Basic Real Analysis
Professor Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture - 53
Change of Variables-Part II

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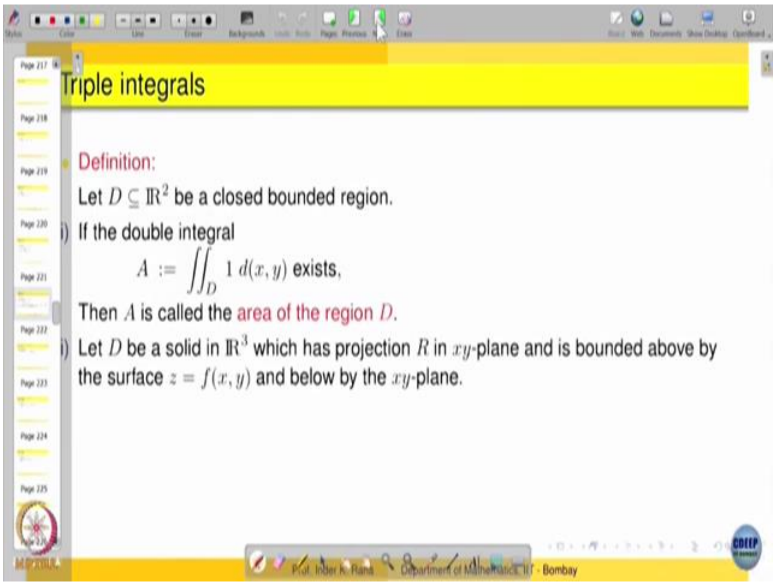

This is okay, double integral when you integrate function 1, what does it give you? The double integral over D gives you the area of the region D, because we are taking smaller triangle, smaller pieces and then integrating. When there is a function of 2 variables that is the area, when it is 3 variables that gives you the volume of the solid bounded by below the curves, below the surface that is equal to f xy.

That is one way of visualizing it. I think these examples are okay. I think these examples you can read, so let me, say what I want to, what is this, multiple integrals, okay. Examples, then again examples are there. Example, I think this will be interesting to see that example.

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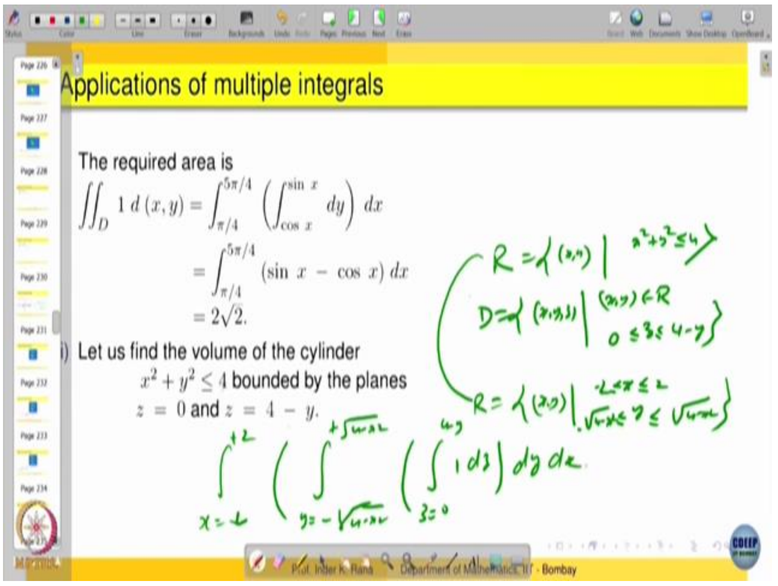

What is that, because. Let us find out the volume of the cylinder bounded by the planes z is equal to 0 and z equal to 4 minus y. So, let us try to visualize it. How does it, object look like? That is why I am trying to do this example. x square plus y square equal to 4, that is a cylinder. What is axis of the cylinder, x-axis because x is independent, X is not mentioned there. So, at any point X, if you want to look at what are the points in the cylinder, then they are x squared plus y squared less than or equal to 4.

That means section, any section of the cylinder is the disc, y squared, x squared plus y squared less than or equal to 4. So, it is a circular cylinder, every point intersection is a circle and bonding. But then z is equal to 0 and z is equal to 4 minus y, so it is a cylinder if we visualize x axis as. Okay, then z is equal to 0 that is a bottom, z is equal to 0, that is a bottom and up to how high it goes, z equal to 4 minus y.

