Basic Real Analysis
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Lecture 48
Optimization in Several Variables - Part III

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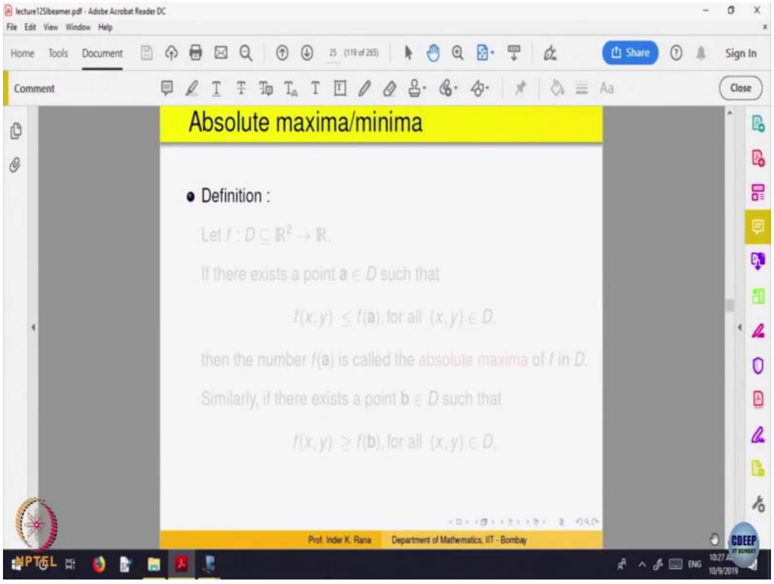

The theory is going perfectly parallel similar to one variable, in one variable we looked at local maxima minima. Then we looked at how do you find out absolute maxima or absolute minima of a function of one variable. So, what was our line of thinking? If the sum point is a point of local maxima or local minima, it is also a point of global maxima and global minima.

So, what we are interested in and that means, we should look at the critical points, the points where either the derivative does not exist one variable or derivative exists and is equal to zero or the boundary points. So, these points look at the values or the function and compare, if the function has absolute maximum or minimum it will be one of those points. To ensure that the function has the absolute maximum minimum you are to justify by some other theorem.

For example, if the function is defined in a domain which is compact closed and bounded in the real line. Then, every continuous function attains maximum and minimum, how to find that, you apply the calculus techniques derivative test and so on. So, in two variables or three variables also the same method is applied, look at the critical points, look at the values at those points and compare, whichever is the largest that will be the maximum supremum, whichever is the smallest will be the in premium provided the function has such things.

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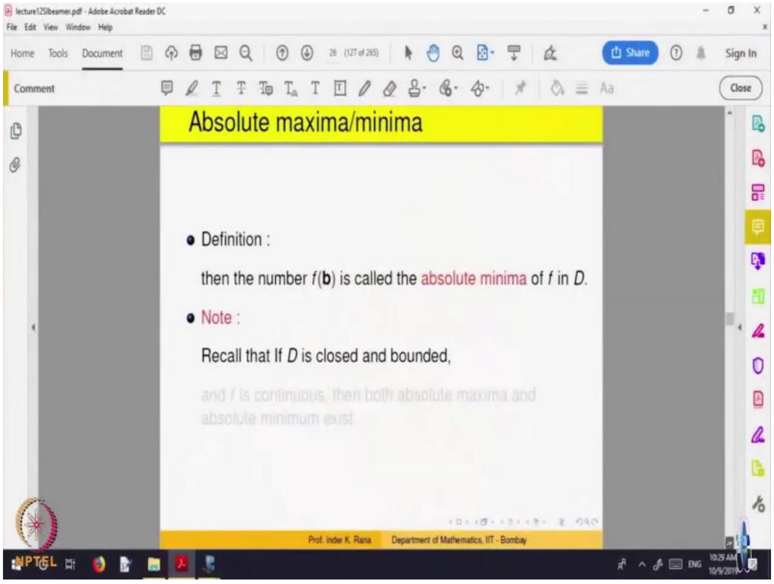

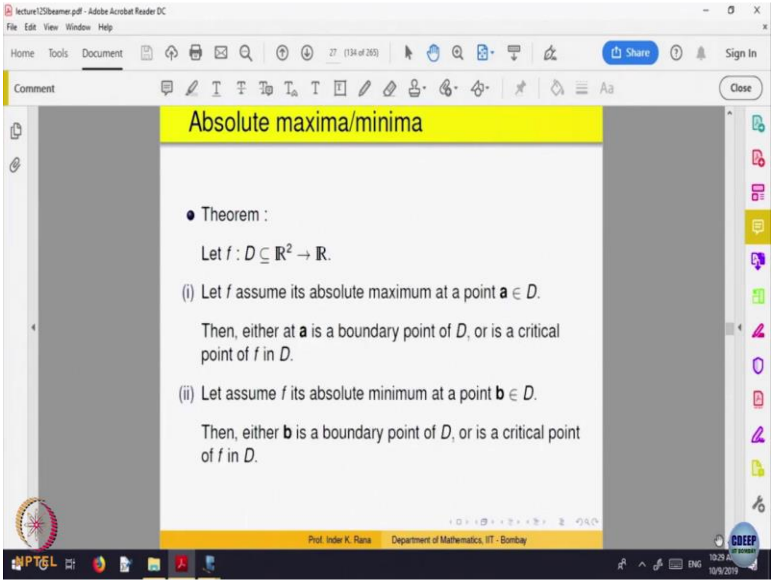

So, justify that, so, let us look at probably some, so these are called absolute maximum or absolute minimum. So, if it has, then either is a boundary point or it is. So absolute maximum at a point, then either it is a boundary point or a critical point, it has to be one of them.

