Basic Real Analysis
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Lecture 52
Change of Variables-Part 1

So we had started looking at double integrals, integral of functions of two variables over a domain.

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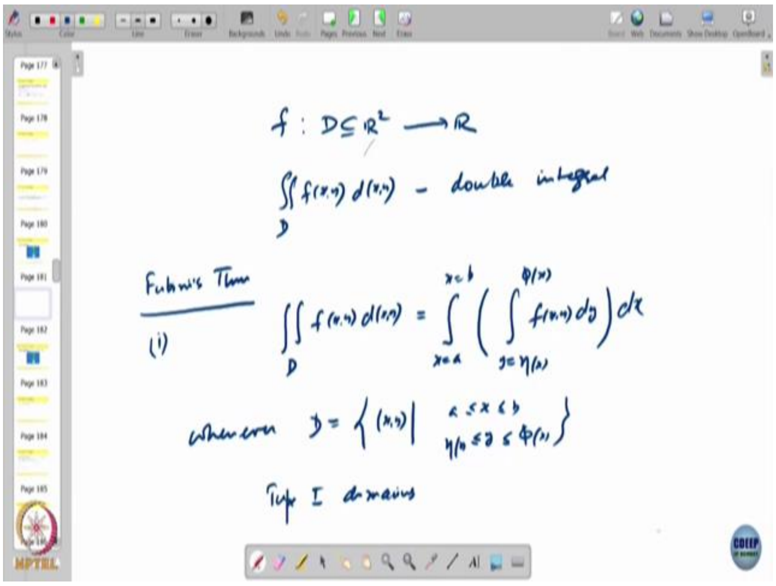

So, we had a function f defined in a domain D in R2 to R and we looked at double integral over D of f x y, d x y. So that was the double integral. So, whenever it exists, we said there was a theorem called Fubini's theorem, that is double integral under suitable conditions there namely the function exists, the double integral exists. For example, when D is closed bounded and f is a continuous function, this will exist and can be given as integral over… okay.

So, let us right case 1, is equal to x is equal to a to x is equal to b, integral y goes from some function eta x to phi x of f x y dy dx, whenever, the domain D can be written as all points x y such that x lies between a and b and y lies between some function eta x to phi of x. So this was type 1, so these are type 1 domains. So, how does one go ahead? Page down?

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Professor: More pages?

Student: Yes.

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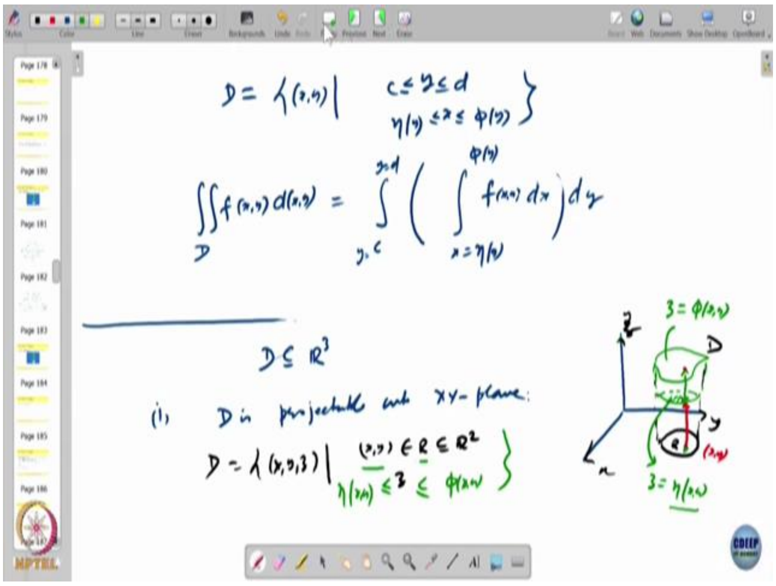

Professor: Okay, you have to add a page, okay. Right. And similarly for type 2 domain, if D is type 2, namely x y such that y is between c and d and x is between some function say eta y to phi of y, then the double integral can be computed over D as an iterated integral, so y is going from c to d, x goes from eta y to phi of y F x, y sorry, dy dx, sorry dx dy. So meaning that double integral for a function of two variables if the double integral exists can be computed by computing one variable integral at a time. So these were called the iterated integrals.

So that gave us a lot of maneuverability of computing double integrals. We must look at what is a situation in three variables. So let us take a domain D in R3. So a domain D in R3 can be quite complicated. So we are going to look at one possibility which says D is projectable on to xy plane. So, we want to look at D is a domain which is projectable into xy plane, so here is, think of the domain. So, that is z, this is x and this is y. So, this is a domain which can be projected onto xy plane.

So, what does it mean? That means, I can look at the projection of this. So, that will be something, so this is my domain D let us call it as a region R, so that means what. So, mathematically means that D can be written as x y and z, such that x, y belongs to a region R contained in R2 and z, so at any point here, so that is x y in the projection my z starts somewhere and goes up to somewhere.

