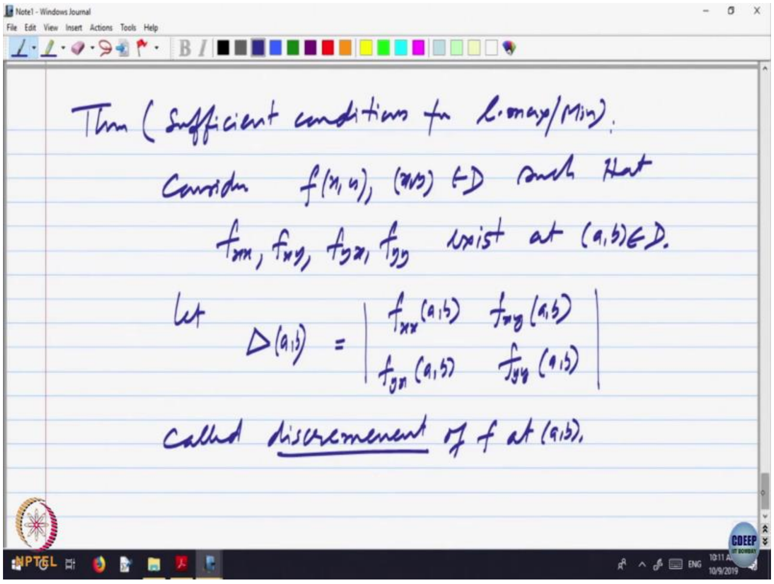
Basic Real Analysis.
Professor. Inder. K. Rana.
Department of Mathematics.
Indian Institute of Technology, Bombay.
Lecture 47
Optimization in Several Variables – Part 2.

So, why I am introducing this partial derivative? Because one can give a sufficient condition in terms of this derivative to say that the point critical point, is a point of local maxima local minima or it is a saddle point. So, let us write that theorem.

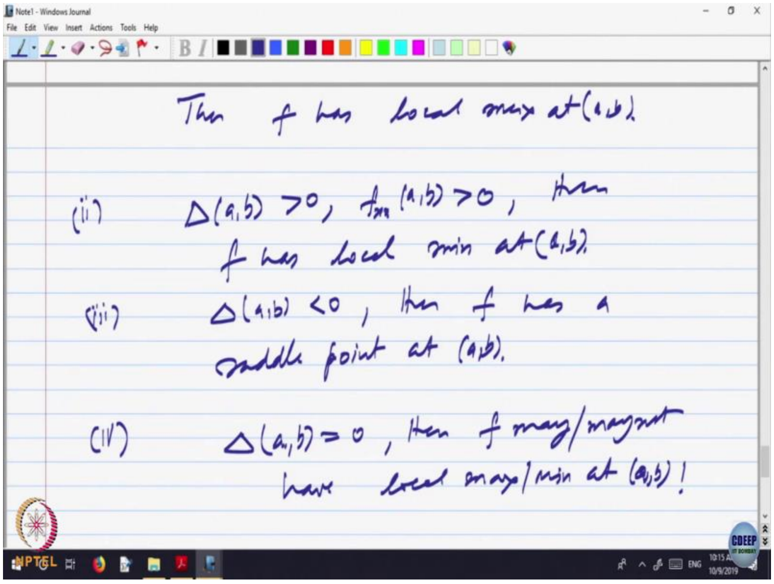
(Refer Slide Time: 00:39)



Sufficient conditions for local maximum, local minimum. So, it says first of all consider the function f xy, xy in the domain D, such that f xx, fxy, fyx and fyy all the four derivative exist at some point a, b belonging to D. All the derivative exist define, so let us look at 2 by 2 determinant, fxx at the point a b f xy at the point a b f yx at point a b and f of yy at the point a b. So, what is this?

This is 2 by 2 determinant, whose entries as the first row is the partial derivative second order partial derivative fxx. Second entry is fxy, f second row is f yx and fyy. So, this is called determinant at the point a b. So called, there is a determinant is called discriminant it is called discriminant of f at the point a b. You will get a number, calculate this quantity you will get a number. So, the test is in this number, so let us write what is the test.

(Refer Slide Time: 03:13)



So one, this delta a b the discriminant at the point a b is bigger than 0 this number you calculate is bigger than 0, and the partial derivative with respect to x at a b is also less than 0. Discriminant is bigger than 0 the partial derivative is less than 0. Then of course, we should have f x at a b equal to 0 equal to f y at, they should be critical points we are looking at critical points.

