Basic Real Analysis
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Lecture 51
Integration in several variables-Part 3

So, let us look at the example, so he says look at the x is bigger than 0, y is bigger than 0, x plus y less than or equal to 1. So, what does it look like?

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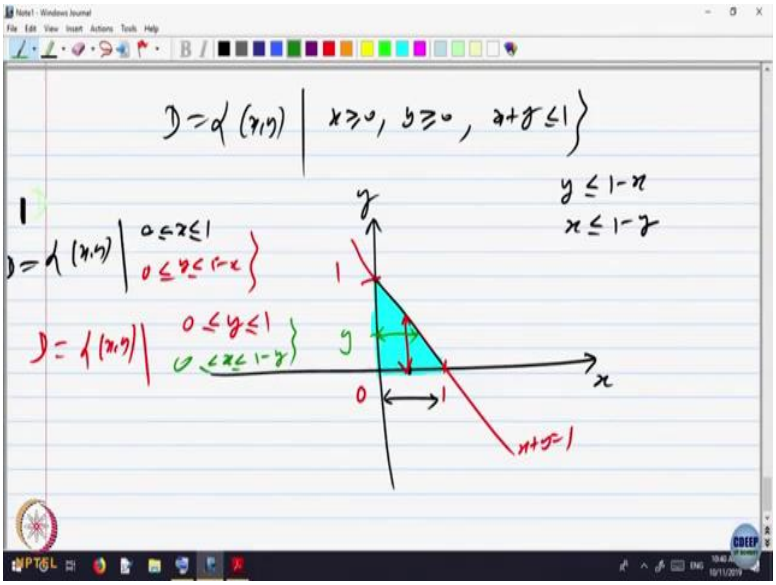

So domain is, domain d is x, y such that x bigger than 0, y bigger than 0, and x plus y less than or equal to 1. So, let us try to picture x bigger than 0, y bigger than 0, so where is the domain? It is in the first quadrant, x is bigger than 0, y is bigger than 0 so on the first quadrant, so y, x, x plus y is less than or equal to 1, what does that mean? x is bigger than 0.

So, you can say y is less or equal to 1 minus x or x is less than or equal to 1 minus y, depending on what is that x plus y equal to 1? That is a line, so that is a line cutting x-axis at 1, cutting y-axis at 1. So, it is this line, so that is this line x plus y equal to 1, so this 1, this is 1, this is 0, so what is our domain? Our domain is precisely this triangle.

So, if you want to write this d, you can write d as so, this d can be written as x, y x lies between 0 and 1 and for every x, what is height vertical line going moving parallel to y-axis it goes from

0 up to so, y goes from 0 to 1 minus x, you have to interpret that line as a function of x, y equal to something, as a function of x, y equal to some psi x.

So, that is I can also write probably this as x, y, y goes from 0 to 1 and if I want to look at every point why, it starts here and goes up to that line and that line remains the same, the limit remains the same wherever be the point. So, that is x between 0 and goes up to 1 minus y, it should be function of y so, 1 minus y. So, it is of type-1 and type-2 both. So, what else f is a function given by so you can integrate that function if you like. We will not go into integration, so f x, y integrate from 0 and 1, here what is fix here?

I am changing that example or what. So, you can integrate so let us integrate as function of, is that okay? Because the function was there are two parts of the function x square plus y square so, integral of x square plus integral of y square. So, that is two parts of the integral, integrate with respect to dx, with respect of y, and then integrate with respect to x. So, the simple example of you can try again yourself later on.