So, let us make this as a, so top is some surface, so let us call it as, the top one is z is equal to a function of x and y, so let us call it phi x y. And the bottom, this one is z is equal to something eta of x y. So, the domain looks like it can be projected onto the x y plane, projection is a region R in x y plane. So, how does the points in the domain look like? They look like x and y belong to R that is a projection.

And for every point x and y the z starts, z starts at this surface that is eta of x y and goes up to the top surface that is for phi x y. So, it is a kind of a cylinder you can think of, some kind of a cylinder you can think of in R3. So, that can be projected on to the, it has a projection on to x y plane. So, when you project it on to x y plane, that is a region R. So, all the points described for a projection, that is projection meaning z coordinate becomes 0.

So, for any point x and y in the projected region R, how high you will go so that you are in the domain. So, when we want to go up, we start at the surface end up to this surface. So, this is the value of z, so that you are inside the region D. So, we start at R is, x y belong to R, so point is here and you go alongside z axis. How high you have to go? You have to start at the surface eta x y and go up to the surface of phi x y. So this is a domain which is projectable on to x y plane.

So what is the advantage of this? Advantage of this is that Fubini's theorem states for R3 the computation says the following.

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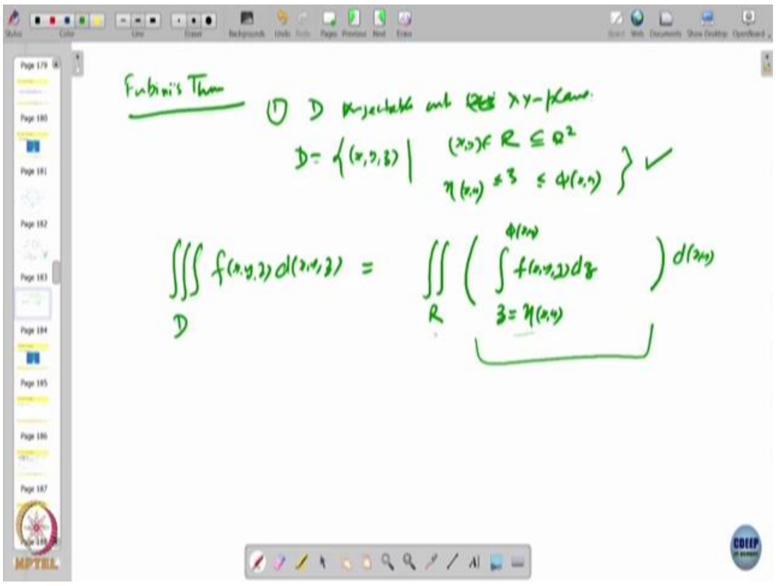

So, Fubini's theorem by D projectable onto x y onto x y plane, that means D is written as x, y and z where x, y belongs to the region R, that is inside the plane R2 and z goes from some function we said say it eta x y to phi of x y. So, with that description of D as projectable onto x y plane, the triple integral of f of x, y, z d x, y, z can be computed as x, y belong to R and z. So, double integral over R d xy and integral f x, y, z with respect to a variable z, z goes from eta of x y to phi of x y.

So, that is what we are saying. The Fubini's theorem says if the triple integral exists, then and the domain is of this type projectable onto x y plane then I can split the triple integral into two parts, integrate the variable z from the bottom surface to the top surface.

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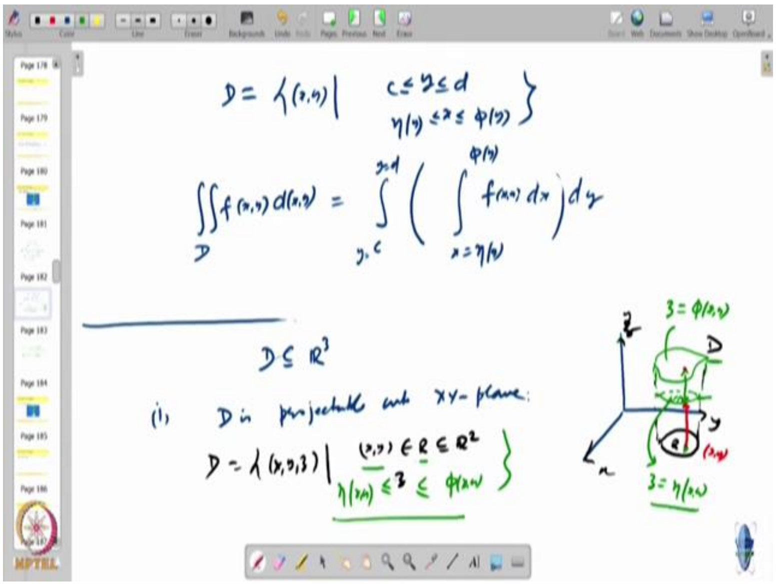

And so, that is this from the bottom surface and the domain, domain is x, y in R. So, we start here at this surface and end up at this surface. So, from this surface, so integrate the variable z. Once you have integrated the variables z, it is a function of two variables, so, is a function of two variables.