So, these are discriminant, so let the first derivative at a b be 0 look at the discriminant, if it is positive and the second derivative fxx is less than 0, then f has local maximum at a b, then the point a b is a point of local maximum. So, this is something like the second derivative test for functions of one variable. Looking at discriminant that is positive, if we look at the second derivative test for function of n variable that was second derivative at that point should be less than 0 in one variable.

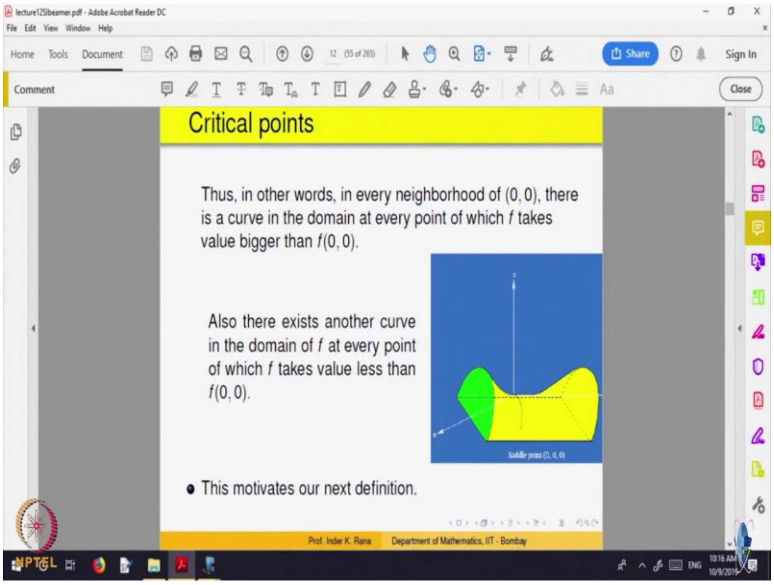
That implied local maximum and same thing is here also. So, that is one conclusion second discriminant at a b is still bigger than 0 but f xx at a b is other way around. So, other possibility is it is bigger than 0 than f has local minimum at the point a b. Again similar to the one variable, if the second derivative test if the second derivative at the point a b is bigger than 0, than the point was a local minimum.

Same thing happening, so this is when the discriminant at that point is bigger than 0. So, what is the other possibility? Discriminant at that point a b is less than 0. In that case, f has a saddle point at a b then it has saddle point at a b. So, look at discriminant if it is bigger than 0 and look at the second order derivative with respect to x, if it is bigger than 0 local minimum less than 0 local maximum, and if the discriminant is less than 0, then you can conclude it is a saddle point.

What happens in case this is the, what is the possibility left this is equal to 0. Then f may or may not have local maximum, minimum. Means what? What does the statement mean? This means that the test you cannot conclude anything the function may have discriminant which is equal to 0 function may have a local maximum at that point the function may have a local minimum at that point we have a saddle point anything is possible or none.

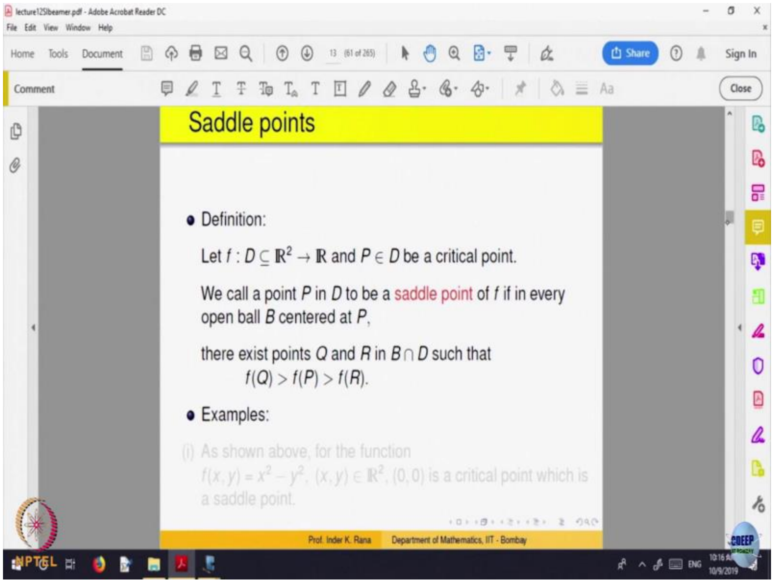
So, then you have to imply some other method of concluding something is like one variable if the second derivative is equal to 0 than you cannot conclude delta function as a local maxima or local minima, you may have to go to third derivative or some other method. So, these are only sufficient conditions to ensure that. So, probably let us look at some examples, to illustrate that it is probably I think because that will endure some computation.

