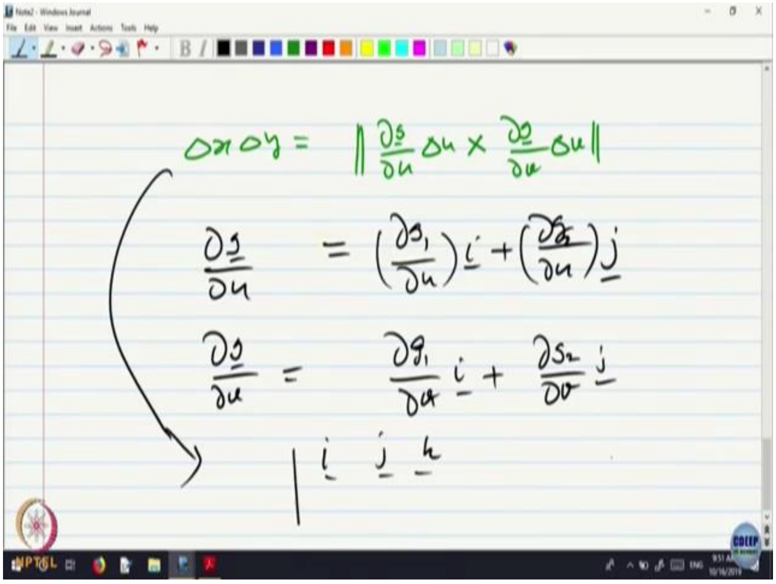

So, what is this equal to, cross product. So, let us first write what is delta g delta u so that is Delta, now first component is g 1, g has got 2 components with respect to u i plus g 1 g2 with respect to u and that is j. This was the component, the vector differentiation, g has got component g 1 and g 2, partial derivative of g 1, partial derivative of g 2 and similarly, this one partial derivative of g 2, g 1 with respect to u i, I have to write the bracket, So, delta g 2 with respect to v so, this is respect to v of j. Then what is this cross product? How do you get the cross product of 2 vectors?

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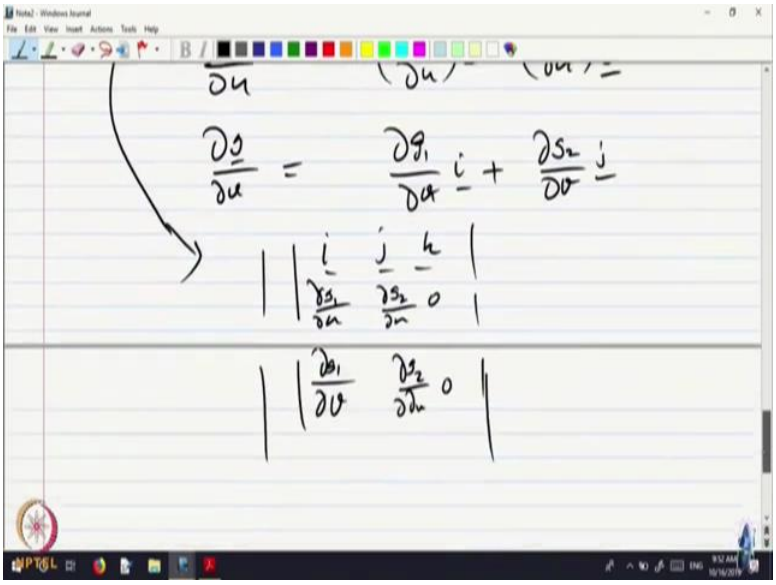

So, that is i, j and k my components partial derivative g 1 with respect to u, partial derivative of g 2 with respect to u, third component is 0, because we are in the plane and so delta g 2

Delta, sorry delta of g1 with respect to delta v delta of g2 with respect to delta v and then 0. So, when you expand this, what you will get is the norm of, we want to compute the norm of this, that is the cross product.

So, we want to compute the norm of this. So, when you compute the norm and simplify, you will get this is nothing but, so delta x delta y.

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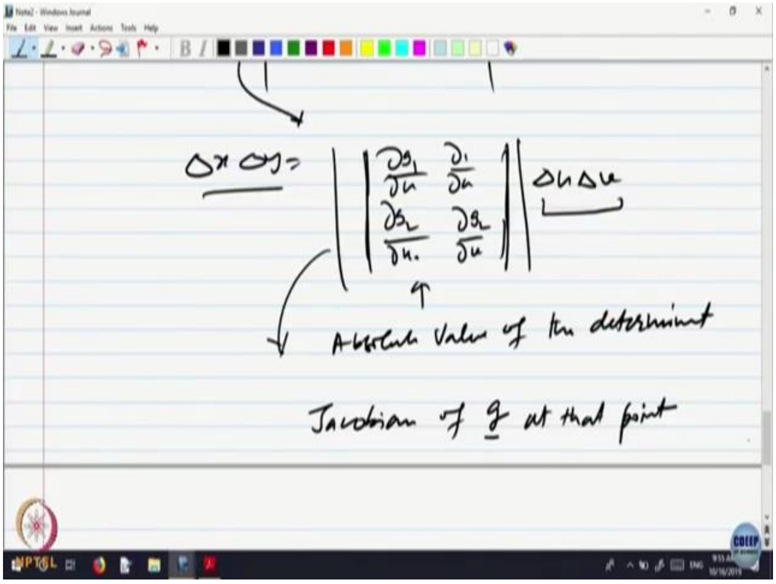

So, this thing comes out to be, these are scalars, so delta u delta v, these are scalar. So, they will come out. So, delta u delta v and you will have partial derivative of g 1 with respect to u partial derivative of g 1 with respect to v, partial derivative of g2 with respect to u, partial derivative of g 2 with respect to v. And this is the absolute value of the determinant. So, you should have, you should have absolute value of the determinant, absolute value of the determinant.

