Basic Real Analysis
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Lecture no. 49
Integration in several variables – Part I

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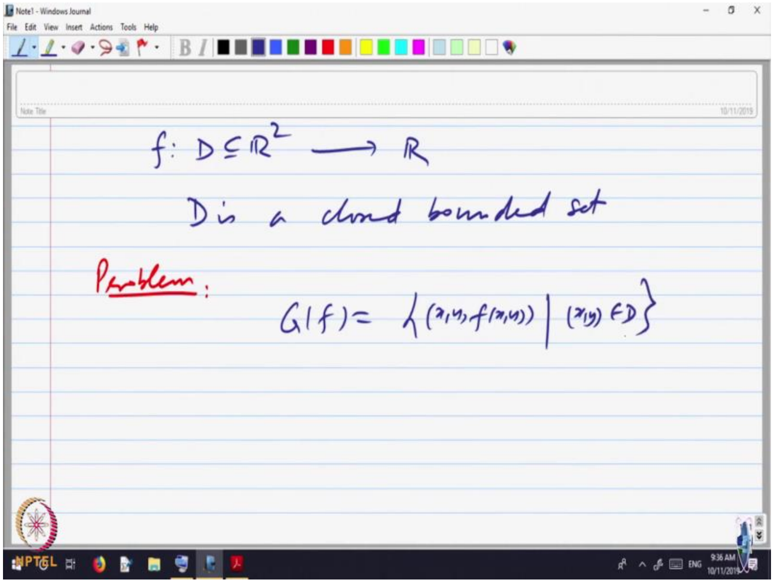

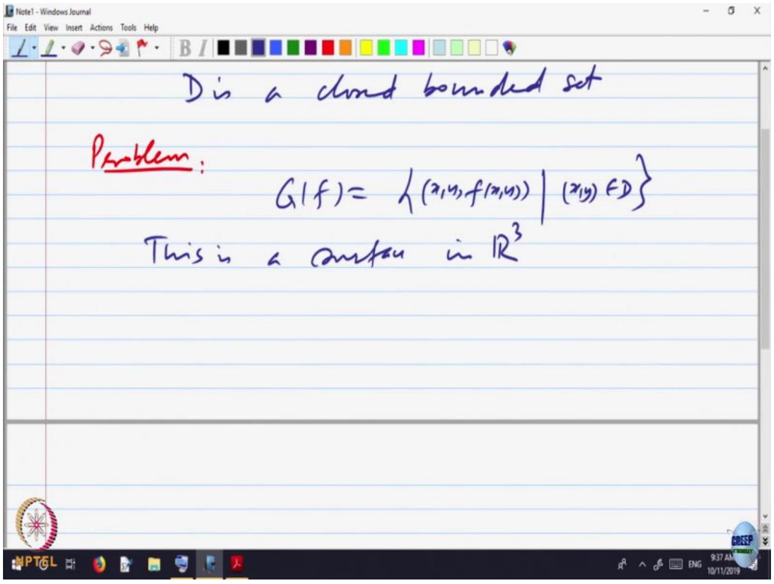

So, let us begin with today's lecture that is going to be on Integrals of Functions of Several Variables. So, let us look at the problem, F is a function defined on a domain D in R 2 to R and we should assume that D is closed bonded set, domain is a closed bounded set. It is something similar to for a function of one variable, the domain was a closed bounded interval. And we will assume F is bounded is not necessarily one can okay right.

So, the problem that we want to talk about is. So, the problem is in one variable, we looked at the notion of area below the graph of a function. So, the graph of this function F is surface. So, what is that surface? So, graph of F is a surface, which is X, Y, F of X, Y. X, Y belonging to the domain D. So, this is a surface in R 3, the graph of a function of two variables is a surface in R 3.

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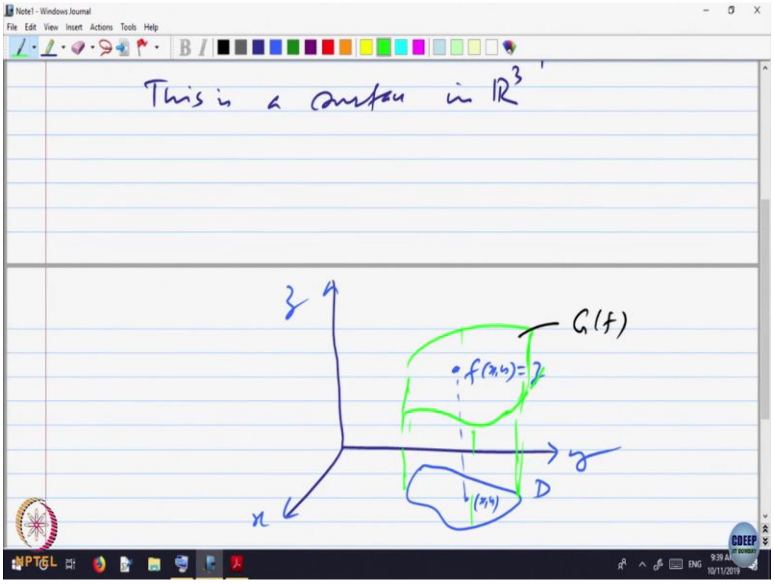