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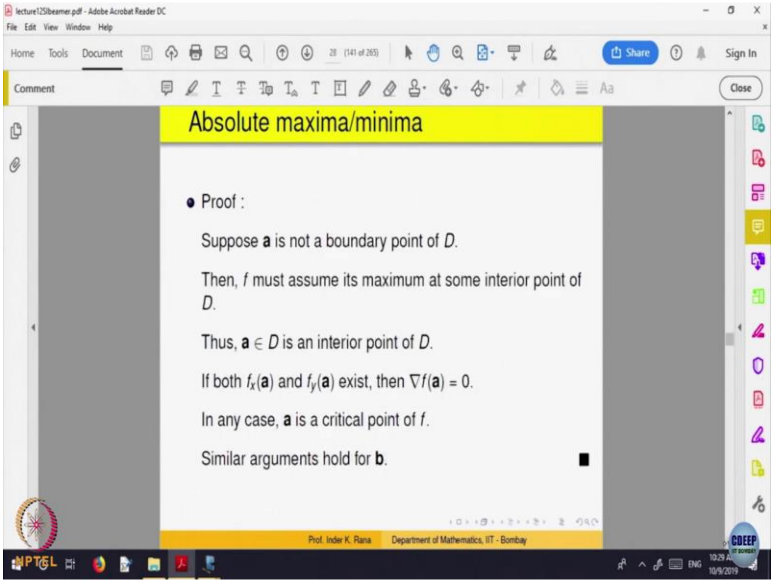

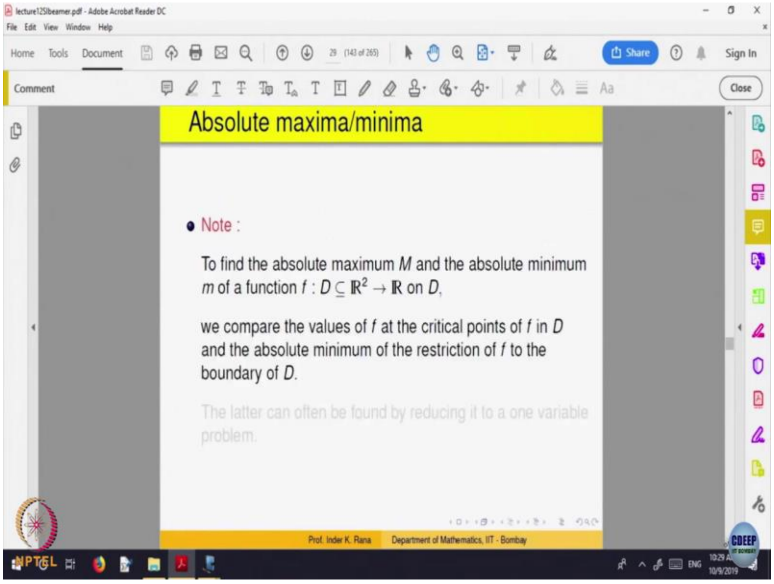

So, there is nothing in the proof, it just say writing that because absolute has to be local. So, so let us look at, so D is a domain where X, Y, is that mod X is less than or equal to mod Y is less than or equal to 2. So, what is the domain? Mod X less than or equal to 2 mod Y less than or equal to 2? What does the domain look like? It is square of what kind? Absolute value, so anyway, is this domain compact? Close bounded no problem, so, it is a compact set.

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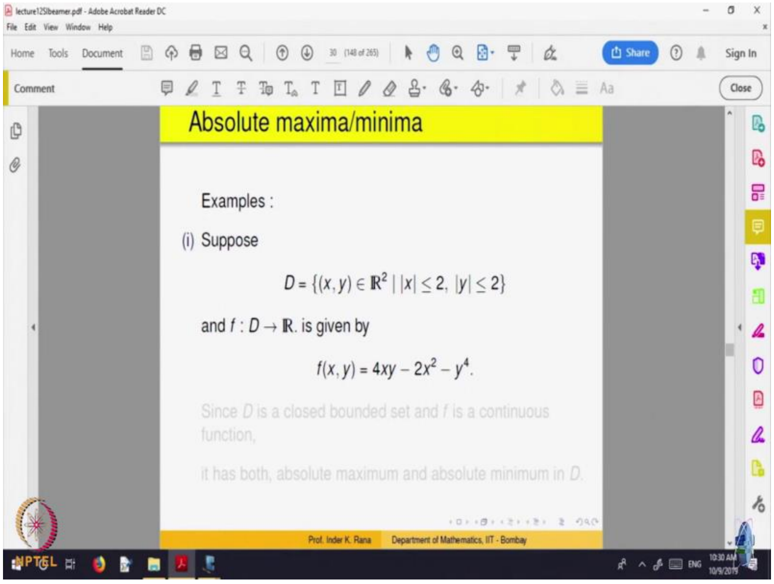

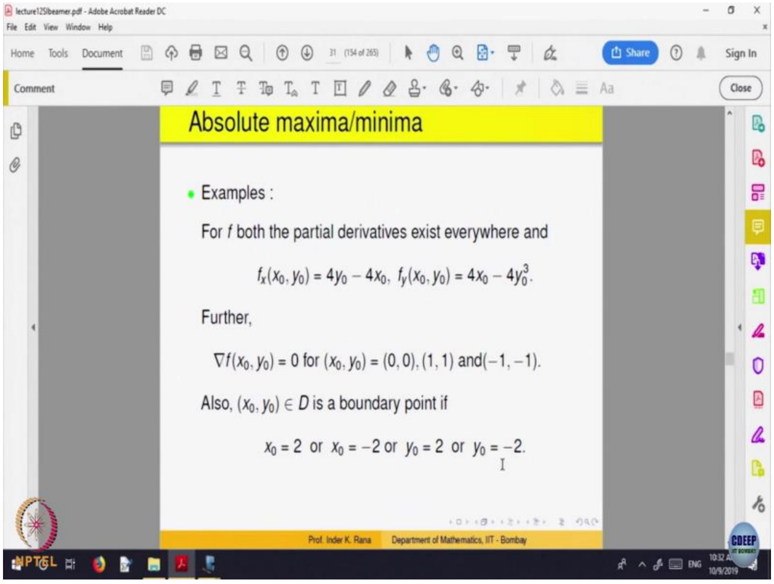

So, let us look at the function F, X, Y is equal to 4 X, Y minus 2 X square minus Y, just now we analyse that function for local maximum and local minimum, but we analyse everywhere, we do not analyse it in a particular part of the plane. So, now we are saying this tick this function in this domain, the domain being closed and bounded. The function should have a absolute maximum and absolute minimum.

So, how do we find out absolute maximum or absolute minimum? You have to look at the critical points, the critical points are the points where the derivative is exists and are equal to 0. So, we found out 0, 0, 1, 1 and minus 1 minus 1, what are the boundary points? Boundary points are where X is equal to 0 and Y is equal to, mod Y is equal to 2, those ones.

So, let us analyse and compare, so discriminant actually 1 to 1 and minus 1. So, let us look at the boundary point, if D is a boundary point, then what should happen? Either X 0 will be equal to 2 or X 0 will be equal to mod X. So, either this side or that side. So, I actually I going up to minus 2 or plus 2, so, those are values of it and similarly, Y mod Y. So, what are the boundary points? When X 0 is equal to 2, Y is varying. X 0 is equal to minus 2, Y is varying and similarly, other two things. So, those are the boundary things. So, we had to analyse what happens.