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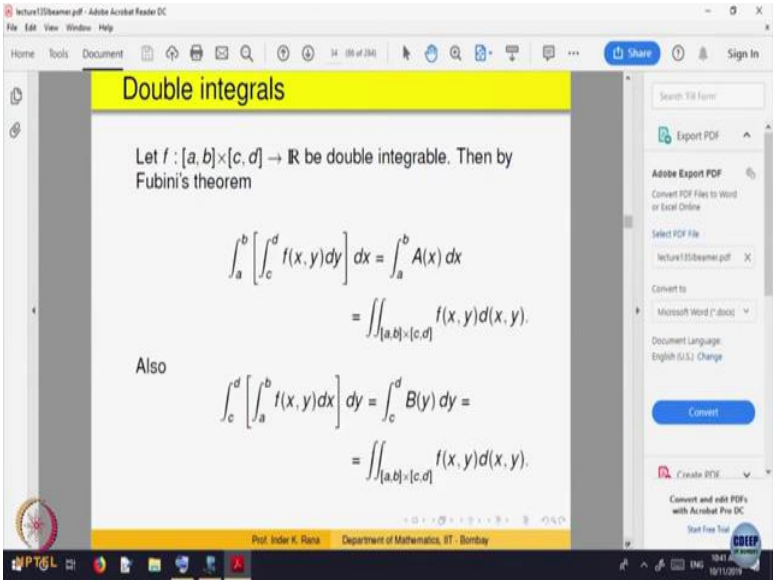

Let us, look at something then by Fubini's theorem this is given by, that is okay for the rectangular region, what do you think for a rectangular region what should be the volume? If the region at the base is rectangle with sides a, b and c, d.

Then, Fubini's theorem gives me this integral if the function is constant function what it should be? A rectangle is both of type-1 and type-2, you can calculate either way, if the function is

constant that comes out integral 1 whether you like type-1 a, b cross c, d so you integrate over c, d so d minus c into b minus a.

So, that is a volume of the parallel (())(5:43) so, that is so let us not go into that.

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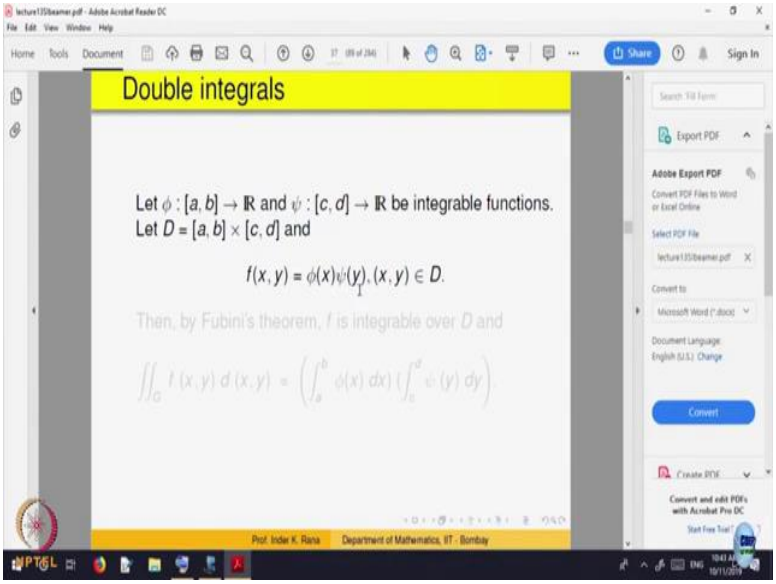

Supposing your function splits into two parts f x, y is a product of two functions phi x and psi y, so when you integrate f x, y with respect to x fix or y fix other function is not going to play any part in the iterated integral, is it clear?

When you integrate f x, y with respect to x, then phi y is a constant so that will come out as a constant. So, the double integral splits into product of two integrals, integral of phi and integral of psi so, that is what it says because, the function itself is splitting into two parts independent parts depending on x and depending on y. So, these are the things. I think let me look at some more interesting example.