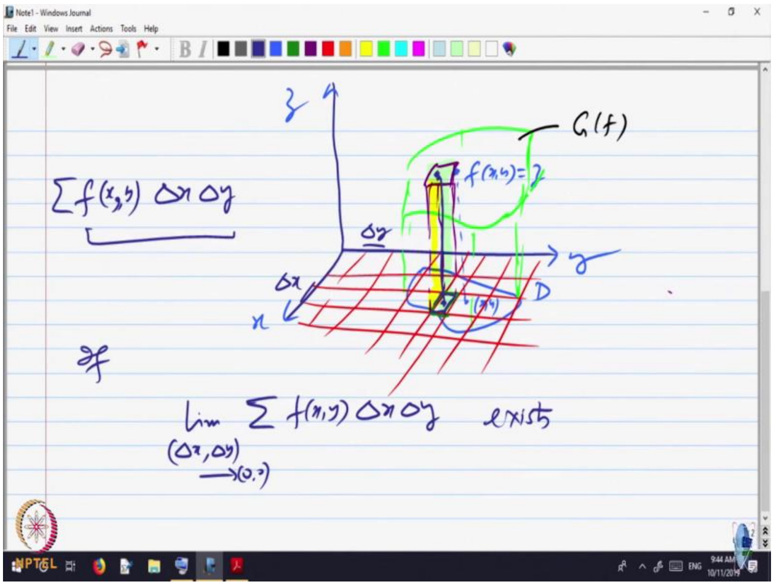

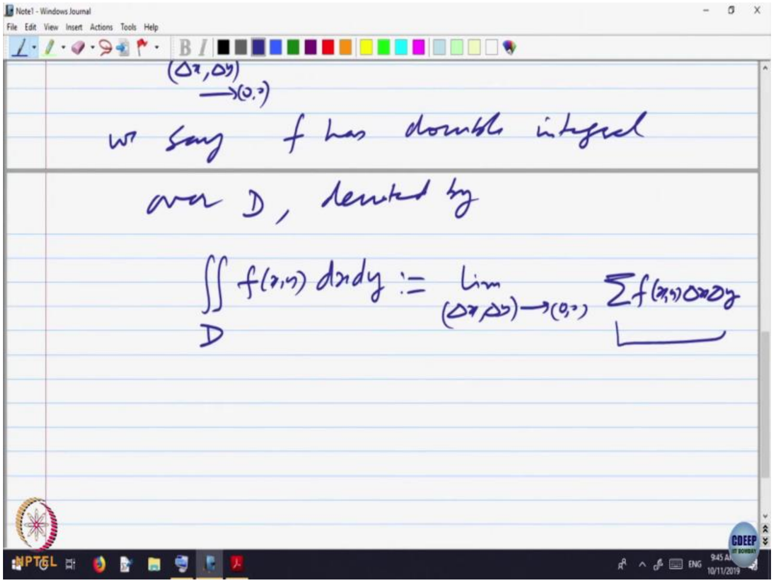

Let us draw a picture of that, so that we sort of understand a bit, there is the domain. So, this is X, this is Y and this is Z. So, this is the domain of the function and in every point here X, Y the function value is some height. So, let us now this is F of X, Y, so, that is equal to Z. So, all, for every point we look at the height that is the value of the function. So, that gives us. So, a kind of, I should not, so that is a kind of surface. So, this is the graph of the function F, the surface.

So, what do we want to do? We want to look at the sort of the region below the graph of this function above the domain D. So, that is in a sense you can think of it as a volume of the solid bounded below by the domain D, top by the surface S. So, you can imagine that to be like this, so it is solid with domain D as the base and top is a surface that is the graph of the function. What is a volume of this solid? That is what we want to give a meaning to.

So, the idea is very much similar to that of one variable, where what did we do? We try to fit in rectilinear figures below the graph of the function in the required area. So, here is being three dimensional will try to fit in parallelepipeds inside this region, right below the graph. So, how does one do that? So, you can imagine, so you can draw lines parallel, so, you can, you can draw lines parallel to the axis and so, that is a partition kind of thing you are thinking of. So, for every, so let us look at a particular rectangle.

Say for example, this is a rectangle, which is completely inside, so that is a base will take a this as the base and try to raise it to the height of the function. So, let us raise it to the height of the function. So, that will look like go up, so that will give you some kind of a, so that will be you can think of this as. So is a kind of parallelepiped with base as this rectangle and height being. So, what should be the height of this? So, let us take a point inside this like very much similar to take a point and take this as the height, so, a point is chosen.

So, if we recall in the Riemann sums, we defined SPF to be equal to sigma F of CI, XI minus XI minus 1. So, that was an area of a rectangle with length as XI minus 1 to XI and height as some point in that interval with that as the height, so same thing we are trying to repeat. So, the main being two dimensional, we cut it up into small pieces, rectangular pieces, so this is a typical rectangular piece. So, let us say this length is delta X and this length is delta Y. So, this is the base area is delta X delta Y.