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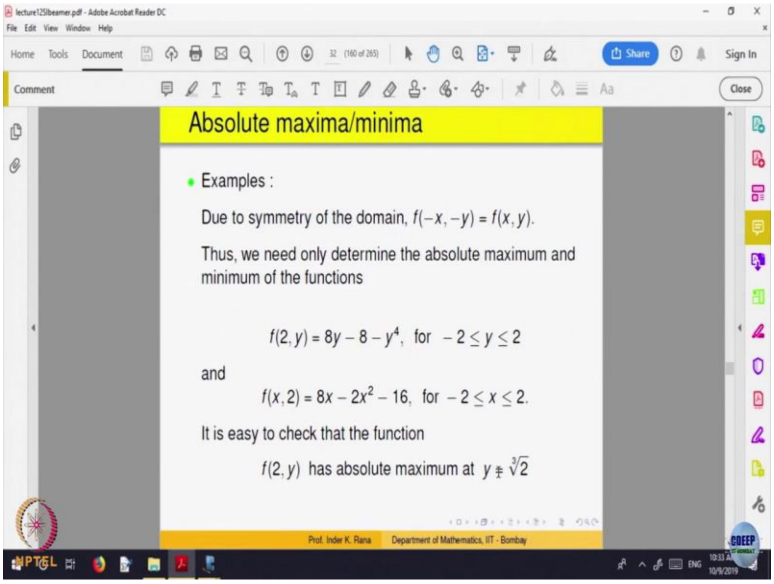

For example, let us look at X is equal to 2, that means, we have to fix X is equal to 2 and let Y vary as a function of one variable, it will comes out to be this. So, on the boundary points look at how the function looks like and then as a function of one variable find out what is the critical points, maximum minimum for that function. So, those are the values of the boundary, function does not remain constant on the boundary point, it changes. So, what is the maximum or minimum at the boundary we have to analyse and then compare.

So, look at a function of one variable, for example when X is equal to 2, this is a function Y varies between minus 2 and 2. So, for one variable I will find out? Derivative. So, this is a function, so if you check for this function, I am avoiding the calculations, it has a maximum at the point Y is equal to cube root of 2, how will you find that? As function of one variable, find out the derivative, derivative equal to 2 then you want to analyse you can apply second derivative test if you like to find out whether it is a point of maximum or a minimum. So, one variable theory will be applied to conclude that it has a maximum value at the point and the this.

So, how do you know compare? Compare the value at 0, 0 compare the value at 1, 1 compare the value at minus 1, minus 1 and the boundary points and see out of this which is the largest.

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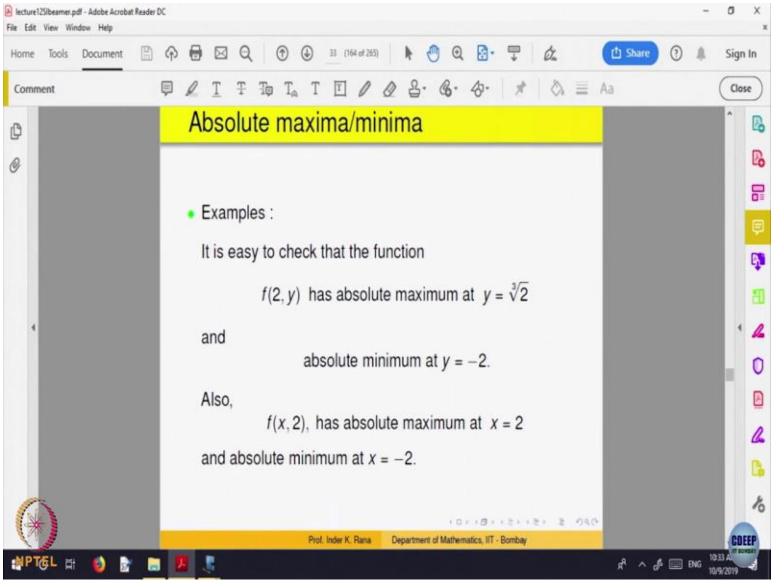

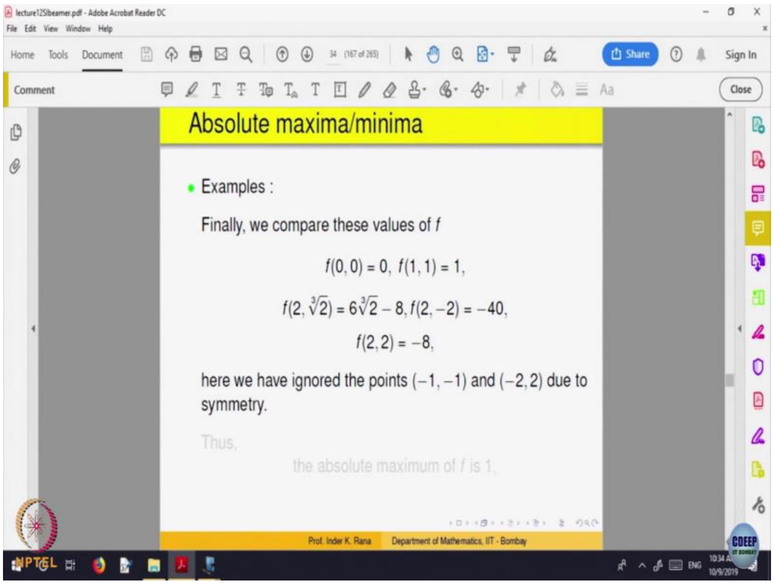

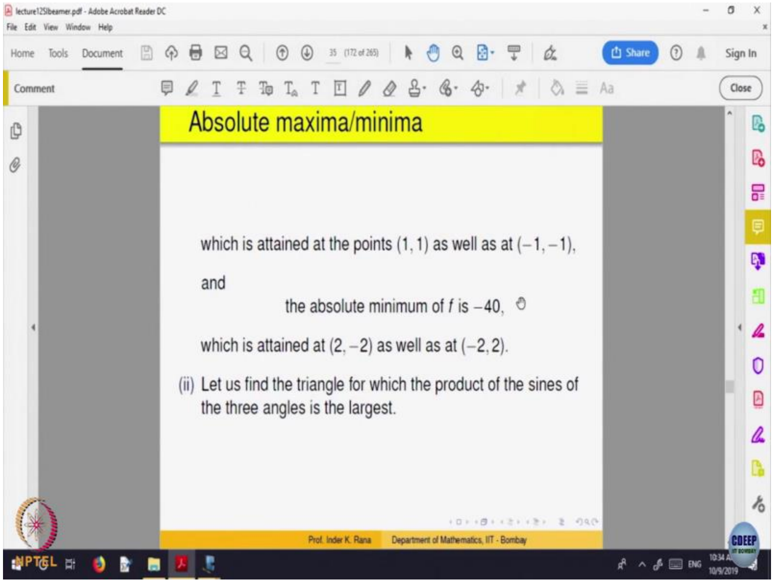

So, if you compare, so you get similarly this is 2 Y, similarly for X 2 you will be doing it, you are fixing earlier fix was X is equal 2, now fix Y and compare what is the value so, that comes out to be minus 8. So, out of all the points the absolute maximum of F is equal to 1. So, all this you have to be comparing, so, for function of two variables you have to compare the values not only at the critical points also bonded, in one variable boundary points maybe at the most two for interval left and right, but here, it will be probably a boundary, you have to see how the function on the boundary looks like, find out the maximum minimum and then compare among all these things.