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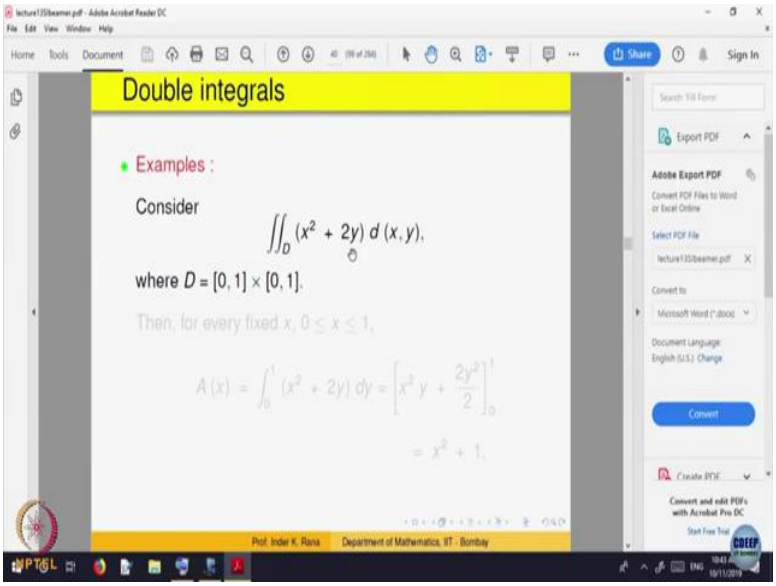

Anyways, these are all simple examples x square plus 2y d x, y over this so what will be the integral, integral of x square plus or you can calculate directly x square plus 2y integrate with respect to x first or y first whichever you like.

So, when you are integrating this with respect to x, it will be a x cube by 3 2y, y is a constant when you are integrating so, it will be 2yx appropriate limits. So, only keep that in mind so, when you are integrating with respect to x it comes out this and then integrate that with respect to x between limits 0 and 1 so, a simple example.

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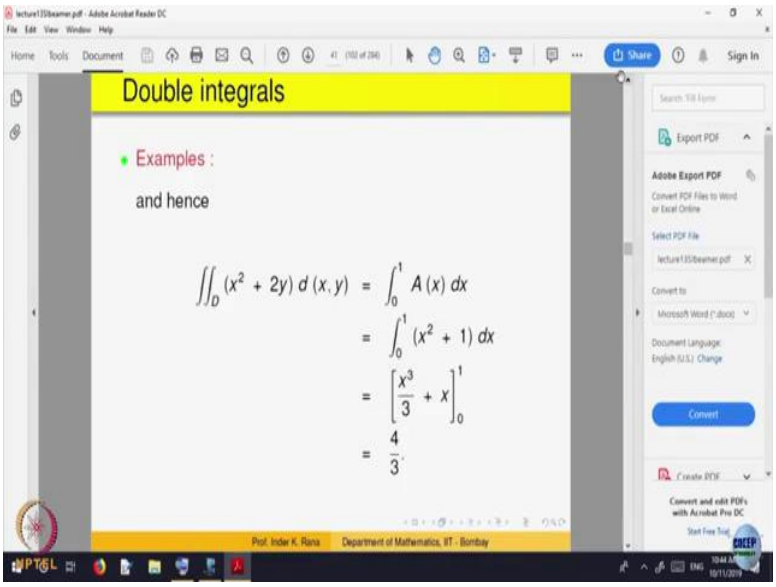

And if your domain is both of type-1 and type-2 Fubini's theorem says you can calculate it either of the ways iterated integral, either we are fixing x first or y first then, the two should be same.

So, your iterated integrations if the domain is both of type-1 and type-2, then the double integral should be same as integrated with respect to as domain type-1 and also equal to integrating as domain type-2 both integral should come out equal, because both are equal to the double integral so, keep that in mind to check sometimes. I think let me look at so, these are all simple examples type-1, type-2 disc and all that kind of thing.

So, I think type-2 I should discuss something more which requires. So, you have to decide upon given the domain whether you want to write it as type-1 or type-2 it may or may not be possible to write every domain as type-1 or type-2 you have to see which is possible and accordingly use Fubini's theorem for that.