So, delta X delta Y there is a area of the best and height is take any point in that and take as the height. So, let us take F of some point, so let us write X, it does not make so let us write X, Y itself, and then summation of over all the rectangles, which are inside the domain. So perfectly similar to the one variable thing we are trying to copy that. So, what we are saying is divide the plane X, Y plane in two rectangles of length delta X delta Y.

So, partition it, look at a typical rectangle inside the domain D, and erect a parallelepiped over that as the base and with the, so height will be choose any point in between. So, when the height of that parallelepiped is the value of the function at that point, so that is the Z is equal to. So that is a volume of the parallelepiped and summit over all possible parallelepiped that will give you approximate value for the volume.

So, what should be our definition of the volume? So limit of this summation F, X, Y delta X delta Y and limit delta X delta Y that should go to 0, 0, the length of the partition should become smaller and smaller. That means the length of the rectangle as well as the width of the rectangle should go to 0. So, if this exists, if this limit exists perfectly similar to one variable, if the limit of the Riemann sums, exist that is called the integral. So, if this limit exists, exists, we say F has double integral over the domain D denoted by.

So, that is denoted by two integral signs because it is a two variable, D is the domain F of X, Y DX, DY. So, what is this? This is a limit of delta X delta Y going to 0, 0 of those sums sigma F of X, Y, delta X and delta Y, summation over those partitions. Just to point out, why will this some exists, this sum will exist because our domain D is a close bounded interval.

So, only finite number of rectangles will be covering it, is that okay? Because close bounded is compact so, all these small rectangles cover it, so, by compactness finite number of them will always cover it. So, D will be covered by finitely many. So, this is a finite some only of those rectangles which are completely inside, from there. So, limit of this if it exists it is called the double integral of F.

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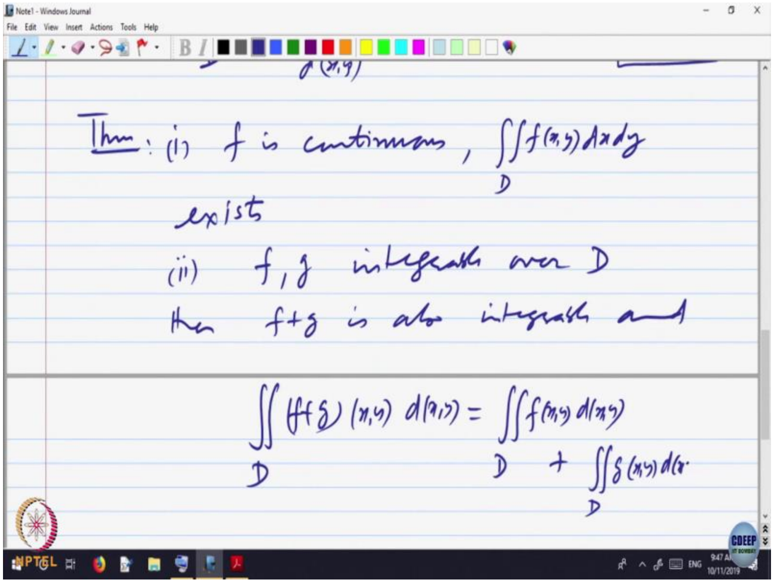

So, we will not go into proving many things, idea is how to compute these things. So, let us write theorems which sort of properties of this, if F is continuous, for a continuous function double integral always exists. One thing more I should point out this sometimes you write DX, DY or sometimes you write D, X, Y both are followed. Both rotations are followed so sometimes. So, if a function is continuous, this always exists. And let us write something more it is a linear function if F, G integrable over a domain D.

So, all those properties are very much there, which are for one variable, then F plus G is also integral and integral, double integral over D of F of F plus G, X, Y, let me write D, X, Y is equal to double integral of F plus linearity property, basically what we are saying is integral like in one variable it is linear.

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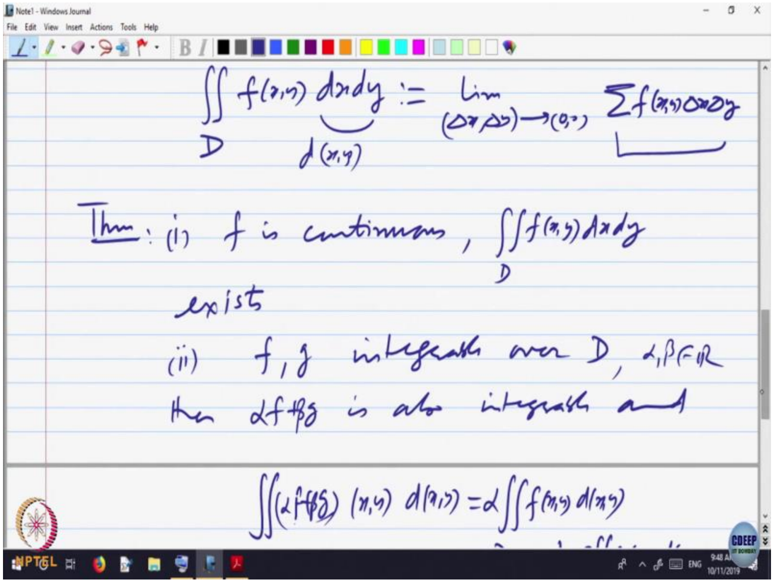