So, we expand it with respect to I, j, find out the norm of that vector. So, that will simply comes out, the absolute value of the 2 by 2 determinant, because there are 0s here. So, when you expand will get only one component. Is it okay? Delta g1, delta g2, okay minus, so, that is the determinant coming here. So, this quantity becomes important. So, the small area element in the transformed thing is scaled up according to this factor.

Like in 1 variable what was a scaling, in 1 variable the scaling is g prime. Here in 2 variables, both the components contribute and that is a scaling. So, you get delta x delta y is, so this quantity is called the Jacobian of the transformation g at that point whichever g at that point, you are looking at a point. So, this is a point at a point you are looking at.

So, basically the idea is small element is transformed like this. So, if you want to look at the full integral, that will be limit of the small elements, so that will give you the change of variable formula. So, let me write the change of variable formula okay. So, in general.

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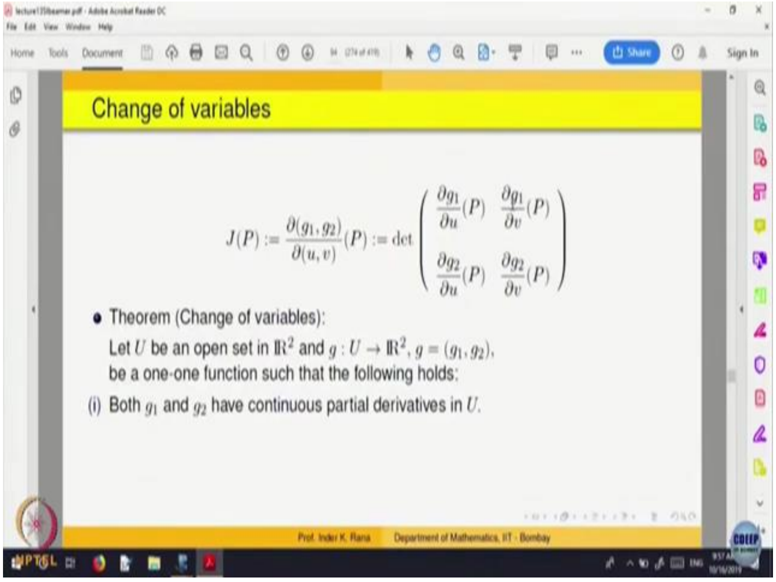

So, this is so this is called the Jacobian which we defined just now. So, Jacobian of the transformation g with components g 1 and g2. The first row is partial derivative of g 1 with respect to the variable u, partial derivative of g 1 with respect to the variable v. So, first row is with respect to g1, 2 variables, g2 with respect to 2 variables determinant of that, that is called the Jacobian. If you want the area that is always a non negative quantity, so you have to take the absolute value of that.

So, now let us for a small element we have seen how the change looks like. So, if we put the small elements, area elements together, we have what is called the change of variable formula. So, it says let u be an open set in R2 and g be a map from u to R2. So, that is, keep in mind that picture.

So, g. So this is a domain and g is a map from here to here, we are assuming this is open, it is an open set, so that you can have increments in all the directions, that is the basic idea. The g has got components g1 and g2, right.

And we are assuming it is a one-one function, this transformation is a one-one function, otherwise everything can collapse into 1 point possibly. The areas can get merged if the g is a constant function for example, then everything will get merged. g1 is constant, g2 is constant, so we want the transformation should be such that from u, v to x, y you are able to go and you are able to come back. Similarly, like in Cartesian coordinates you are able to go to polar coordinates and from polar coordinates, you then come back to Cartesian coordinates.

So, it should be one-one map. Otherwise there is no meaning of saying how does the things change? Both g 1 and g2 are continuous partial derivatives, that is just to say that when you multiply by this kind of Jacobian, everything is continuous and you are able to integrate. So, that is a mathematical condition you have to put. The partial derivatives are continuous of both components g 1 and g2.