So, is it okay clear? Theory goes parallel to one variable, only the work involved is more because the boundary in two variable maybe a triangle, maybe a circle maybe or maybe cut complicated one. It will quite complicated one you may not be able to find what is the maximum minimum, some other way one has to apply, is that okay? Process is clear, rest is only competition. How do you compute? There are some more examples.

Now what we? Next we want to do is sometimes one has to find out maximum or minimum function with a constraint. Here we are looking at the function define a domain. But what does it mean saying that I want to find out function is defined in a domain but I want to find out the maximum or the minimum with a constraint?

So, for example, you may be interested in knowing that is a problem with space scientists face. A shuttle is entering into the earths circle atmosphere, a space shuttle is falling is normally done, nowadays all space shuttles, when they come back to earth, what is happening? They enter the gravitational force field of earth and they may sort of path along which the drop somewhere in a sea or somewhere.

Have you ever wondered why they take a drop in the sea? Why should space shuttle be manoeuvre to fall in his water body, a sea, normally seen all the fall in a sea. That there are two reasons for that, one if it is a hard surface, then you have to control the speed it should land very smoothly slowly, gravity will be increasing the speed you have to decelerate somehow. So, you have to add on mechanism that is one, and that is why probably our mission slightly failed in the end because we could not decelerate it, landed a very harshly there was no sea.

But other advantage is when something is falling on the earth, its surface gets heated up, because of friction, because of the air friction, the outer surface starts getting heated up. And within says something with such a mass is coming down with such a velocity, it gets so heated up, then it has to be cooled down. So, water is a natural coolant, it falls in sea and then naturally it cools down to some temperature.

So, the point what we are trying to do analyse here is, the body of a space shuttle is something, that is a surface , there is a surface and at a particular time point there is a temperature on the surface. So, what is the point on the surface where the temperature is maximum, we want to analyse? So, we are analysing temperature on that body. So, it is a function of how many variables? X, Y and Z the position of the surface. And time point T, at time point T is the temperature as time changes even, if the point X, Y is not coordinate in the same temperature changes, so it is a function of four variables, X, Y Z and T.

But those, so you want to know what is the temperature at that point? So, temperature is a function of four variables, X, Y, Z and T. We want to know the point on the shuttle, where the temperature is maximum. So, is a function of four variables but we want to put a constraint on X, Y and Z that X, Y and Z should be on that surface, so that is a constraint, it should satisfy the equation of the surface. So, that is a constraint. We do not want to, as it falls temperature goes on increasing, probably somewhere, but we want to know on that surface, what is the temperature?

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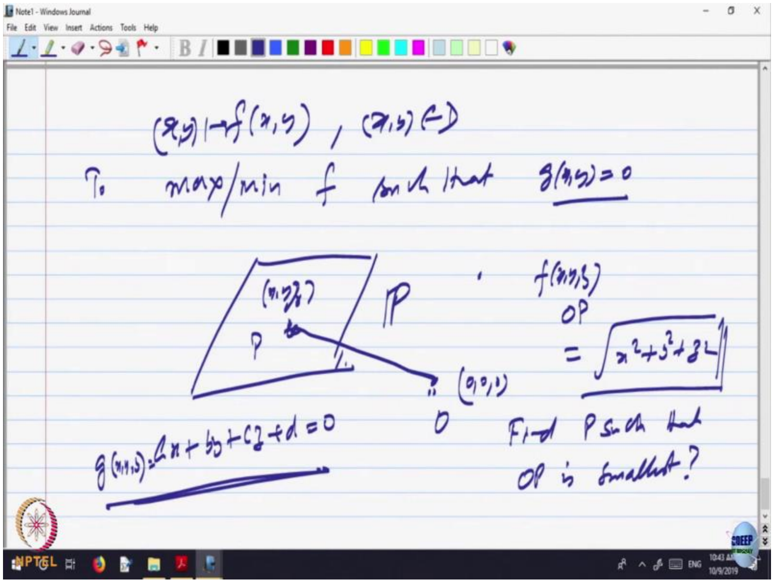

So, in general, this is a problem F, say X, Y is a function, X, Y belonging to, so the domain D oh sorry, X, Y, X, Y goes to, X, Y belonging to D. So, problem is to maximize minimize F such that G, X, Y is equal to 0 there is a relation between X and Y. For example, let us look at a plane, P is a plane. X, Y is a point or let us look at the, let us look at to be every simple, let us look at the point 0, 0 origin.

So, for any point X, Y on the plane, it has a distance, any point X, Y outside the plane, any point X, Y on the X, Y and Z, say let us the three variables X, Y and Z. So, this is origin and this is a plane and let us call this point as something or let us call it P. So, OP so, what is OP equal to? Distance, distance formula, X square plus Z square, square root that is a distance.

So, the problem is find P such that OP is smallest, so this is a problem such that OP is smallest. So, what are we going to look at? So, we are trying to. So, this is my function F, X, Y, Z that is a distance. So, I want to minimize this distance but I want to minimize. Where is X and Y? X and Y lie on the plane, the plane may have some equation.

So, let us say this plane has a equation. So, what is the normal equation of a plane? Equation in R 3 a plane. So, it will be AX plus BY plus CZ plus D equal to 0, is it okay? So, this is my G, X, Y, Z. I want this function FX, YZ to be minimized with a constraint that the point lies in the plane that means, it should satisfy this. . So, (()) (15:55).