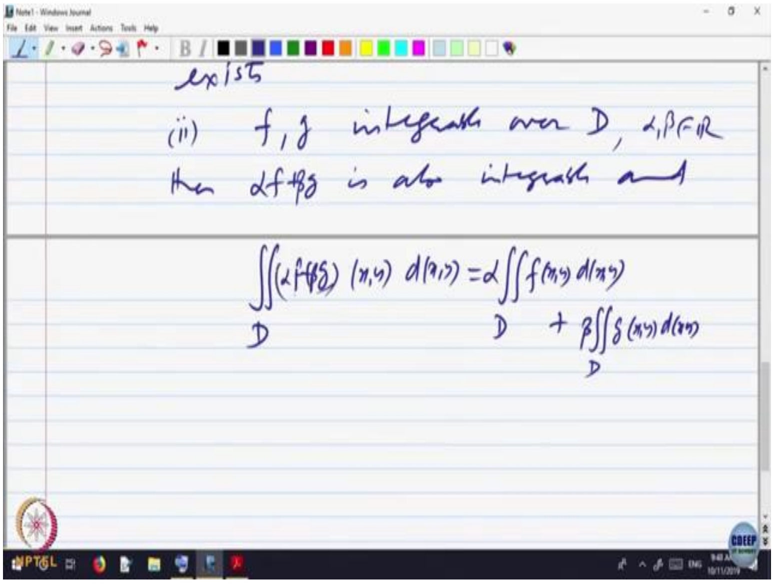

And actually, you can, if you like, you can generalize more F and G integrable alpha, beta real numbers, you can take a linear combination alpha F plus beta G that also is integrable. So, then it will be alpha F. So, let me just write that also, because why to repeat alpha F plus beta G, so, that is equal to alpha times this and beta times this, scalar comes out from the integral.

Actually, because the we are defined it as a limit. So, that is not very difficult to see because if alpha is multiplied in that some, then alpha will come out. So that will be alpha times the limit. So, they are easy to verify, but we do not go into the proofs of these two.

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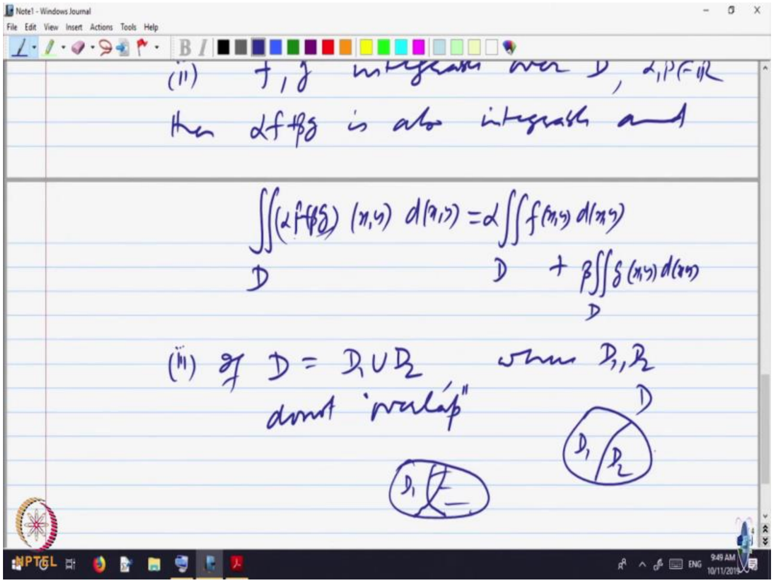

Let me write one more, so that we do not have to write again and again. So, third all right. Suppose, if you recall in one variable supposing a function is integrable over a domain, over interval A, B and you cut it into two parts A to C and C to B, then it is also integrable over the sub intervals and the integrals add up to give you the integral over the big interval. So similarly, here if D is equal to D 1 union D 2, where D 1, D 2 do not overlap.

So, one has to intuitively understand what is meant not overlap? We are not saying they are disjoint, but there is no common portion of them. So, if it is D you can cut it into two parts D 1 and D 2, then D 1 and D 2 do not overlap.. So, though it is written as a set theoretic union will emphasizing the role. For example, if you have something like this and this is D 1 and you take this portion is D 2, then they overlap. So that kind of thing is not allowed, the only thing common is the boundary points kind of thing.