It is a function of y only, x is independent. So, what will a triple integral look like? Projection is x squared plus y squared less than or equal to 4 equal to 4. So, that is of region R, so region R is x y, such that x squared plus y squared less than or equal to 4. And what is the region D, volume of the cylinder? So, what is the D, so D is x, y and Z such that x, y belongs to R. And why does z vary? z goes from z is equal to 0 and top is 4 minus y.

So, this part, we can write, this R if you like you can write it as x y type 1 if you want to write, x goes from minus 2 to plus 2 and y goes from y squared is less than or equal to 4 minus x squared. So, the positive part will be square root of 4 minus x squared minus less than or equal to positive part 4 minus x squared. So, when you want to integrate, we will be integrating x between minus 2 to plus 2. Integrate y goes from minus square root of 4 minus x squared to plus square root of 4 minus x squared. And Z goes from 0 to 4 minus y 1 dz dy dx. So, that will be the integral. All right.

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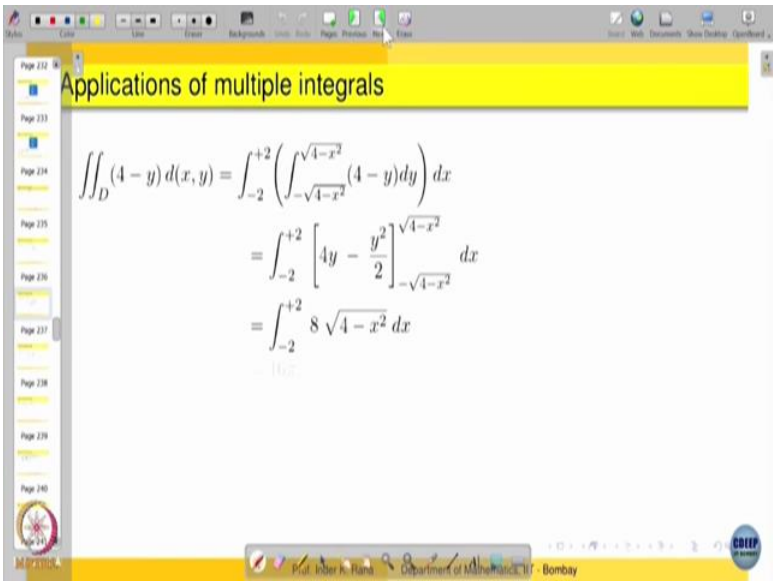

So, that is what it looks like, dy or dx whichever way you want to write, so you can integrate Okay.

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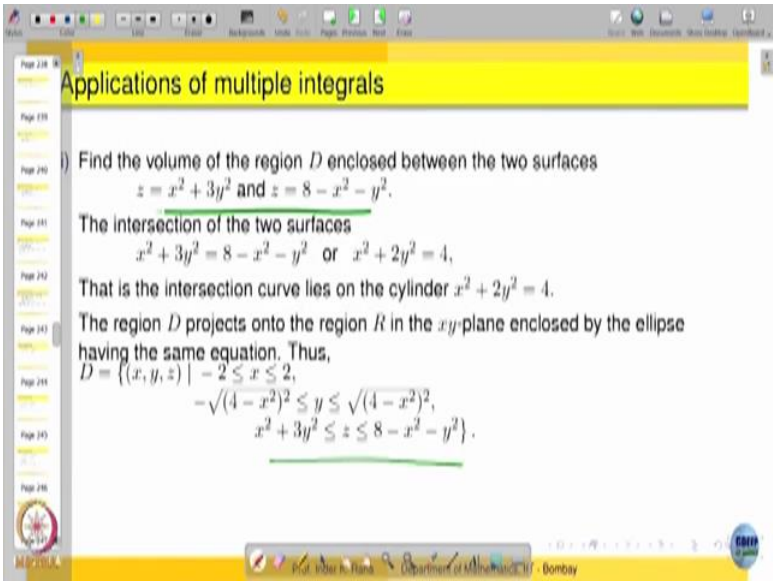

Let us do one more. Find the volume of the region D enclosed between 2 surfaces this and this. Z is equal to x squared plus 3Y squared and z is equal to 8 minus x squared minus y squared. In these 2 surfaces, so it is like one surface, another surface at the bottom and the region enclosed. So, how do you find out the projection onto xy plane? It is the intersection of 2 surfaces, one surface and other surface intersecting, so there will be intersecting curve. That will give you the projection on to the xy plane.