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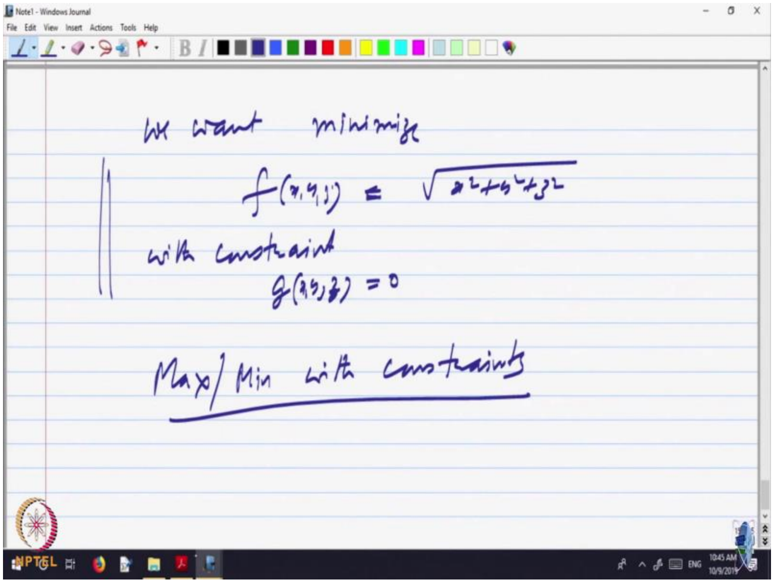

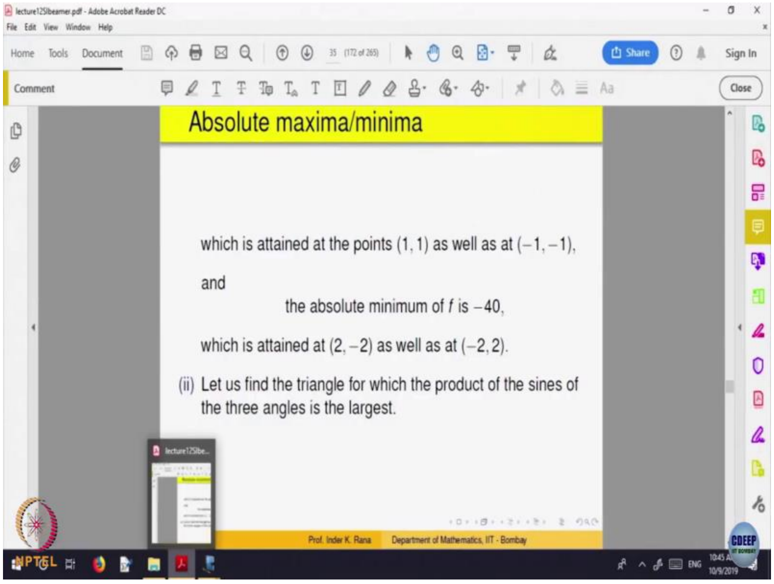

So, this means we want to, so in this problem minimize the function F, X, Y, Z which was equal to square root of X square plus Y square plus Z square with constraint G, X, Y, Z equal to 0, that means there is a relation between those points X, Y and Z. G, X, Y, Z is equal to 0. So, that is a constraint.

So, this problem occurs not only in practical situations in many practical situations, this also occurs in probability and statistics when you want to do statistical inference and you have estimates, likelihood estimates kind of a thing coming, there you want to maximize or minimize error with respect to some constraints. So, this will come back in case you are doing some courses in probability and statistics also. So, this is the kind of thing we want to do.

So, these are called maxima minima with constraints. So, these are called problems of maxima minima with constraints. And for that Lagranges proved theorem, so will not go into the proof of the theorem, we are going to the consequence of that theorem, how that method called Lagranges method of constraint maxima minima is used will look into that. So, let me look into that.

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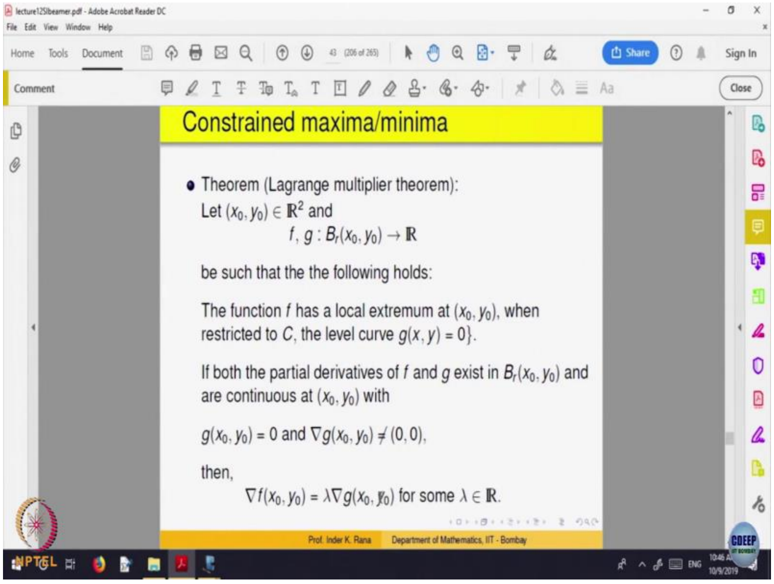

So, here is what is called Lagranges multiplier theorem, I just stating it so that it is clear F and G are two functions defined in a neighbourhood of the point X 0 Y 0. So, F is going to be the function which is going to be maximize or minimize and G is the constraint that is going to be, such that the following holds the function F has a local extremum at X 0, Y 0 when restricted to C, the level curve G, X, Y equal to 0. So, that constraint, so G, X, Y equal to 0 will be a curve, in the domain, is it clear? X and Y are related with each other.

So, Y has a function of X can be computed properly. So, that is a level curve. So, saying this is the constraint, so, the conditions are if both the partial derivative of F and G exists and are continuous in that neighbourhood G X 0, Y 0 is equal to 0 and derivative, this is a gradient if you remember what is the gradient? Partial derivative of G with respect to X comma partial derivative of G will spread to Y that vector is not equal to 0, then this relation must be satisfied. So, it is a necessary condition.

So, F is the function which is to be maximize or minimize, G is the constraint. So, he says then there must exist some lambda, such that this is equal to lambda times this. Now, if you look at this partial derivative gradient of F, it gives you two, is a vector, is a vector equation.

So, what is this vector equation? Partial derivative of F with respect to X is equal to lambda times partial derivative of G with respect to X.

Second component equation is partial derivative of F with respect to Y is lambda times partial derivative of G. And what is the third equation? Third equation is G, X, Y equals 0 that is another relation. So, three variables, X 0 is not known, Y 0 is not known, lambda is not known and you got three equations. So, you have to solve these three equations and find out the points X 0 Y 0 that will give you constraint maxima and minima.