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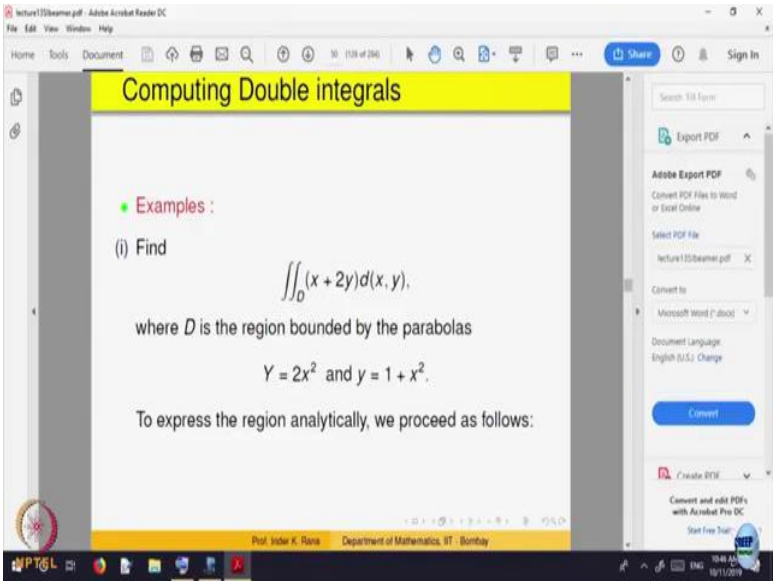

So, we have seen examples of those domains let us, look at this where d is a region bounded by the parabolas 2x square and 1 plus x square.

So, now your curve sketching of one variable will come in handy, what are these parabolas and what is meaning of saying is a region bounded by the parabolas.

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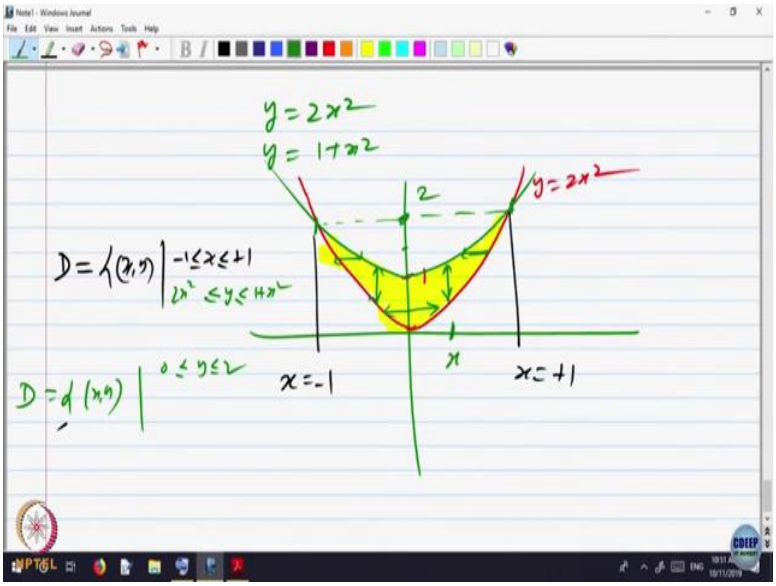

So let us look at one is y equal to 2x square that is one, and the other one is y equal to 1 plus x square so, what are these parabolas? y equal to 2x square that is our standard parabola y equal to x square anyway two times. So, let us draw that so it will look like something like this, what is y equal to 1 plus x square? I have copied, 1 plus x square.

Is again y equal to x square essentially, but y is shifted by 1. So, when x is equal to 0, y is equal to 1 so this is. So, if I draw this parabola now, I am saying it should look like this, it should intersect the other parabola y, see if we look at y equal to 2x square for the same x, y is varying twice, here y is only x square so that is increasing at double the rate. So, what is the region bounded by so, what you are looking at? We are looking at this point.