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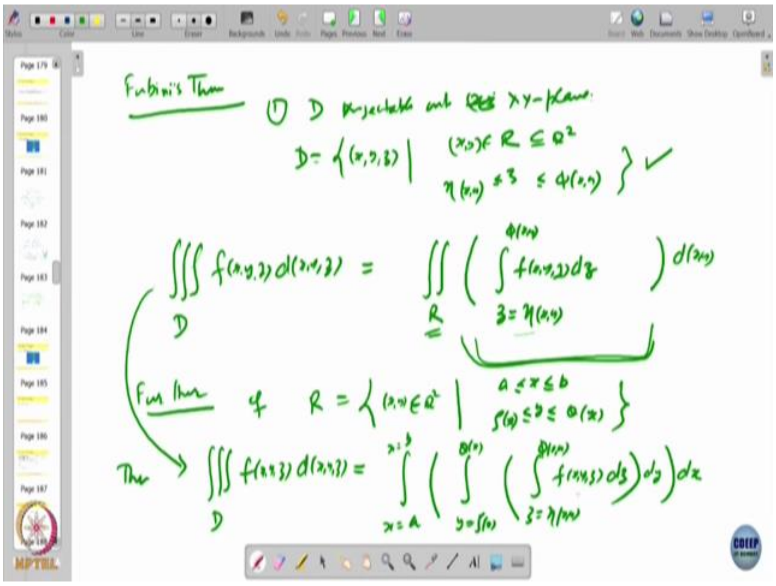

So, that says that this as a function of two variables can be integrated over R with respect to x and y and that will give you the triple integral. So, we are splitting the triple integral into two iterated integrals, one is a double integral, other is the ordinary integral of 1 variable. Now, exploring further this region R, which is a projection into x, y plane also is of either type 1 or type 2. So, let us write further.

So, there we will be using the further if this R, the region R which is in x, y plane, x, y belonging to R2 is of type 1. So, what is type 1 in the plane, x lies between a and b, y lies between some points, let us call that as psi of y and say theta of x, sorry and theta of x, that was type 1, x lies between a and b and y goes from 1 lower curve to the upper curved. So, lower curve is psi X and upper curve is theta x.

Then what does this triple integral look like? So, this triple integral over D f x, y, z, d x, y, z can be computed, now, this double integral when R is of this type, I can write it as integral x goes from a to b, y goes from psi x to theta x. So, this was a double integral, this was a function. So what is that function? So, that function is z equal to eta x, y to phi of x y, f x, y, z dz, here is with respect to dy and then with respect to dx.

So, what we are saying is the triple integral, if your domain is special, first it is projectable on to x y plane, then I can split the triple integral into two parts, one is the double integral over the projected region R and the third variable that is z, that is the ordinary variable, integration of one variable. Further if the projected part looks like domain of type 1 in the plane, then you can use earlier Fubini's theorem, that double integral will look like x goes from a to b iterated integral x a to b, y goes from some functions psi x to phi x theta x and the double integral.

So, this means that if the surface, if the domain D is projectable on to x y plane and the projection itself is of type 1, then the triple integral becomes integration of 1 variable at a time. So, it will be with three integrations which will be coming, with respect to x, y and z iteratively. So, that is if the projection R is of type 1, supposing it was of type 2 that is also possible, then what will happen?

Then this one will split according to the type 2, we will be integrating y first and then with respect to x. So, this is when the domain is projectable onto x y plane. So, two possibilities, projection onto XY plane is possible and the projection is either type 1 or type 2. So, you get two iterated integrals possibly. If you come if your surface, if your region R is projectable onto say x z plane then, again two possibilities will arise, and similarly on to the y z plane.

So, in all there are six possible, iterated integrals possible for a function of three variables on a domain D, possibly you can write. So, let us try to do some examples on this.

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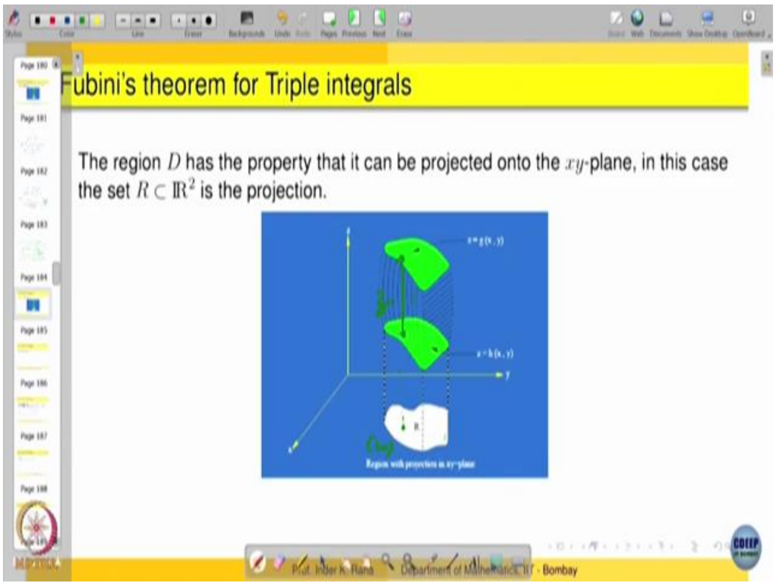