So, how do you find the intersection, that curve? That is where the 2 are intersecting, so z is the same for both. So, z of one surface is equal to z of other surface. So, when we equate these 2, so you get x squared plus 3 y squared is equal to 8 minus x squared minus y squared, will get that surface. So, that is a that is a curve where the 2 intersect and that is a projection. Are you able to visualize? Yes, imagine 1 cup like this, another cup like this intersecting somewhere.

Okay, that is a solid, so when we project it, this region will get projected onto xy plane, that is a region R. And what is the solid, for every point it goes from the lower surface to the upper surface. So, you can write down the integral very easily. So, project onto the region in the xy plane. So, and that you can write as, that will be an equation in X square and Y square only.

So, that is type 1, you can write it as x goes from minus 2 to 2 that curve projected, y goes from 4 minus x squared square root to plus lower part to the upper part the projection only. And the surface so, that is the from this to this. You have only to find out which is a lower surface, which is upper surface out of the given ones, which one is the lower which is the upper because you have to go from the lower limit to the upper limit.

So, we have to find which values of z gives you the lower, which values of z will give you. Okay, so you put some values and analyze. That is how triple integral. There is computation part, so I leave it okay. So, it gives you the volume of the solid okay. Other way around many integrals are possible. So, you can write this as type 1, type 2 and so on. So, let us not go into all this, again something similar.

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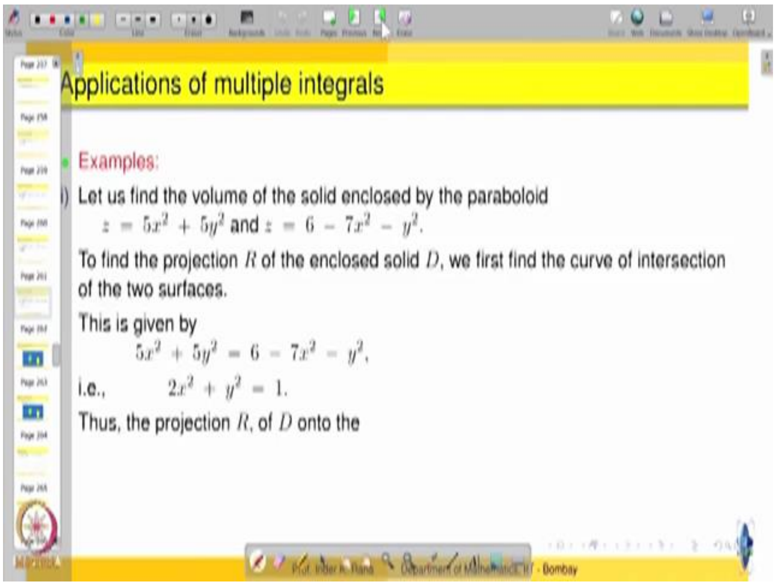

So, remember z is equal to this, z is equal to this, when you find the intersection of the 2 what is a common thing, this equal to, so that gives you 2 x squared plus y squared equal to 1. So, what is that look like, x y equation in x and y only, there is a curve where they intersect. So, that means that is a projection on to the xy plane, so that is an ellipse, right. 2 x squared plus y squared equal to 1. So, that is a projection. So, that ellipse, you have to draw that ellipse and see what is the major axis, minor axis and analyze okay.

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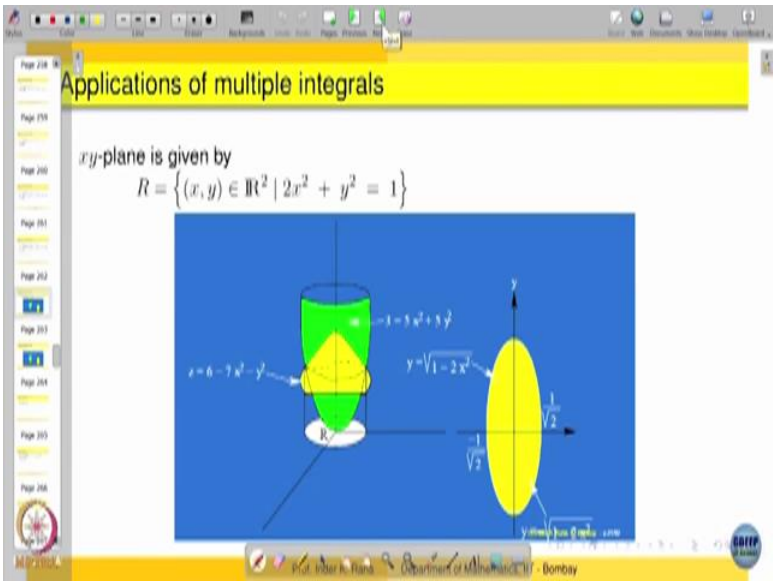

So these are all. You should sit down and analyze these things. So, that is what it looks like okay. So, because, so that will be the common part and okay.