(Refer Slide Time: 07:40)



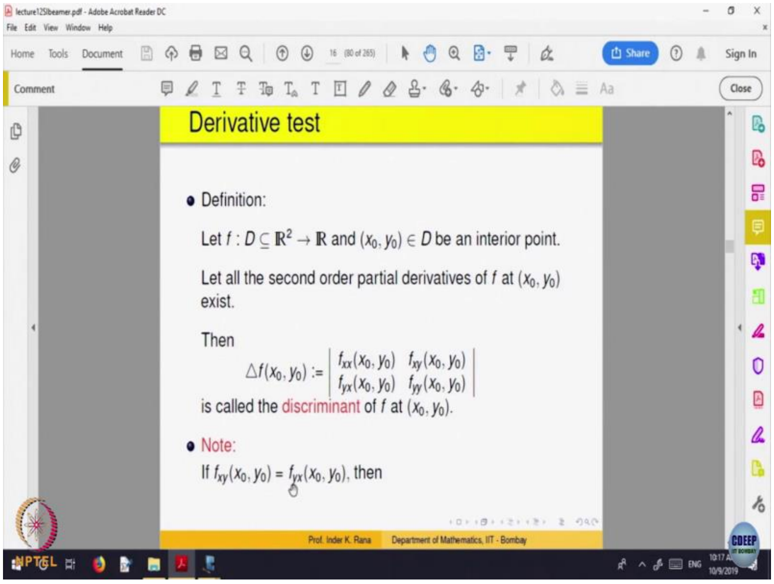
So, let us we are, whatever I have said is there, so let us look at some examples. So, this is what saddle point, sort of the saddle looks like is a better picture, so you can have a look at. So, if you go along this there is a maximum and along this there is a minimum. So, that is saddle point.

(Refer Slide Time: 08:12)



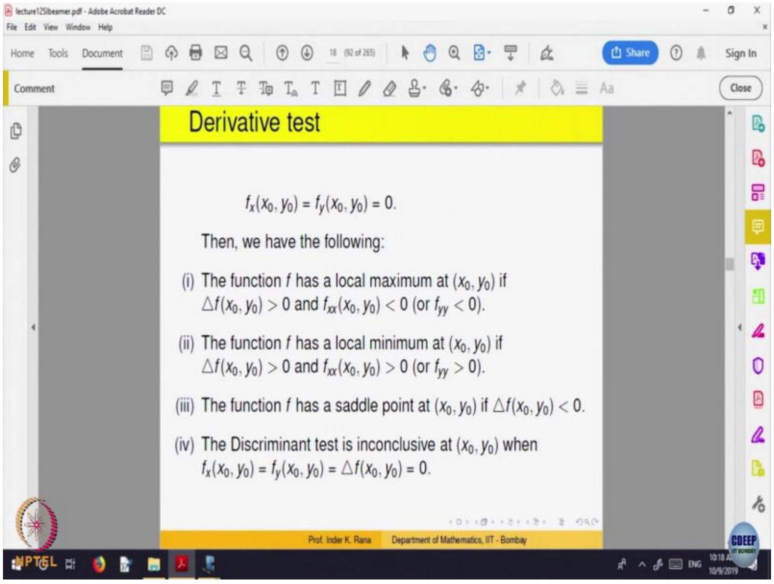
So, the definition once again saddle point P is belonging to the domain is a saddle point. If there is a point where the value is bigger than the value at that point and also there is a point where the value is smaller.

(Refer Slide Time: 08:43)



So, let us looking at the theorem and the consequences there is a discriminant we define. So, it is a 2 by 2 determinant f xx fxy fyx and fyy. So, they may not be 0 always but sometimes you are lucky and they are.

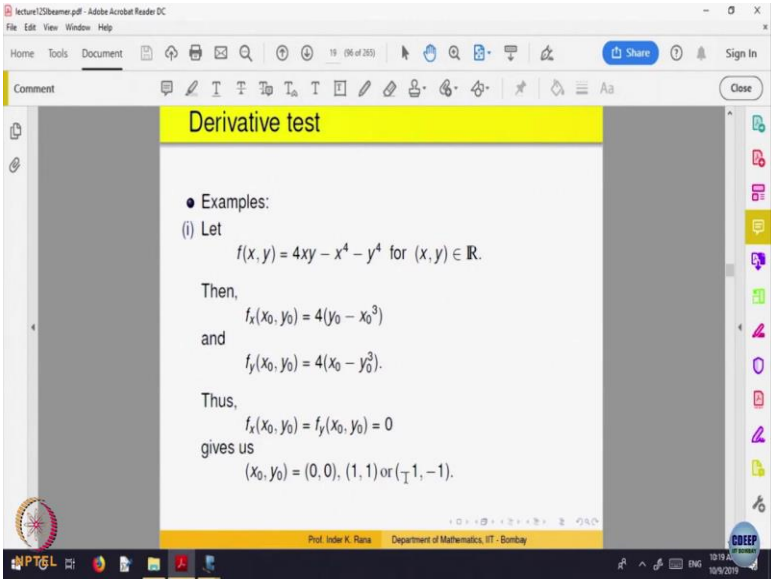
(Refer Slide Time: 08:56)