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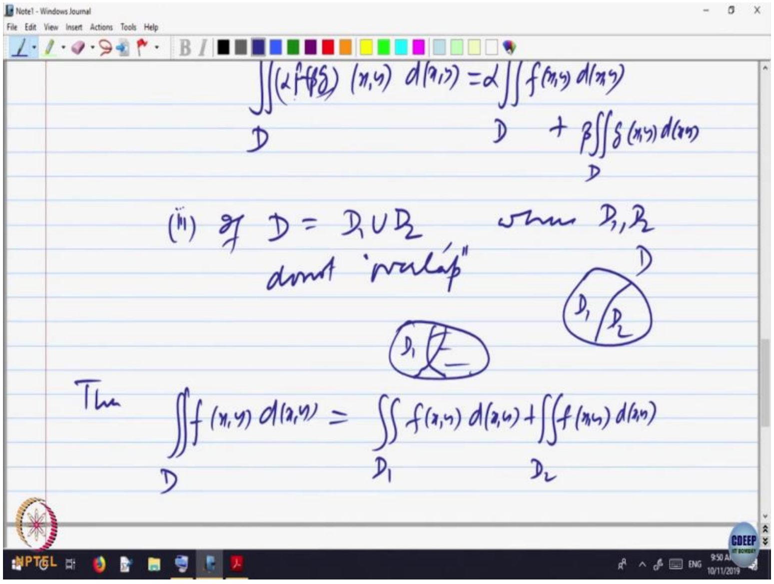

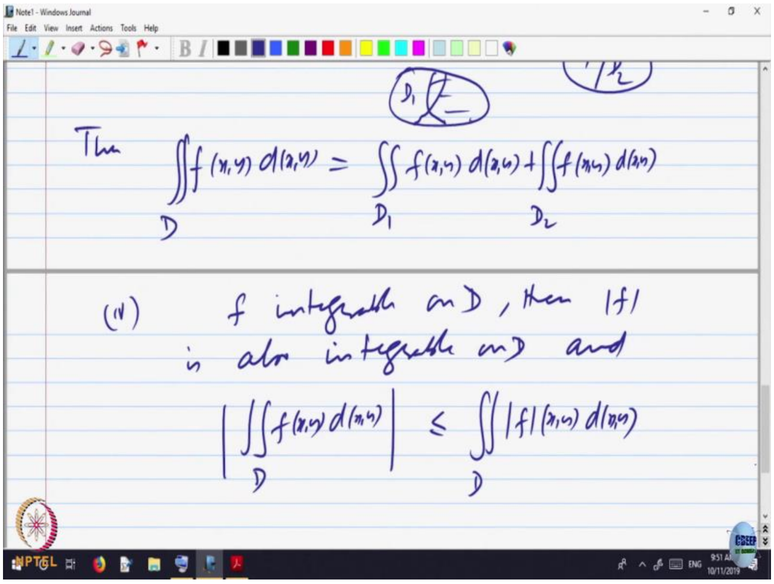

So, intuitively understand if the overlap do not overlap, and D is cut into two parts then integral of F over D is same as integral over D 1 plus integral over D 2 F of X, Y. So, it is like, if the domain is cut up into two parts which do not overlap then the integrals add up. . I think more or less these are good enough for properties, one can write more for, let me write one more so that it looks very much F integrable on D, then mod F is also integrable on D.

And F, X, Y. D, X, Y that is a number, so take absolute value of that that is less than or equal to double integral of mod F, X, Y, D, X, Y we had set it all these properties for integrals of one variable also. So, the same properties hold and we shall not in totally they are quite we will not approve all these properties, but we will use them to when we come to computing these integrals.

So, the basic problem that we want to intuitively integral is the volume below the graph of a function for two variables. The problem is how does one compute this integral? How does one compute the integral of integrable function? If we recall for one variable most or almost all computations of integral are given by the fundamental theorem of calculus. If you know that the function integrant has antiderivative, then you can compute directly the integral, by looking at the values at the end points.

Here what we want to do is and if you recall, we had tried to for a function of two variables, we are tried to shift the problem from two variables to one variable and analyse the problem there. So, here also will like that, to for a function of two variables, which is integrable, we will like some tool which helps us to integrate this by using integration of functions of one variable. So, given a function of two variable, if I fixed one variable, then it becomes a function of one variable. So, can that be used to integrate the function of two variable, so that is a important result.

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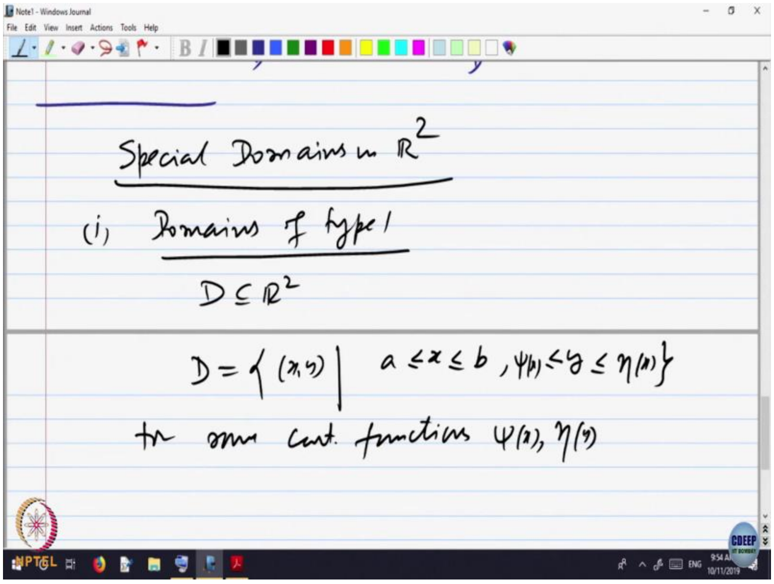

So, for that let me introduce something that depends on the nature of the domain. So, let us write what are called special domains in R 2. So, I think it is good to categorize them, so let us domains of type one. What is domain of type one? So, this is a subset of R 2 of course, the domain looks like all X, Y such that X lies between some numbers A and B. And Y lies between a function so, let us call it as psi X and say eta X, where for some you can say continuous functions psi X and eta X.