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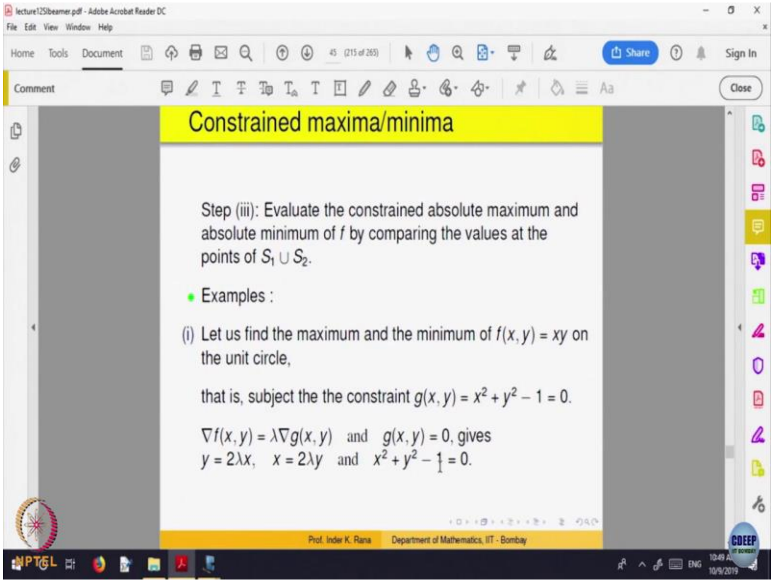

So, let us look at one example at least to this illustrate. So, let us look at the function F, X, Y equal to X, Y. On the unit circle, we want to maximize that means we want to find X and Y the function is domain is everywhere, function is defined for all points in the plane but we want to know what is the maximum and the minimum value of the function on when X and Y satisfy the equation of the circle.

So, on the unit circle, so what is the constraint? X square plus Y square equal to 1, so, G, X, Y is X square plus Y square minus 1 is equal to 0, so that is a constraint. So, what are the equations that you will be solving? So, gradient of F is equal to lambda time gradient of G that is one vector equation. So, that means, Y partial derivative of the respect to X that is Y, partial derivative of G with respect to Y that is 2 Y, so, FX is equal to lambda times GX FY is equal to lambda times GY the second and the third one is the equation of the constraint that is a circle.

So, these three questions have to be solved minded they are not Linear equations. So, it is not linear algebra being done because these equations this is linear, this is linear, but this is not linear. So, those techniques are linear equations may not work. So, somehow you have to the basic idea is you can remove lambda from first two equations probably find a relation between X and Y, put it in that equation and find Y and then find X. So, that is how you do it. So, this is, this can be a problem sort of how to find equations, find solutions.

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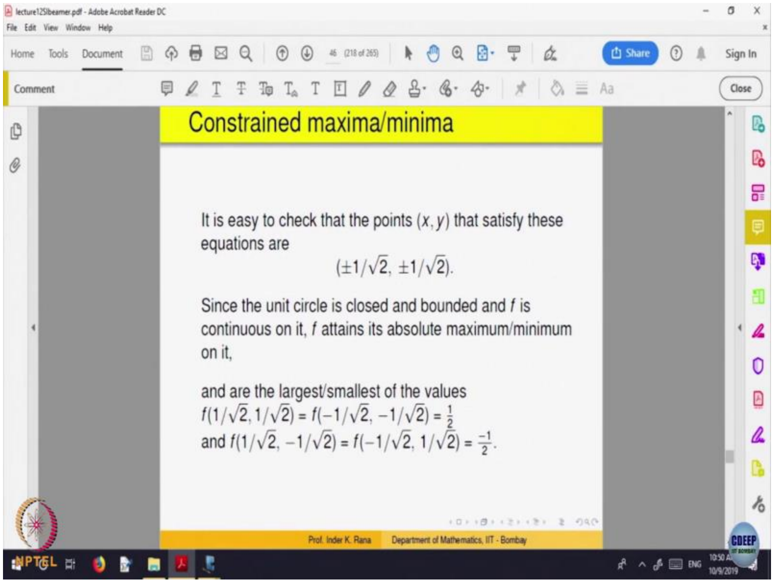

So, it comes out that these are the possible points. So, once you have found the points the values of the points. Now, how to find the maximum or the minimum? Look at the values at these points and compare which is the largest which is the smallest so, that gives you. So, once you do that you get these are the values. So, 1 by 2 is the maximum and 1 by minus 1 by 2 is the minimum and they are attained at more than one point. So, this is called Lagrange multiplier method for finding constraint maxima and minima.

Sometimes the constraint can be more than one, sometimes the constraint can be more than one, F, X, Y you want to maximize minimize with the constraint G 1 and G 2. Then, one more variable will enter into picture. So, you will take because your constraint, if you want F, X, Y equal to 0 also G, X, Y, G 1 equal to 0, G 2 equal to 0 that means have a linear combination of that should be equal to 0. So, you can make it as a one constraint but one more variable enters into picture. So, that is with more than one.

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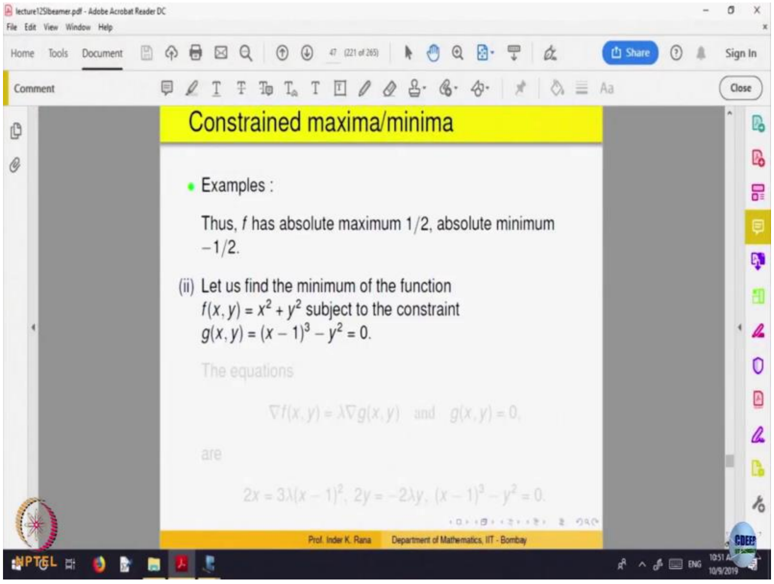