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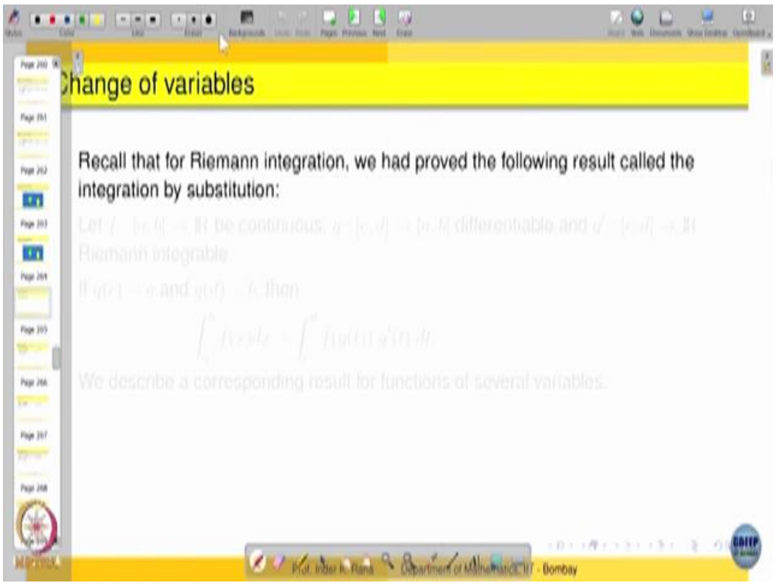

Next, what we want to do is what is called a change of variable formula. So, that is what we have started looking at last time, okay.

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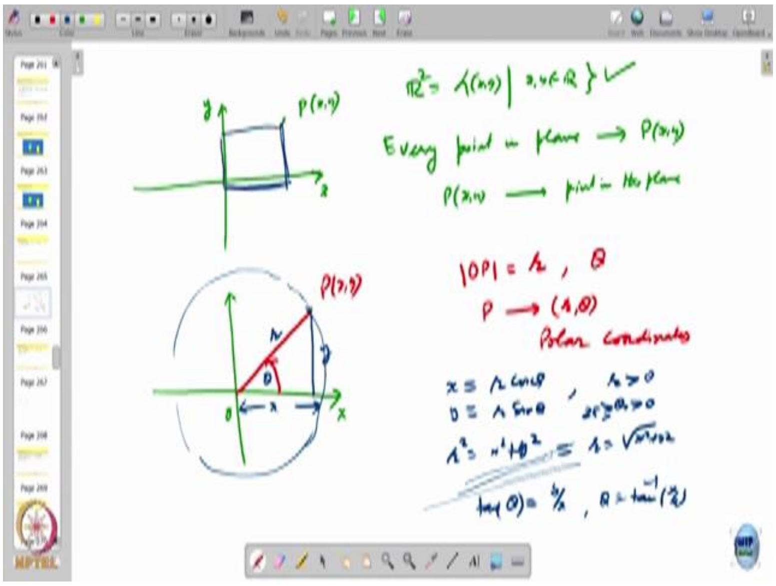