So, this is a region, so this is a domain d, is it of type-1 or type-2 or neither? If it is of type-1 then I should have x lies between something so, it looks like x lies between this and this, so I have to find this points of intersection. So, what are these points how do I find the points of intersection? 2x square equal to 1 plus x square, that means x square equal to 1, so x is equal to plus minus 1. These are the co-ordinates of the points of intersection the x coordinate.

So, x lies between minus 1, plus 1 and for every x, where does the y vary? So that is a question we have to analyze. So, for every x here is the why, so it goes from the lower parabola to the upper parabola so, y goes from for every x, y goes from the lower parabola that is, what is the lower parabola? y equal to 2x square, 1 plus x square so type-1 I can integrate and be happy about it whatever the function may be.

Let us try to write it as a type-2, can I write this as type-2? All x, y so, I have to find out what is this value of y, so what is this point? What is this point y equal to 2, so y goes from 0 to 2. Now if I want to find out want is a limits of x, I have to go parallel to the x-axis because x I want to verify so, for that for any point in between this if I go, I go out and then I come in again. So, up to some part it starts here and ends here and again starts here and ends here, if I go here then it starts same and end same.

So, it does not look like a nice idea to write this as a domain of type-2 you cannot, but you can if you like, you can try to cut it into many pieces and do it. So, when you want to integrate over such a domain, you should decide to write domain as type-1 and use that to calculate the integral, so, that choice you will have to make.

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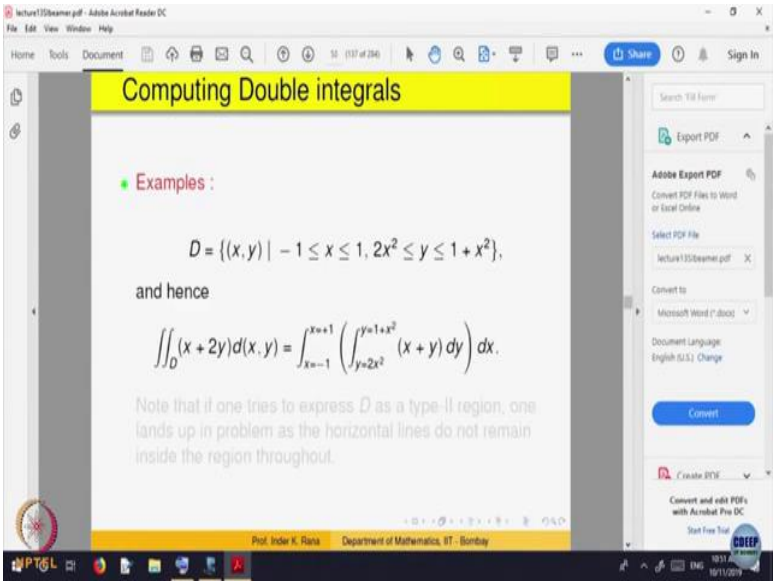

So, you define so all this is describe here so, when you want to say whatever functions f x plus y, so integrate 2x square to 1 plus x square integrate with respect to x is outside, so this type-1 we have written minus 1 to plus 1 so, that is.

So, we try to write it as type-2 we will have a problem and so on. So, more examples of double integrals so, I think other examples you should try to do it yourself.