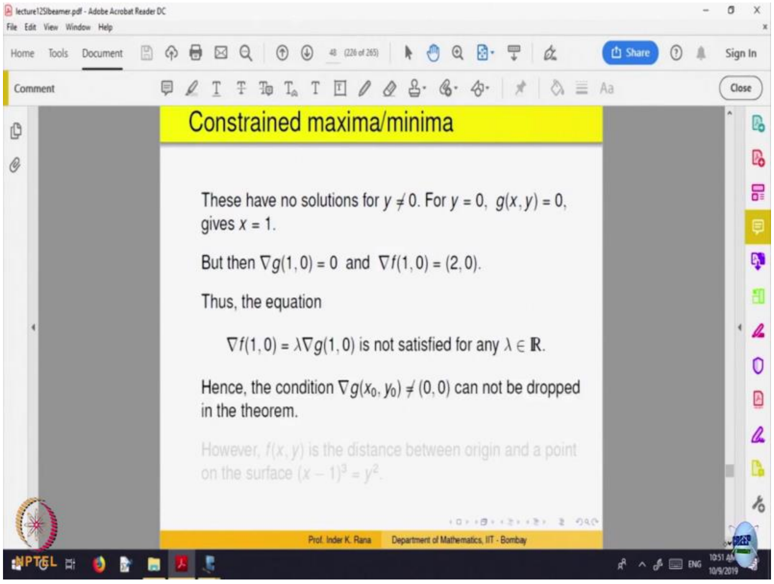

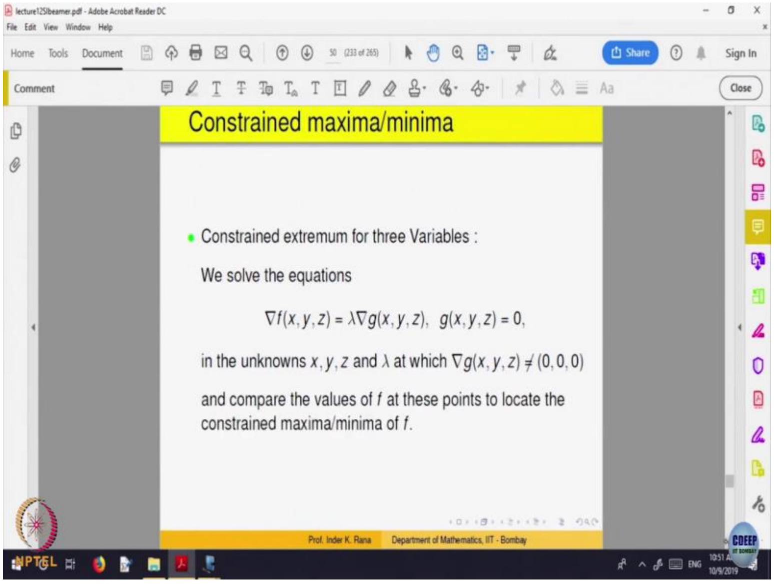

So, let me just state that the examples two variables, these were three variables, number F equation will become three vectors lambda. So, four equations and four variables, so that is what but constraint is still one.

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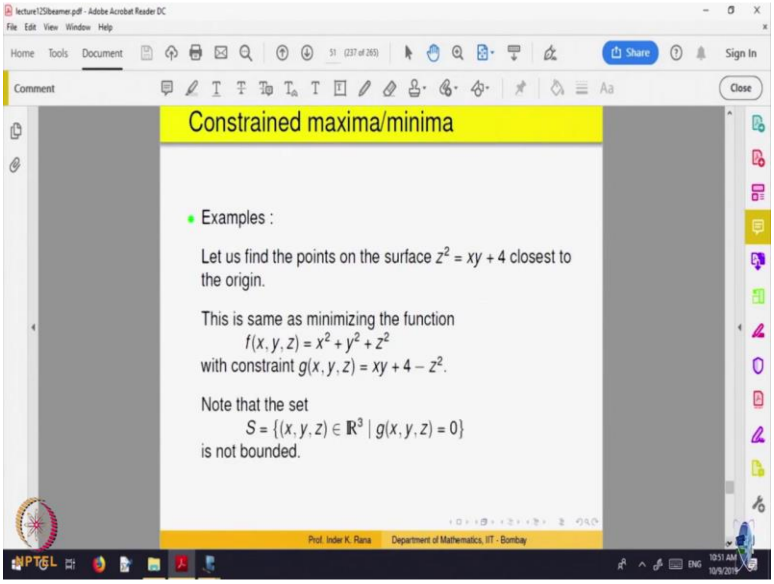

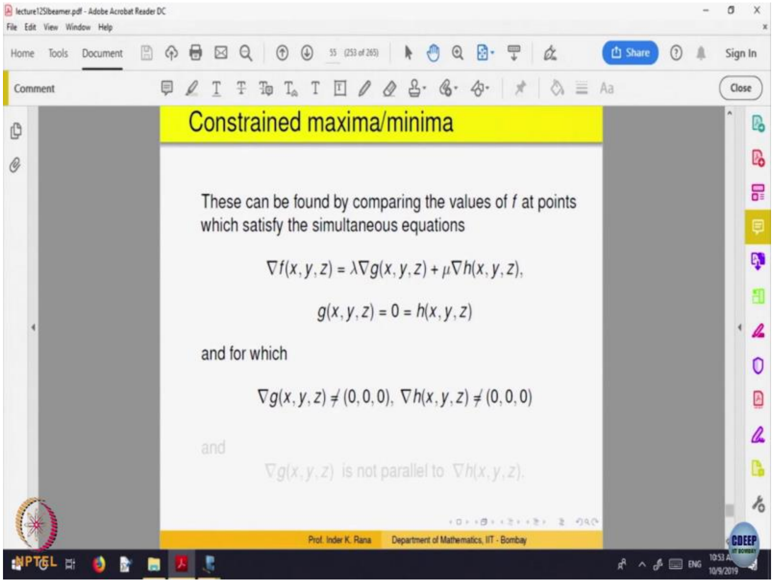

So, let me look at multiple constraints yeah, here is a multiple constraint. G and H are two constraints, which are applied to the function F. So, what we are saying is you can look at a linear combination of these two, as if there is one constraint. So, look at lambda and mu to be scalars, variables of core. So gradient F is equal to lambda times G plus mu times H. So, this is again a vector equation.

So, how many? Three components X, Y and Z. So it will give you three equations, fourth one comes G equal to 0, fifth comes from H is equal to 0 and how many variables to be solved X, Y, Z lambda and mu. So, the problem becomes slightly more complicated that is all, but it can be done. So, and many times finding solutions of such things, you may not be able to find exactly solutions. So, there is something called numerical techniques for finding solutions. So, if you do a course in numerical techniques, you may come across these things.