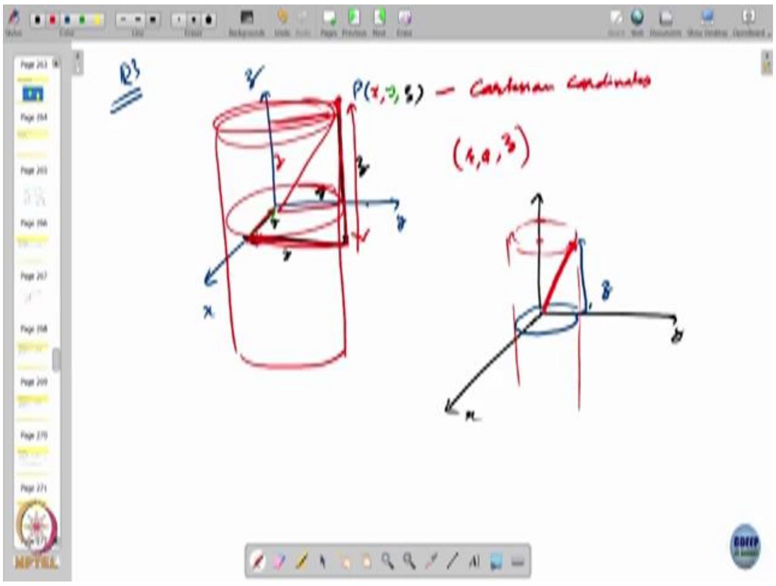

Namely in the plane, we said given a point you can draw coordinates x and y. So, R2 which is all as a set x, y, such that x and y belonging to R. This notation comes because every point in the plane that is a geometric object, a plane. So, what is the way of describing it? Every point in the plane gets associated with the point P with coordinates x and y and every point P with coordinate x and y will give you a point in the plane.

So, the geometric object which is a plane can be described analytically as. There is a way of locating a point in the plane. And we said there is another way of locating a point in the plane, you can have a reference line x and take a point P. How do I locate the point P, you can find out what is the distance of that. So, this is a point P. So, what is the distance OP? So let us call this distance is equal to some r okay.

And from the x, from this reference line how much you have to turn to go to that line that is the line OP, so that is angle theta. So, every point P can also be represented if you know what is R, what is Theta. So, these are what are called polar coordinates. So, these are what are called the polar coordinates. So, if this point has got Cartesian coordinates x and y, then what are the polar coordinates? What is the relation if I want to transform Cartesian coordinates to polar coordinates, so what is the relation?

So, this is my, if I want to translate that, so, this is my y and this is my x and this is theta. So, that is x is equal to, x is equal to if this distance is r, so what is x is equal to and what is y equal to? r cos theta and r sin theta, where r is bigger than, if P is not origin. So, then it is R is bigger than 0 and theta is between 0 and 2 pi. So, that is a relation. So, you can write x is equal to this another way of writing okay. So, that is the relation between x and y.

So, you can, if you know theta and r, you can find x and y, if you know x and y you can find r and theta. How do you find r? So, r squared is equal to x squared plus y squared, R is positive. So, implies r is equal to positive square root of x squared plus y squared and theta is y by x, tan theta is y by x. So, you can find out theta is equal to tan inverse y by x. So, that is the relations between x and y and r and theta. You can go from one coordinate system to another.

Let us look at some more. So, this is R 2. Let us look at in sorry okay. Let us look at R3 what is possible. So, we have got x, y and z. By the way, this meant, so these are polar coordinates and I should have said that for every point in the plane, you get r, Theta a pair and every r, Theta gives back you a point in the plane. So, that is a one-to-one correspondence geometric object and that. And to visualize this, try to visualize that, at this point if I take a circle of radius r, then all the points on this circle will have same r. Distance is same, only theta is going to change between 0 to 2 pi.

So, you can imagine in the Cartesian coordinates you are looking at. So, you are looking at this as the corner of a. This is a corner of a rectangle. So, you can imagine the whole of R2 as built up above rectangles. Okay. Here, you can imagine whole of R2 as built up of all circles. Every point in the plane will lie on some circle, that will determine its R and how much, where on point on the circle that will give you the theta. So, visualizing R2 as concentric circles, filled up of concentric circles.

So, what we want to look at is this one now, so Cartesian coordinates. So, let us look at, there is a point P with components x, y and so x, y and as well as say z. So, how do you find this geometrically, how do you find these points? So, let us draw a perpendicular okay. This is my z, projection on to the xy plane, x is equal to 0, y equal to 0, what is z. And having reached z you should move either in the direction of x or in the direction of y.

So, you can move in the direction of x or in the direction of y, that means you will get these point. So, this is your y and this is your x. So, how do I reach a point? Move along, so that gives you z and then move along the y axis and then move along x axis. So, this is same as y, is it okay? If you like you can remove that point, you can remove this, so let us say that is z it is black, green is y, so this is green, that is y and red is x.

So, from any point, move along z axis, and then move along y axis and move along x axis. So, you can see that you need 3 directions, you need 3 directions to reach a point, from the point to the origin and same backwards. Given any point you can reach by traveling along this. So, that is why it is R3, it is called 3 dimensional, you need 3 dimensions and a reference point. So, these are the Cartesian coordinates.