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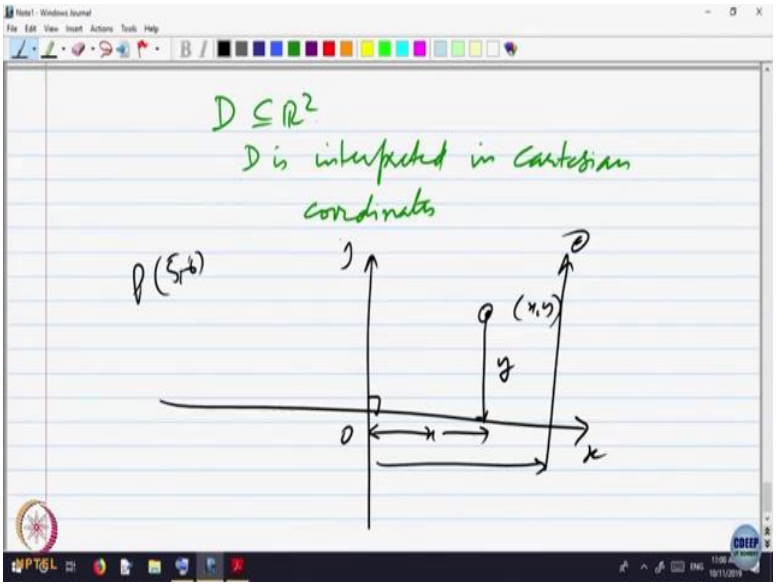

Now, I will like to spend some time on so, this is when your function, when your domain d contained in R2 and d is interpreted in Cartesian coordinates R2 which 2 in the plane there are different ways of locating a point, you can locate a point differently, why are coordinates defined at all, what is the need of having coordinate geometry? Because, it helps you to locate a point in the plane.

For example, in the line what is the need of putting a 0 somewhere and 1, 2, 3, 4, so on? Because, calibration of the line helps you to locate a point, the point is minus 5, that means what? That means somewhere you have marked a 0, some point as a reference point as 0 if you move on the right side you write plus or you move on the left side you write it as minus, just for your convenience and then you mark equal units that on the right side if you go 5 units then you are reaching a point which has got coordinate plus 5 assume only left side.

So, on the line marking those things are putting coordinates for every point on the line, it is very interesting story if you will imagine planet which is all linear everybody lies on the line, everybody stays on the line, and two persons meet on that line somewhere and one person invites other one come to my home in the evening, so he ask where do you live?

So what shall he tell? How does the person tell the other person where does the other person live. Imagine a road which very long there is no sign post anywhere, anywhere, so you do not know which side the habitation is, which is on the this side or on this side, which side you should start moving like a hero is lost on any road he does not know which side to go so, that is where these mile post know those help.

If you walk somewhere and you find it says 20 kilometer on this side will be a city called so and so. So, on the line, on a line land if we want to give every point a address you have to choose one reference point call it 0. Every city has a Gantaghar, reference point. So, that is a reference and what you do, from the reference point it also gives you a direction on the line one side you can call it positive, other side you can call it as negative that is mathematical way of saying.

One side you can call as north other side you can call south, whatever you like, you can one side you can call it "Ram" other side you can call it "Sham", does not matter what name you give because, we are doing mathematics we say on this side, what is left and right on the line depends on way you are standing and way we are looking this way, this is left, this is right if you look other way around that is left, that is right.

So, you have to say this side is called plus, this side is called negative whatever it is, and now every point get say coordinate, how many units markup that is coordinate geometry in line you need one reference point and one direction that is one reason why line is called one dimensional, only one direction. Now, imagine a plane, imagine a plane on which no sign post, nothing is visible anywhere and the same problem, how does two person tell each other where to come, to come to my house, he says where is your house.

So, to give an address to every place, every point on the plane you need first a reference point and then one direction is not enough, because one direction will allow only you to move around one line to get out this you need another direction so, you need two directions one direction is given by one line, the second direction is given by another line passing through the reference point but, not parallel to this because, if it parallel it is not giving you any direction neither coincident with it then, again no direction is given.

So, you need two intersecting lines which are not coincidental or parallel and then, move along one direction some units, move along other direction, some other units and you reach a point, you do not need rectangular coordinate system, for co-ordinate geometry you do not need, you only need two directions that means two lines and a reference point that is why the plane is called two dimensional.