So, this is what it says. So, this is just recalling what we have said. So, this is a domain which is projectable onto x y plane. So, this is the region R which is a projection, say every point you go from this lower surface, that is written here as H and G, so from here to here, can you see the pointer moving? You cannot see the pointer moving, but I do not know whether you can write on top of it or not. Let us see, so if you take in move it, good, additional thing. So, as you go up, you will start here and you will go up to here.

So, that will be your z. So for point X, Y in the region R. You want to go, it is projectable onto x y plane, so you are moving along z axis. So z goes from this region to this region. So, this is for the triple integral.

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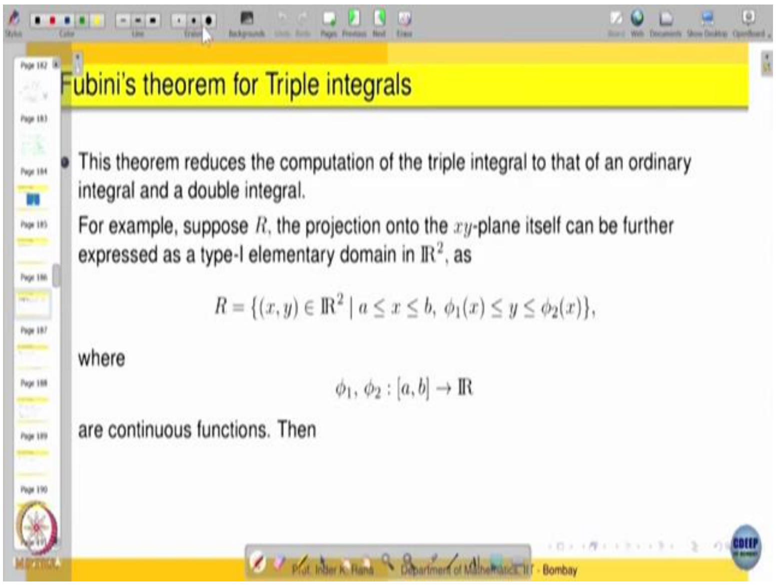

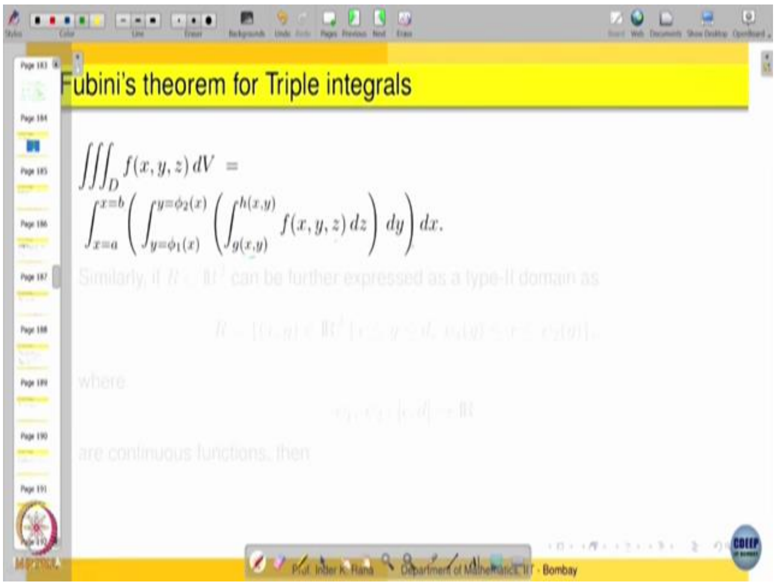

So, it says, if you can write it this way, projectable onto the x y plane, so you call it as type 1 elementary region. And so, this is what x goes from a to b projection is again of type 1. So, x goes from a to b, y goes from phi 1 to phi 2 and z goes from lower surface to the upper surface dx dy dz. So, suitable conditions have to be put, that namely this phi 1 phi 2 are to be continuous, G is to be continuous and so on. So, we will assume all those nice conditions hold, that all the iterated integrals exist and are equal to the double integral.