So, if these are the Cartesian coordinates what other ways? So, in the Cartesian coordinates geometrically what we are doing? We are looking at a box whose one side is x, other side is y, third side is z and this is a corner of that box. So, we are looking at parallelepipeds. One corner is at the origin, other is some other corner, that is a diagonal one. And imagine the whole space being filled up with these parallelepipeds.

So, that is a Cartesian coordinates. Another way of visualizing this would be, let us look at this point as a point on a cylinder, as a point on a cylinder. This is the point on a cylinder of, circular cylinder of radius, what is that radius it is at some height z. Now, how do you look at this point? How high you go on that cylinder, how high you will go on that cylinder and on the cylinder where will the point be. Where will be the point on the cylinder? I want to locate all the points on the cylinder.

So, all the points on the cylinder are located by the height and the distance. So, z is as it is, okay, x and y. So, if I look at this, this is a circle and it is a point x squared plus y squared equal to something. So, you get polar coordinates. So, x and y, so polar coordinates r and theta. How do you get r and theta? How do you get r and theta? So imagine this is here now, that point is lifted up or down depending on z. Okay, so this is the radius. So, what is the radius?

Once you have. So, let me draw it again so that you are able to visualize. So, this is sorry. Okay. So, if I have. So, that will give me surface height is z. Let us say this, what color I should use, okay black. Let us say it is too big, so that is the point. Okay. So, let us I am just revising it again.

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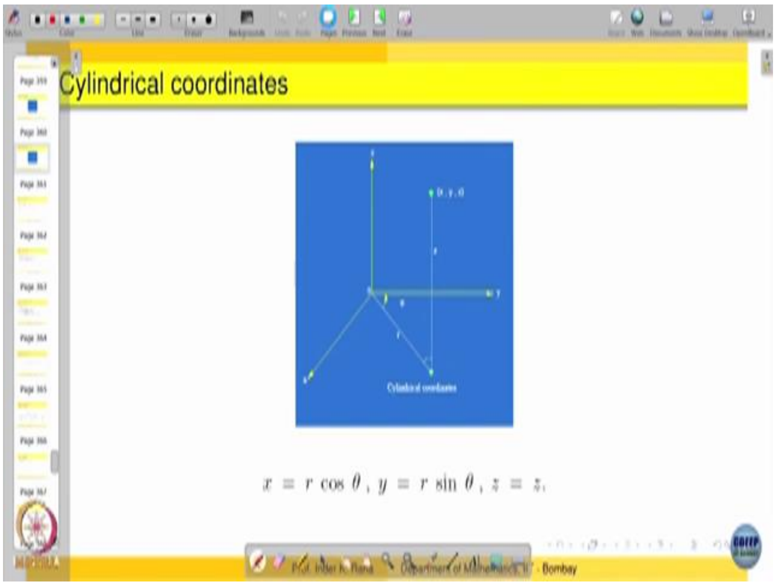

What the coordinates, if this is the point x, y and z okay. If you draw the perpendicular that will give you the z coordinate. So, I want to look at, I think this is a nice place to insert back that page.

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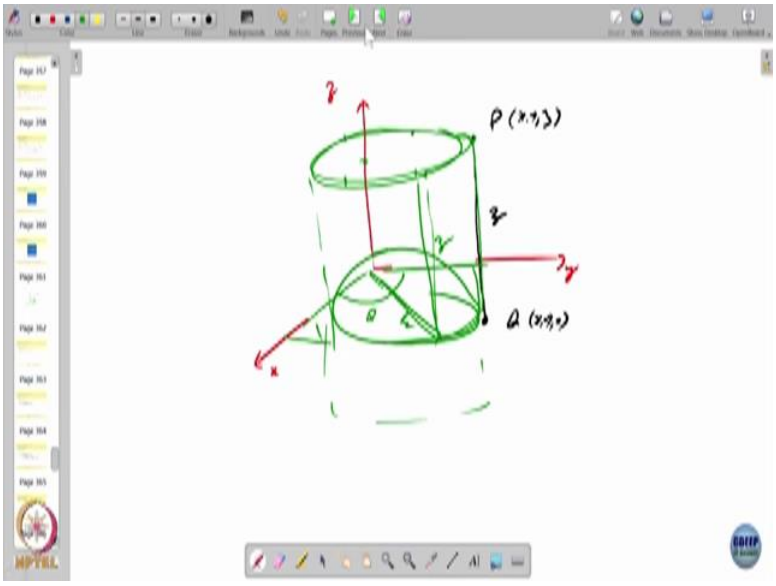

So, here is x, y and z. So, this is a point P, with x, y and z. So, how do you find the z coordinate, you take the perpendicular here, so call it as point Q, So x y and 0. So, this is the z coordinate. Now what we are trying to do was on, if that point has to go on a circle somewhere, so that will be a part of a cylinder, if you look at that way, that will be a cylinder.