So, here is a test. If at a point first derivative of 0 that is necessary condition anyway point to be local maximum or minimum we are analyzing the critical points what we can say further, then if the discriminant is bigger than 0 and the second derivative xx is less than 0. Which will also imply actually that fyy either of them can be looked at then the function as a local maximum, and similarly the discriminant is bigger than 0 and the second derivative with respect to x is bigger than 0 both are bigger than 0 than this is a point of local minimum.

And if the discriminant is less than 0 than it is a saddle point equal to 0 you cannot conclude anything the test fails you can say, that is in inconclusive. So, let us look at some examples to illustrate this point.

(Refer Slide Time: 10:05)

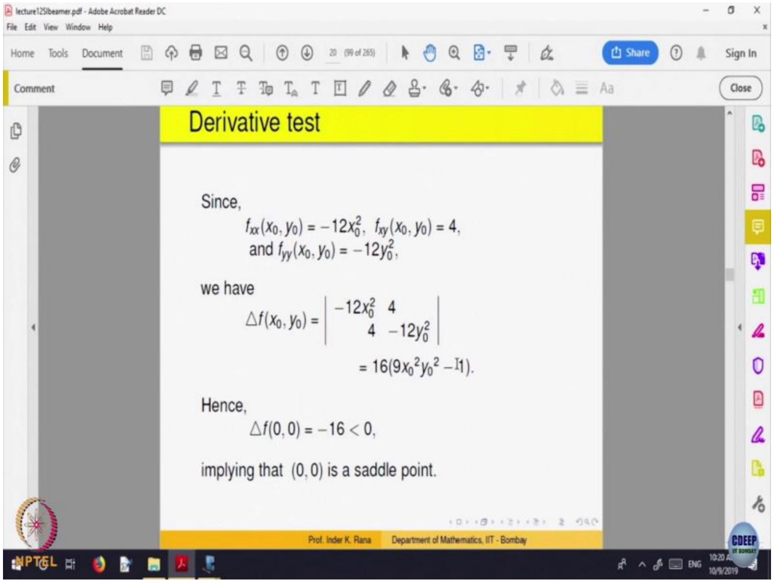


So, let us look at this f xy is equal to 4xy minus x4 minus y4. So, the function is continues everywhere it is differentiable everywhere, no problem it is a polynomial function of two variables. So, what are the partial derivatives? The partial derivative with respect x, so it will be 4y minus 4 x cube, so that is with respect to x and similarly, with respect to y re-symmetric so you will get the partial derivatives.

So, both equal to 0 you have to solve those two equations. So, when you want to solve these equations, this equals to 0 this equal to 0 simultaneously solve these equations. So, you will get points where the function can have local maximum, local minimum. So, this gives you three points. Namely we are possibly the function can have local maximum, so how are these three points obtained?

By solving these two equations, so we are not spending time on solving that you will have to do that. So, now the point is at 0, 0 at 1, 1 at minus 1 minus 1 whether the function has local maxima, local minima or saddle point or you cannot say anything directly you may have to look at something. So, let us try to see whether the test is applicable or not. So, let us compute the discriminant.