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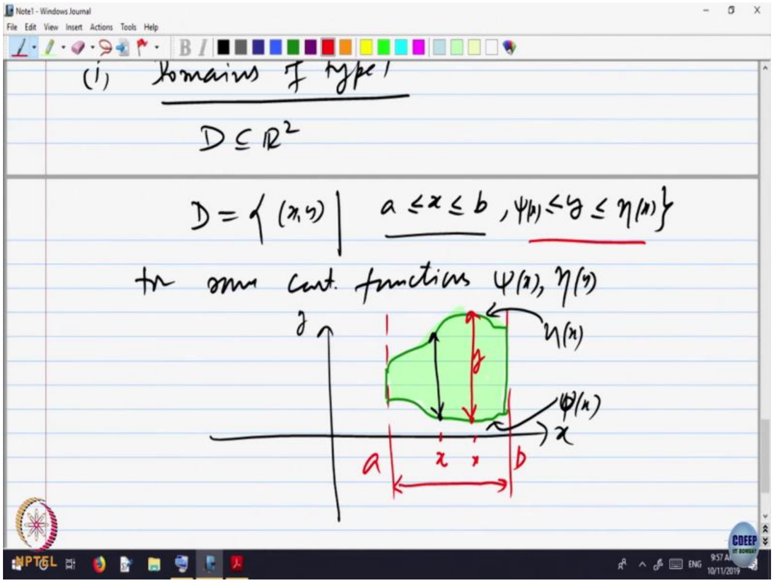

Let us try to visualize this what, what I am saying. So, these are part of the domain. So, let us look at, so this is Y and this is X. So, when I want to say that this is all set of all ordered pairs X comma Y where X lies between A and B, so there is a bound for X. So that, so let us understand that. So, here is the say line A and there is a line B. So, domain is in between these two lines, and what are the, so X is somewhere in between. So, here is a point somewhere in between X, for that point where does Y lie? Y lies between some function of X of that point to some other function of X.

So, let us, let me try to draw and then illustrate. So, it is like so, this is what my domain looks like. So, this is my domain, the domain looks like this. So, it is bounded by X between A and B, and for every X between A and B, where does the Y variable go? Its starts here and goes up to, so this is Y coordinate to be inside the domain. So, the top one is a function of eta X, so this top is eta X and the bottom one is psi of X..

So, for every point, whatever point you choose, say point X is here, if you draw how much is a Y coordinate varying for a point X inside A, B. So, that you are in the domain. So, go a vertical line, you want to know the Y coordinate how much it is. So, as soon as you start, so, from this point you go up to this point, so that is a Y coordinate. So, all these points on this line are inside the domain. So, we are describing the domain, what are points in the domain? X should lie between A and B and for every X the Y lies between some function of psi x and some function of eta X. So, such domains are called domains of type one.

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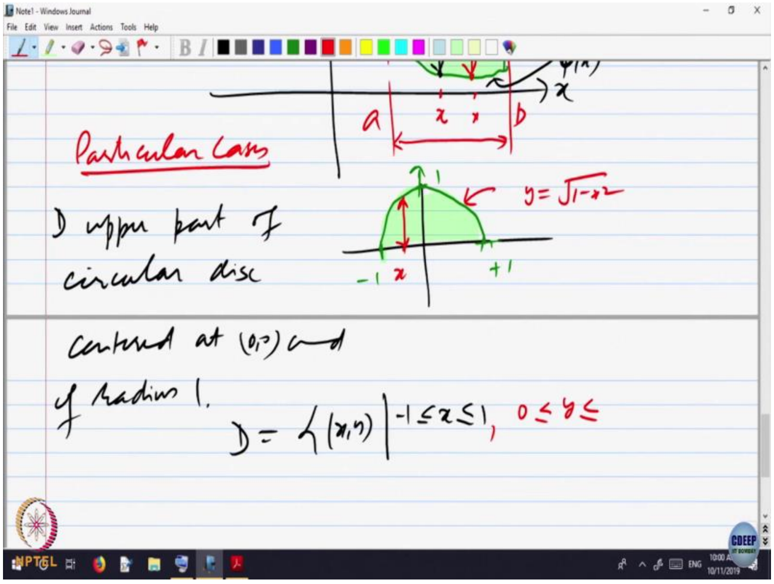