So, to find out this point, whether it is here or here or here or here, if I take the projection of this, okay, so that would be a circle. It does not look like a nice circle but let me draw it better probably.

So, this I should know what is the, if this is the point which is coming here, then I should know how much is that angle. See the point is this point P is determined by this height and how much I have to rotate, how much I have to rotate on that cylinder. Points on a cylinder are determined at what height you are okay and how much you are going around the cylinder.

If I want to determine all the points on the cylinder then I shall know how high I am so that is z coordinate, up or down, that is z coordinate and how much I should rotate on the cylinder to locate it.

So, locating it means I have to find out what is its radius okay. So, what is, this angle is, say x axis angle is theta. So, coordinates will be r and theta. What is r, what is the height r, what is the radius r? So, this is z okay.

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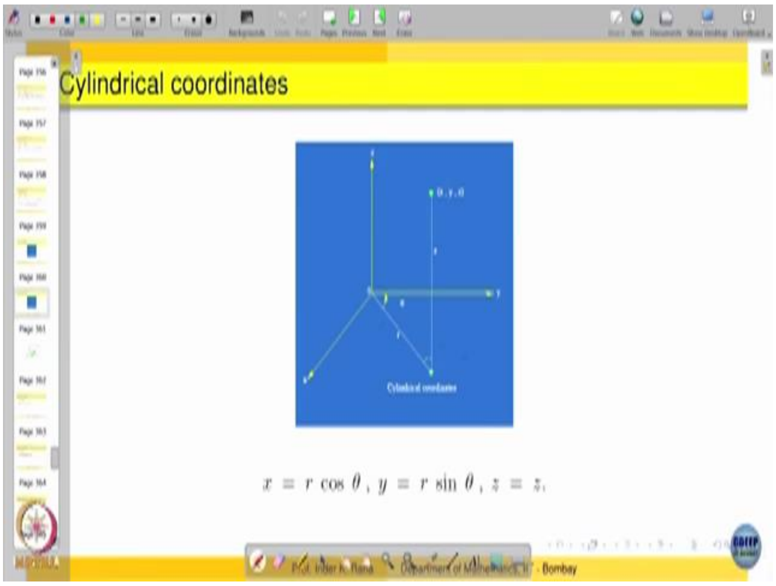

So, this is z okay, angle is theta. So, what are the coordinates if this distance is r, then it is r cos theta, r sin theta, z is equal to z, polar coordinates on the circle basically okay. So, that gives you cylindrical coordinates. So, these are called cylindrical coordinates. So, you can locate points in the play in the R 3 by looking at points as points on concentric cylinder with the same axis, okay.

So, where is the next? Okay, next is here. Yeah, so that is called cylindrical coordinates. One more coordinates I would like to introduce in R3 before I go to change of variable formula. So, let me write that. I will revise it again next time.

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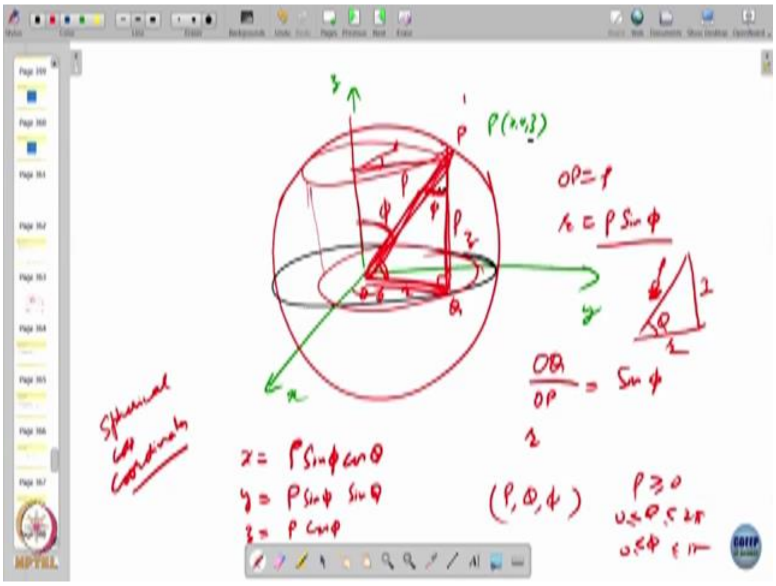

So look at x, y and z, so here is a point P. Okay. So, one way is, Cartesian coordinate is by looking at parallelepipeds and looking at the corner. The second was looking at the cylinders, concentric cylinders with expanding radii. The third is once again let us look at this distance, so that is OP, P is the point. So, now in the polar coordinates we looked at a circle but in R 3 let us look it as a sphere. So, let us look at this R 3 being made of concentric spheres of increasing radii okay.