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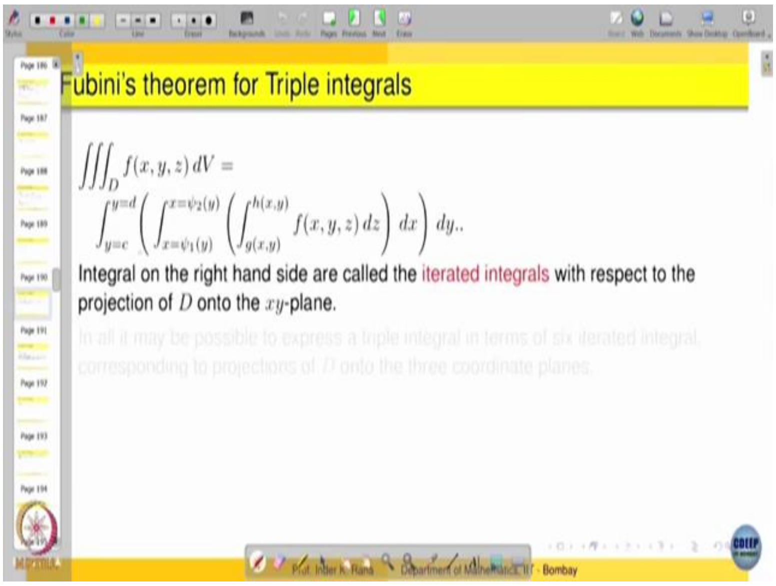

So let us see, so similarly if it is, the projection is of type 2 and then it will be Y going from C to D, x going from lower curve to the upper curve psi 1 to psi 2 and then z going from lower surface to the upper surface.

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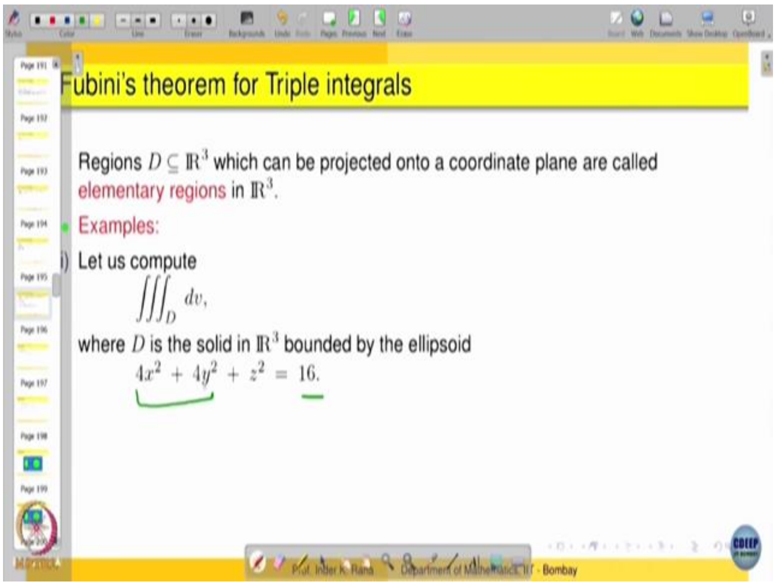

So, let us do one example, so let us see this example. So, we want to complete triple integral of the function, 1 where D is a solid in R3 bounded by the ellipsoid. So this is the ellipsoid, ellipsoid is 4 x square plus 4 y square plus z square equal to 16. What does that ellipsoid look like? What is the ellipsoid? How do you visualize the ellipsoid? If you take z is equal to 0, what will that give me? Z equal to 0 will give me the projection of this solid onto x y plane. And that looks like 4 x square plus 4 y square.

So, that is 4 x square plus 4 y square equal to 16. And similarly, if you want the projection on to other, we will put y equal to 0 and projectable. So, this is a solid which is projectable onto all three coordinate planes x y, y z and z x. So let us project it onto the x y plane, that means z is equal to 0, you get 4 x square plus 4 y square equal to 16. So, what is that? So that is the region R, that is a projection. So R is 4 x square plus y square equal to 16.

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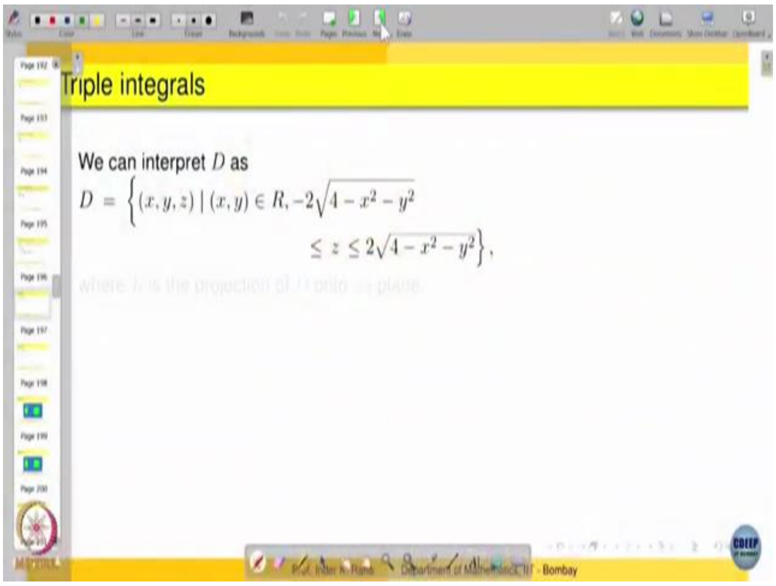

So that gives me, so, I can write it as x y belonging to the region R, region R was, the region R, that is 4 x square plus 4 y square less than or equal to 16, inside of that. And that itself I can write it as, there is a circle, I can write type 1 or type 2, 4x square plus 4 y square is equal to 16. So, what would be that? So, x goes from, so there is a region R. And that itself I can write it.