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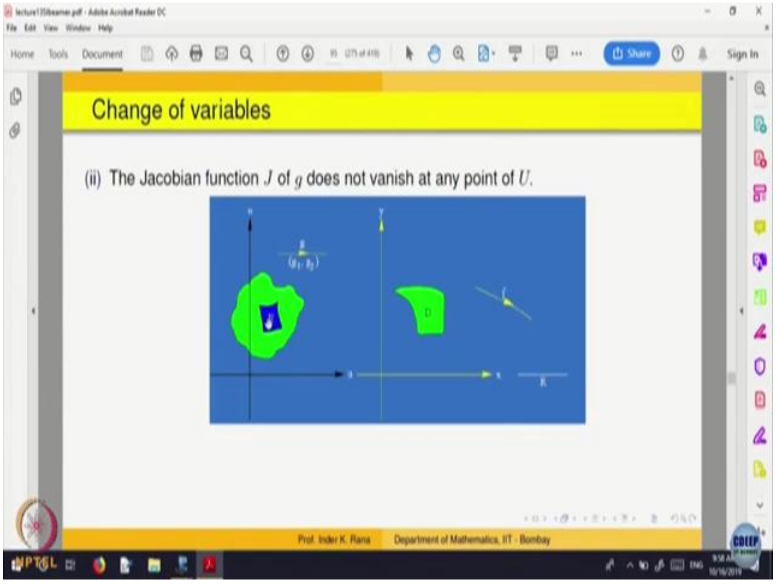

So, with these conditions, so this is what the picture looks like, a small element goes there g is the map.

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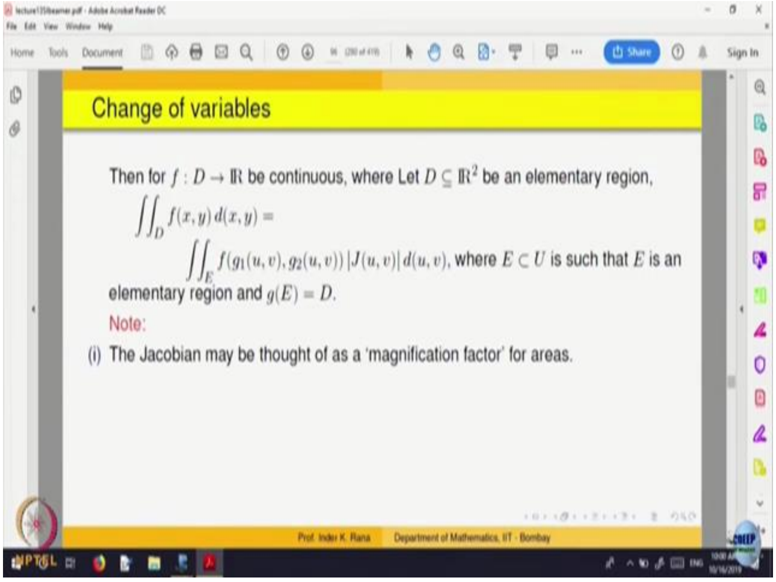

So, what does the statement say? It says, if we want to integrate the function f over a domain D in xy plane, x, y is range, that you are integrating a function. So, you make. So, it says you make a change of variable, call x 1 as g 1 of uv. So, you transform the variable x to u and v by the transformation x is g1 of uv, y is g2 of uv, if you make that transformation, then this function will be a function of the variable u and v.

So, we are saying how does the integral of f over the domain D in xy plane looks like integrating this changed variable thing in the uv plane. So, for that first you have to transform the region D, how does the region D look like in the uv plane? So, say if that is E, then you have to multiply small element du dv by the Jacobian, absolute value of the Jacobian, that is what we saw.

So, what it is saying is that this dxy, the small element, area element changes to Jacobian absolute value of the Jacobian times du dv, that is the magnification that is coming. In 1 variable it comes g dash, in 2 variable it is a Jacobian of that transformation that comes. So, one can think of Jacobian as the magnifying factor, the small area dx dy, that is getting magnified as Jacobian of u and v times the changed area. Times delta u delta v.

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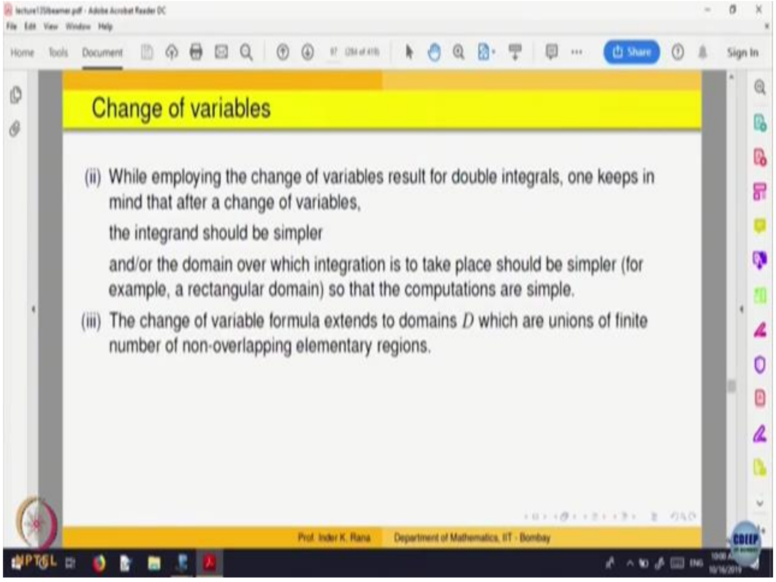

We can extend it to domains where known overlapping and so on. I think the best thing is to look at some examples. We will come to polar coordinates also.