Only rectangular because, it helps you to compute things easily, the distance between two points is given by the Pythagoras theorem very easily, if the axis are not perpendicular to each other you have to apply trigonometry that is a birth of trigonometry, why trigonometry is

required? Anyway so, what we are saying is what we have interpreted d as, we have interpreted d in the plane by coordinate system that means x coordinate and y coordinate, and what is this? So, we have got two lines, reference lines.

One is called x-axis other is called y-axis they are perpendicular this is the origin. A point how do I locate a point? You draw a perpendicular call it as y, see how much it is from other one x so, this is got x, y these are the coordinates. So, this locates the point given a point in the plane, given two axis and a point of reference you can write down the coordinates. So, every point gets a unique coordinates and conversely. How do you get so, it is one to one map every points gets coordinates and every coordinate gives you a point.

So, you supposing if 5, 6 I say a point is 5, 6 coordinates that means you go 5 units here and then go 6 units you will get a point. Every point gets coordinates; every coordinate ordered pair gives you a point.

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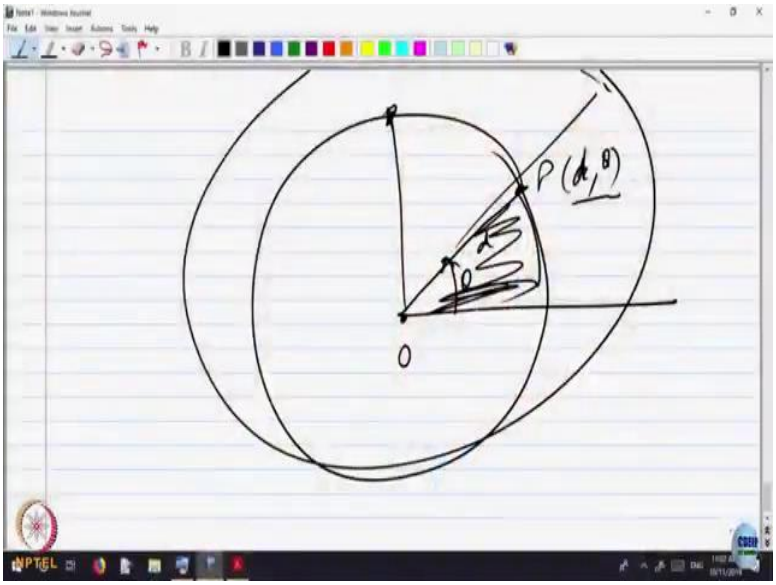

So, points in the plane can be located in one to one way by Cartesian coordinates but, this is not only way here is another way, I can locate this is only for my imagination.

So, let us put a reference point, imagine they are not there so, let me remove them so, that you do not feel, so I want reference point is required. Now, after this reference point, this is the point I want to locate, this is a reference point, how do I locate it? Without rectangular coordinate system or coordinate system at all. First of all let us see, how much is this distance

d, p is at some distance, how many points are there at distance d from origin? There are infinite. So, these are all points that is a circle.

Now, this point I am looking at, why I am looking at that point because, I want to fix this point so I look at a reference line and look at how much is a angle I make so, this point is known how much is a distance d, how much is a angle theta, if the angle is theta is here then, this is a point I am locating. So, every point at a distance d is uniquely known by the angle it makes with a reference line. If it is some different point, then distance is different angles are so, all the points in the plane are located by treating them as points on a circle of some radius and how much angle it makes.

So, d and theta you again you need two reference things, how much is a distance, how is the angle and these are called Polar coordinates, angle comes into picture. So, we will see next time how do you evaluate a integral when the domain is based describe by polar coordinates because, supposing you want to integrate over this sector then there is neither type-1 or type-2 you have to make efforts.

So, we want to change the variables from Cartesian to polar and see how does the double integral change evaluation of double integral change. So, we will look at what is called change of variable formula for double integrals from Cartesian to polar, let us stop here.