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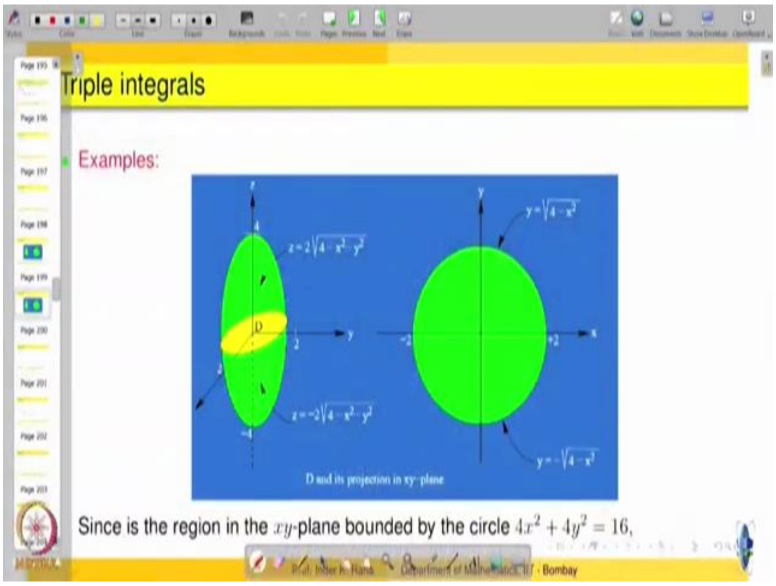

So, that will be a region R in terms of, so that is a is a circle. So, we can write type 1 or type 2, 4 x square plus 4 y square equal to 16 that is x square plus y square equal to 4. So type 1, where does x vary from the circle? Minus 2 to plus 2, lower part of the circle to the upper part of the circle. Or if you like, you can write it also as type 2, where y goes from minus 2 to 2, x goes from the left side of the circle to the right side of the circle.

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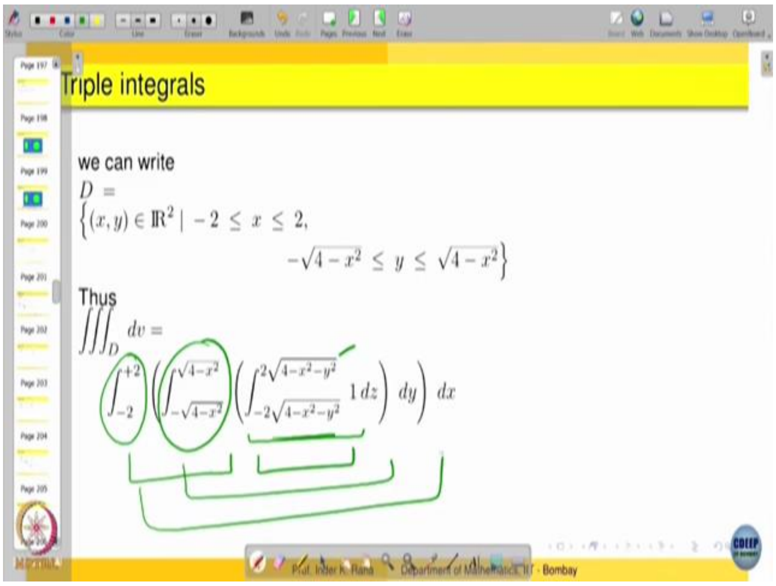

So if you do that, then you can write your, so x goes from minus 2 to 2, that is a circle. And where does y go from? Upper to the lower. So what is the upper circle? So upper circle is 4 minus x square positive square root upper part lower part is minus of that. So what does your domain look like? Domain looks like x goes from minus 2 to 2, y goes from here to here. And where does z vary from? When you project, you get the circle. So what is the lower part of the circle, that is a lower part of the ellipsoid to the upper part of the ellipsoid.

In the projection to be inside the domain D, we will start with the lower part of the ellipsoid to the upper part of the ellipsoid. So, what would be the lower part of the ellipsoid, z in terms of y, X and Y, so z square equal to, so what is the equation of the ellipsoid? So, 16 so you divide it by that, so we will get z in terms of, I think it is quite clear. There is a lower part of the ellipsoid, z square, and there is upper part of the ellipsoid.

So this is a region ellipsoid, when projected onto x y plane to points inside the ellipsoid, can be described as x goes from minus 2 to plus 2. So this is x goes from minus 2 to plus 2. And where does y vary from? For every point x, y varies from minus 4 minus x square to 4 minus x square plus square root. So lower part of the circle to the upper part of the circle, that is the region R. So this is describing the region R the projection onto x y plane.