So, if I look at that, then what will be this? So, here is here is a sphere okay. So, look at this point, look at this distance, from origin that is r. So, we can imagine that point in R3 line on a sphere of radius r. But there are so many points on the sphere, which point we are referring to, how do I locate that point? Okay. So, locate at that point, let us look at. Cut by a circle of that height. Cut the sphere by a plane of height z okay.

If you cut it by z, then if I can tell you what point on that circle it lies, that would be okay or another way could be a easier way probably could be, let us just look at, this is a sphere, this is a point P and that is Z axis. Another way that could be to locate that circle would be. Imagine this to be a rod which is fixed at some angle phi with the z axis, z axis vertical one here is the rod which is going.

How much this rod can rotate, that will give you points on the circle. So, how much this rod can rotate that will give you points on that circle of that height z. Is it okay? So, how are the points located here on that? How will these will be. These will be located by how much is the angle and what is the radius of that circle. Polar coordinates on that circle, the easiest way of locating the points on the circle is by polar coordinates.

And how do we determine the polar coordinates, by finding the radius of the circle and how much you are revolving. So, if I bring it down here, so this point is z, this is r, this distance is we will call it let us say rho. So, rho is the distance OP is equal to rho, this is r and what is this angle? If this angle is phi, what is this angle? That is 90 minus phi. If this is phi this also is phi, vertically opposite angles, parallel lines.

So what is this r equal to? In terms of phi you can describe, what is this r? This is a right angle triangle, this is r, this is phi. So, what is r equal to rho, this is a right angle triangle, OQP is the right angled triangle, this is 90 minus phi this is phi, so this is the height. So, rho sin phi, is that okay, height OQ divided by OP is equal to sin of the angle. This is the origin, so OQ is r, that is equal to rho sign phi. So, if this is rho sin phi this r and this angle is theta, so what is x coordinate, polar coordinates?

R cos Theta, but what is r, it is rho sin phi, so it is rho sin phi cos Theta. And what is y, that was rho sin Theta. What are the polar coordinates r cos Theta, r sin Theta? What is r, r is sin phi sin theta? And Z, what is z equal to? This is z in terms of rho and phi, it is rho, this is z. This distance is rho and this height is z. So, this is rho cos phi, is it okay?

In that right angle triangle that height was, this was rho, so this was rho, this was z and this was theta and this was r. So, we get relation between x y and z and how much distance you are? How much the rod is rotated from the z axis, that is angle phi, and how much to locate that point Theta, how much you have to go on that circle. So, 3 things determine your position of a point and conversely, if I give you these 3 things, I can locate a point.

So, rho, theta and phi, where rho is bigger than or equal to 0, Theta is between 0 and 2 pi and phi, what is the angle Phi, that is between z axis, so how much is the angle possible with z axis? 0, you can go down, so 0 to pi. So, Theta is, phi is between 0 pi. So, again 3 things determine the points on R3, namely rho, how much it is away, how much you are deviating from z axis and how much you are rotating around x axis. So, these are called spherical coordinates. These are called spherical coordinates.

So in R3, the are 3 types of coordinate possible, one Cartesian coordinates, by looking at the corners of all parallelepipeds. Second is by looking at a point as a point on the cylinder, concentric cylinders or third is by looking at concentric spheres. Okay, and there is no wonder that this spherical coordinates are very useful when you want to describe the earth. Any point on the earth you want to describe. Earth is a fixed radius, so rho is fixed. Imagine a sphere for fixed radius, so what are other things that will determine how much you are away and how much you are rotating.

So, these are what are our longitudes and the latitudes of any point on the surface of the earth, which determine your location on the surface. So, if you go to Google and you want to find your location, it will tell you in terms of phi, longitudes and latitudes, how much you are away from and the base how much you are rotating. So, these are very useful in modern navigation, modern location of points on the sphere. So, and we will see how they are useful in our mathematical thing also.