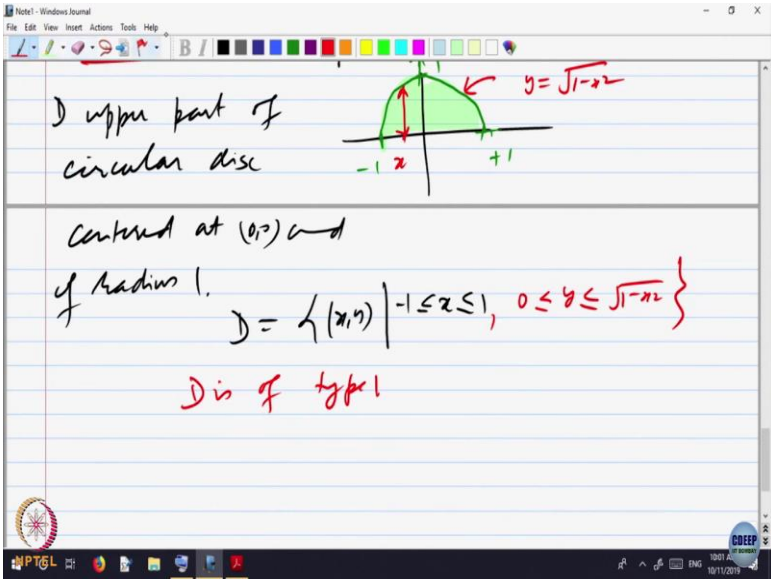

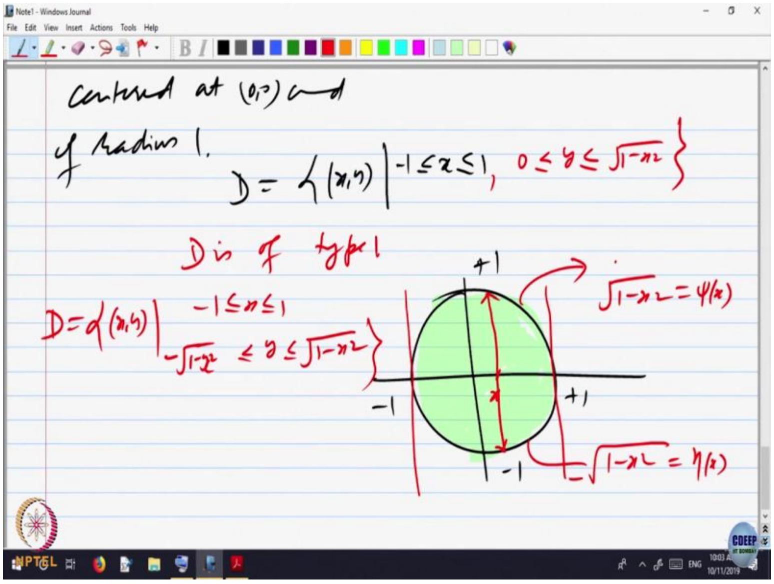

So, let me give some more special examples particular cases of this. So, let us look at particular cases of this. So, let us look at. So, let me look at the domain D which is let me write it in words and then describe it. This is upper part of circular disc centered at 0, 0 and some radius at 0, 0 and of radius 1. So, it is, what is inside a disc of upper part of the circular disc, centred at 0, 0. So, what does this look like? So, this looked like radius is 1, so minus 1 to plus 1, this is one.

So, it is this part of the, very bad circle but it is okay. So, this is the domain we are looking at. What does that domain lie between? Can I say the domain D, this domain D is all points X and Y, such that X lies between some limits? So, X lies between minus 1 and 1. If X is outside this, then no vertical line is going to intersect the domain. And for any such point inside X, to be inside the domain I draw the vertical line. So, this is, this is going to start at the X axis and going up to the boundary of the circular disc.

So, what is? Where does it start? Starts on X axis. So, Y lies between 0, and where does it go? This part, so what is the equation of this part? That is Y is equal to square root of 1 minus X square, which now the circle is X square plus Y square equal to 1, we want to interpret Y in terms of X. So, Y is square is equal to 1 minus X square but the upper part of the circle. So, Y is equal to positive square root of 1 minus X square. So less than square root of 1 minus X square.

So, its domain, I can describe it as this, so D is of type 1. What about if I take the full disc? Is it of type 1, can I say have taken the half disc? So, let us take the full circular disc. So, geometrically that will be we take this disc, so minus 1 plus 1 minus plus 1 and minus 1, so, inside of that. So, is this the whole disc type 1? Let us write, it lies between two lines again, it lies between two lines that is X is equal to minus 1 and X is equal to 1. So D is equal to X, Y say that minus less than X less than or equal to.

Where does Y? So, take a point X, draw vertical line and see how much it varies. So that goes between lower part of the circle. So, what is a equation of the lower part? So that is minus 1 minus Y square, X square less than or equal to 1 minus X square. So, this part is 1 minus X square with a negative sign because it is below and this is with the positive sign square root of 1 minus X square. So, this is my, the function, psi X and this is my function eta X.