And then, looking at to be z, where in the region, is you take a point in the region, how high or how high, how much higher, how much lower you should go to be inside. So we will start at the lower part, go to the upper part. So this is the lower part and this is the upper part. So what we have done is a function of three variables, instead of integrating it three variables at a time, altogether, we have split, integrate with respect to z, integrate with respect to y and then integrate with respect to x..

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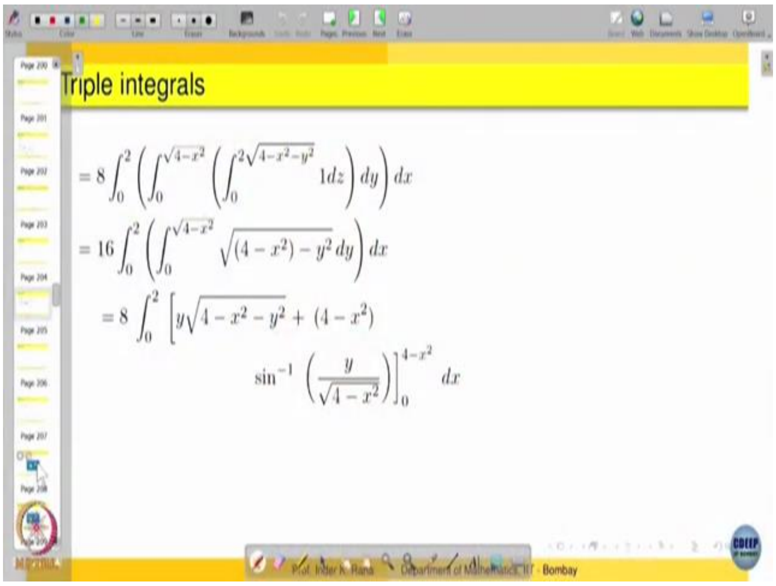

So, this is integrating one variable at a time. So that is the illustration of Fubini's theorem. So, where we were. So, you can integrate all that and finally, your answer comes out something like this, so let us not bother about sin inverse and all that thing.

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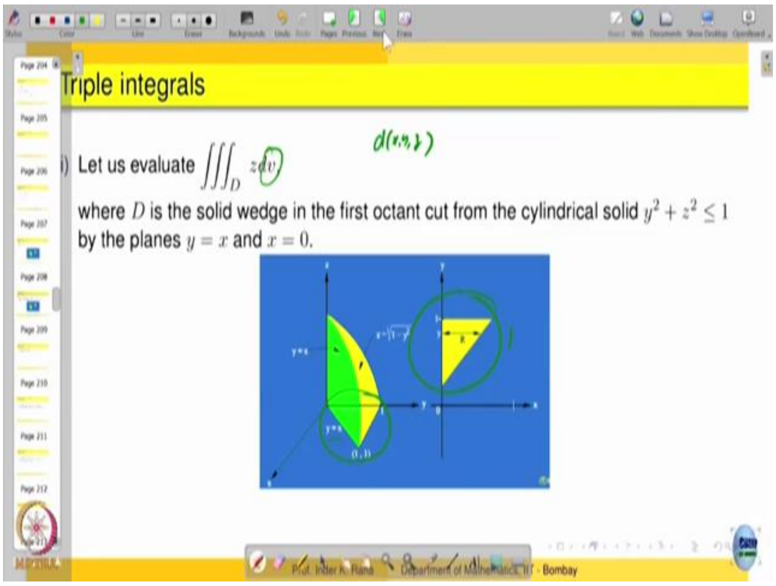

Let us look at this. Let us find out what is the, what is integral z dv, see d x, y, z, that is also written as in short as d of small v, volume basically, it comes from that. So, this dv is same as d of x, y and z. Where d is the solid wedge in the first octant, here your imagination also will come into picture. In the first octant cut from the cylindrical solid by the planes y equal to x and x is equal to 0. So, how do we visualize that? This is a solid wedge, in the first octant cut from the cylindrical solid.

So, the is cylinder y square plus z square less than or equal to 1. What is that cylinder , if you want to just look at the cylinder y square plus z square less than or equal to 1? It is a three dimensional solid cylinder, three dimensional solid, cylindrical solid. So how do you visualize? It is a cylinder with axis going through x axis. It is symmetrical around x axis. x is independent, so the axis goes minus infinity to plus infinity.

And if you look at any point y square plus z square less than or equal to 1, at any point x, how does y and z vary? Is a circle. So sections of the solid are circles y square plus z square less than or equal to 1, cylinder, you are able to visualize. In that cylinder, we are looking at first octant, that means we are looking at only a part of the cylinder when x is bigger than 0, y is bigger than 0 and z is bigger than 0, on that part only.

So you can see here, so there is upper part only we are looking at in this picture. Where it is bounded? It is bounded by the planes y equal to x, if we do not bind it then it will be in finite. So, y the equal to x? What is a plane y equal to x, y equal to x is a plane, z is independent here. That means what? It is a plane which meets x y plane in the line y equal to x.