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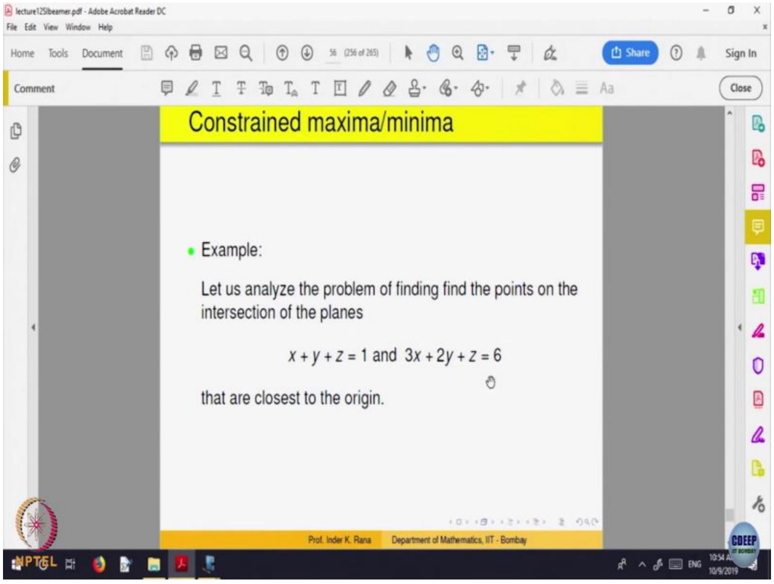

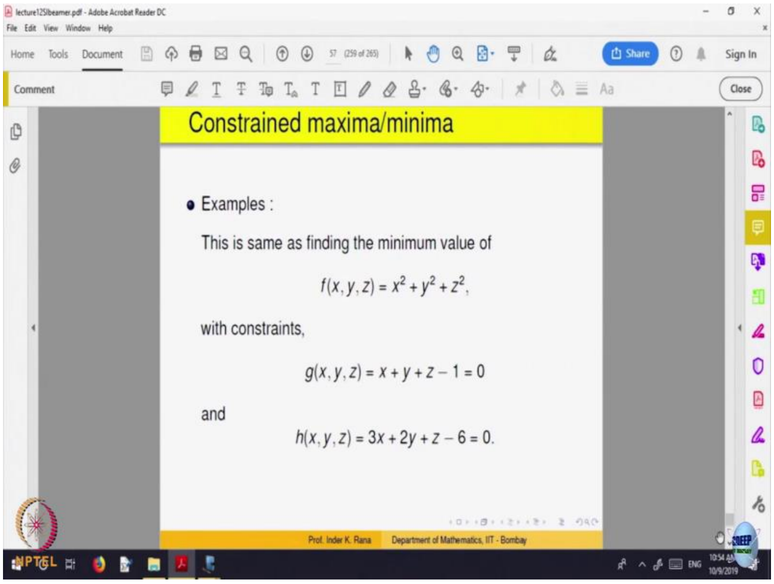

So, for example, here is the analyse a problem of finding the points on the intersection of the planes. There are two planes when they intersect what you will get? When two planes intersect what you will get? You will get a line, so essentially you want a point on that line, which is closest to the origin. So, one method could be, you find out the intersection of the two plans as solving system of linear equation, find out that linear equation that line and then closest to origin. So, reduce, but why to do that much you can just look at.

So, this is the constraint, what is our distance formula? X square plus Y square plus Z square, square root that is a function F to be minimize with the constraint G that is the first one H plus Y plus Z is equal to 1, minus 1 equal to 0, H three H plus two Y plus Z minus 6 is equal to 0. So, these two constraints. So simple problem look goes to Lagrange multiplayer. So, you do that, so with respect to these constraints, and then you can solve them.

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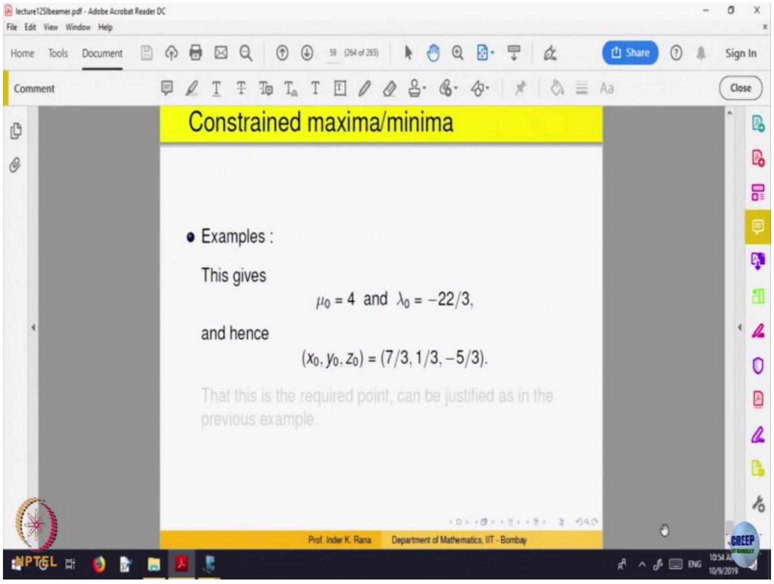

Problem becomes slightly more involved, because two constraints are there and nonlinear things may come into picture. So, find X, Y and Z put in that equations and solve and so, your ability to solve those equations, you get two equations in lambda and mu, their method of solving and then solve those two equations, get values of lambda and mu and then find out the point. So, the basic idea is as the constraint increase, you can take a linear combination of them to be the constraint. So, this kind of things will come into picture for you.

So today, we have just tried to look at maxima minima problems of several variables. And, theory goes parallel to one variable, essentially that find critical points was though the points are the points where either the, partial derivatives do not exist or derivative, partial derivative exists and are equal to 0 or the critical or the points which are the boundary points later. And there are possibility of points out of these critical points, points with neither maxima or minima probably can be what is called a saddle point.

So to analyse them, you have derivative tests, secondary derivative test discriminant, discriminant positive. Second derivative FX X less than 0 local maximum, derivative discriminant positive. Second derivative less than, bigger than 0 local minima. Discriminate equal to 0 saddle point, no okay good. So less than 0 saddle point and equal to 0 it is inconclusive, bigger than 0 less than 0 you can conclude but equal to 0 you cannot conclude, you may have to go directly and the method is in the domain try to look at points, some curves along with it could be maximum along with it could be minimum to give you the value.

And then same method applied to with constraints, it gives you FX Y to be maximize minimize with respect to constraint G, so, we had to solve the equation F is gradient of F is equal to lambda time gradient of G and if more variable, more constraints than lambda and mu come into picture. So, let us stop here.