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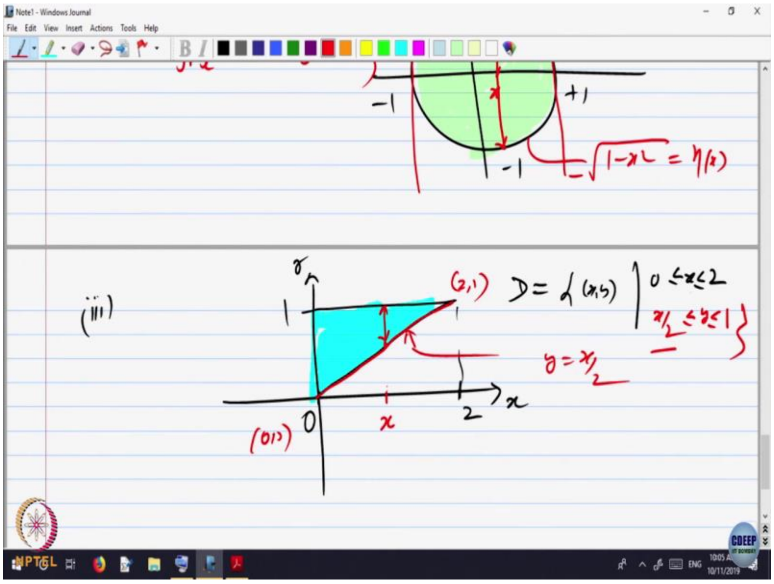

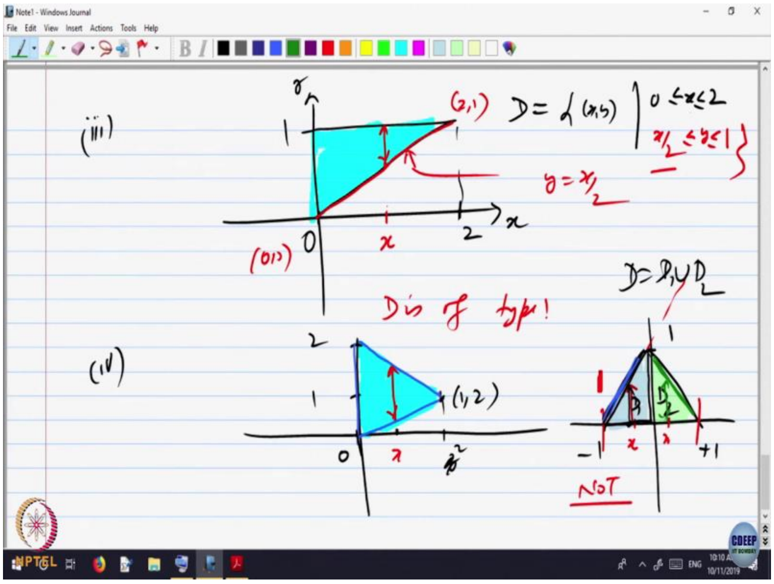

So, let us look at one or two more so that. So, let us look at one more. So, let me just draw, so let us look at, so this is the domain I should tell what is what? So, this is 0, let us call it as 2, let us call it as 1. So, what is this domain D? Can I say it is X, Y such that X lies between something, X lies between 0 and 2, so 0 and 2.

So, here is 0, here is 2, so it lies between these two and for every point X in between 0 and 2, where does Y lie? So, it starts here and goes up to here. So, we have to know, we have to find out what is the equation of this line? So, what is the equation of this line? X is equal to; X is equal to Y? What is this point? This point is 0, 0. So, what is equation of this line? Is not Y equal to X? It is yes, Y equal to 2 X? No X, Y 2.

So be careful, okay. So, this equation is Y is equal to because when X is equal to 2 Y is equal to 1.. So, this goes up to 1 and starts at Y equal to X by 2. So, that is my domain, clear? It starts at this line and that line is Y equal to X by 2 goes up to the line Y equal to 1. So, that is the domain. So, this this D is of type 1. Let me do one more, here is 2, here is 1, and here is 2.

So, the domain is and this point is 1, 2, is clear this is of type 1? Yes, X lies between 0 and 2. And so this is 2 and for any point in between X you start at this line and go up to this line.. So, the lower limit for Y is the line joining 0, 0 with 1, 2 and a parameter is a line joining with 0, 2 and so you have to find those equations and write down.

Let me along with it draw another one. So, let us look at a variation of this. So, let us look at this minus 1 plus 1 and this is 1. So, that this a domain. So, can you say that this domain is of type 1. Again, X lies between, so, X lies between minus 1 and 1. Now if you take a point X in between, it starts at 0 and goes up to this line.

But if I take a point X here, then it does not go to the same line right, it goes to some different limit. So, for same X, I cannot say the lower limit and the upper limits remain same wherever with a point B, the function changes. So, this is not a domain of type 1, is it clear? This is, so this domain not this is not of type 1, because I cannot write it as all points X comma Y.

So, that X lies between some limit and Y lies between some function of X lower limit bounded by some upper limit of a function of X. The functions change, if I like, I can cut it into two parts. I can cut this domain I can call the left side, this part as D 1 and let us, so let us call this part as D 1 and this part as D 2 because Y axis is the line which is changing the nature of the value of Y. So, then I can write as D as D 1 union D 2, they are non-overlapping intervals and each is of type 1 because on the D 1 it goes from up to here.

So, this is a limit and for D 2 the limit is this line, X goes from here to upper one, regressions can be written down, is it clear to everybody? The domain is not of type 1 , but I can cut it up into two parts, such that they are union is of union is a whole domain and each is of type 1. So, that will be useful when you want to compute things.