So, in the in this line, so this is a plane going up, y equal to x. Are you able to visualize? Y equal to x is a plane in R3. In coordinate plane x y it is a line, but if you visualize it as a object, as a set in R3, then it is a plane. And what does it mean? It is cutting the line, it is cutting XY plane at the line y equal to x, so it is going through that, so it is a plane.

So, that is one side of the plane, bottom is x y plane anyway, we are going up only. And then and x is equal to 0. So, what is x is equal to 0? That is a y z plane, x is equal to 0, the y z plane, y and z vary. So, this is the wedge. So, this wedge is projectable onto x y plane, this wedge is projectable onto x y plane. And where is the projection? Projection is this triangle bounded by y equal to x. So, this is the projection. So, that is visualized here in this side, on the right-hand side, that is a projection.

Is it clear? The wedge, when it is projected onto the plane that will give you a triangle. So, how do I write that now.

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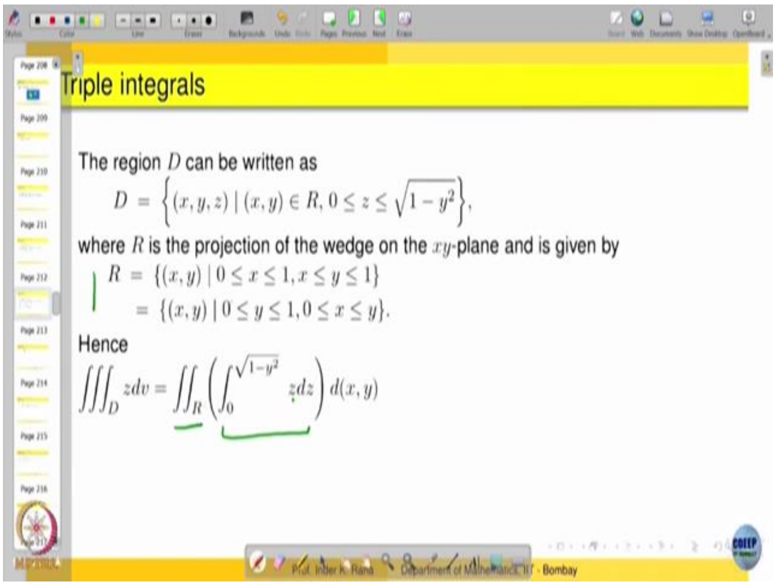

So, x, y, z, x and y in the projection z goes from bottom that is x y plane 0, how high it will go, up to the part of the cylinder, so that is z less than or equal to 1 minus x square. So and the projection that R, there is that triangle, so let us write down the x between 0 and 1, y goes from the line y equal to x, it was at the y equal to x that line up to y equal to 1. So, it starts and that triangle it goes up to there or you can write other way around if you like.

The projection here is both type 1 and type 2 or you can write y equal to 0 to 1, x goes between 0 and x less than or equal to y either way it is okay. So, you can write down the integral by using the Fubini's Theorem, it is a double integral over the region R which is a projection of z, the function z integrated 0 to 1 minus y square. So, that is the inside thing, that is operation is R which is type 1 or type 2.

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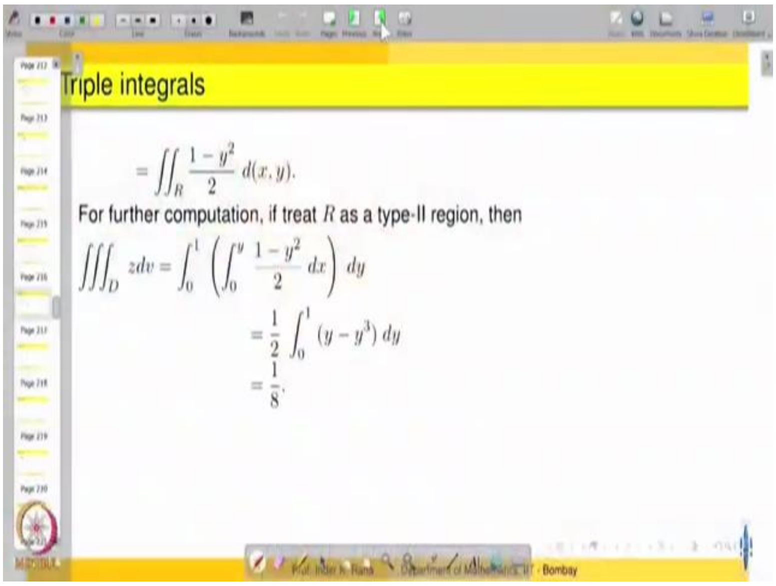

So you can further write it as, you can further compute 0 to 1, 0 to y, that integrated z integrated out is this and you integrate it out. So basically, the idea is the region in R3, you have to visualize whether you want to project it onto the x y plane or y z plane or z x plane, which one it is easier to visualize. The same wedge if you try to project it onto the y z plane, that looks like very weird one and you cannot describe it as going from one point to another, so that is where your visualization power will come into picture.