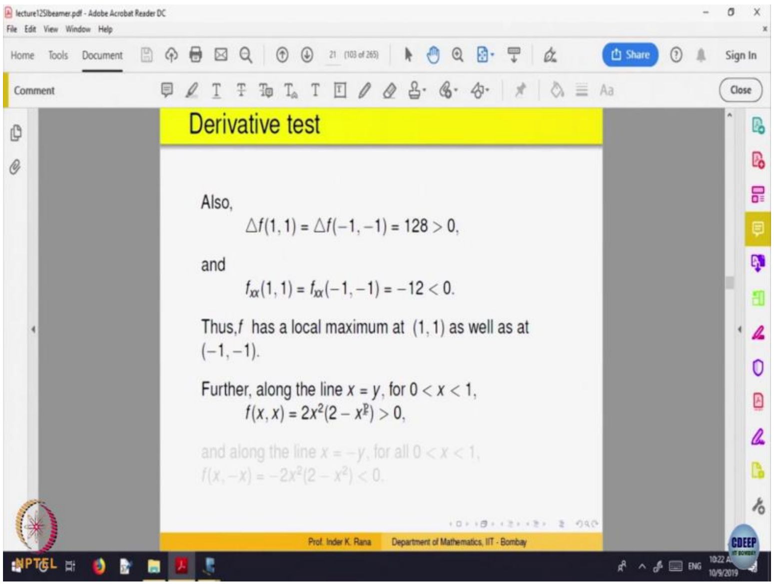
(Refer Slide Time: 11:39)



So, if you find out the second derivative the minus 12 x square f xy is 4. So, that gives me this is a second derivative, sorry this is the discriminant at any point. Now, we have to compute it at the point separately for 0, 0 1, 1 and minus 1,1. So, let us look at the point 0, 0 when x is 0, y is 0. So, these two terms are 0 so that give you a negative term. Minus, so that is negative minus 16. So, directly you can say 0, 0 is a saddle point.

Discriminant you compute, so this is the computational application of the 0. Similarly, you computed 1, 1 so put x is equal to 1 y equal to 1. So, what happens to this thing? x is equal to 1 y equal to 1 that is 9 minus, so that is bigger than 0. So, discriminant is bigger than 0.

(Refer Slide Time: 12:40)



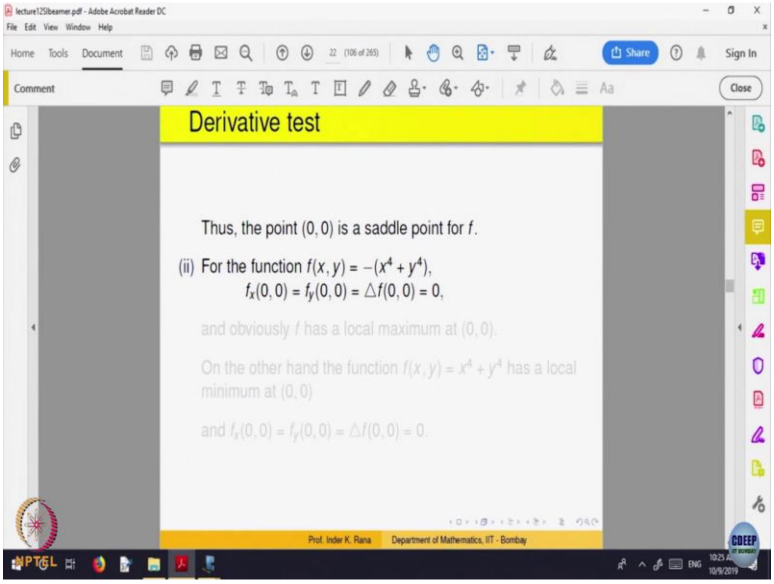
Now you have to look at f xx at that point, so you compute f xx at that point and that turns out to be minus 12 that is less than 0. So, what is the conclusion? Discriminant is bigger than 0, second derivative f xx is less than 0 this is a point have local maximum. So, this is a point of local maximum and similarly for the other one because it is product, so it will not change actually, the product will not change. Similarly, you can check that at the point minus 1 minus 1 also discriminant is bigger than 0 and f xx or f yy is also less than 0.

So, both the points are point of local maximum, so test applies to both of them. If you like just look at the point 0, 0 was a point of local minimum or no local maximum or no, it was saddle point according to the test. Let us just see whether we can avoid using the test. So, look at the line y equal to x that is passing through 0, 0 we are looking at the point 0, 0. So, look at the point passing through 0, 0, what is f xx? That is this quantity which is always bigger than 0, in a neighborhood of 0.

So, and similarly if you look at x minus x that is negative. So, along the line y equal to x the function remains positive. So, close to 0, 0 you can find points where the value of the function is positive at 0, 0 the value is 0. So, and along the line y equal to minus x close to 0, 0 you can find points the value is negative. So, by definition itself you can conclude that the point 0, 0 is a point of, is a saddle point for the function, you do not need to apply really the test.

But test gives a nice straight forward application but you can look at. So, how do you analyze something test may fail, so, then how do you analyze whether, is a local that is a saddle point or local, you may have to go to the definition straight away. See in a neighborhood how does the function behave.

(Refer Slide Time: 15:07)

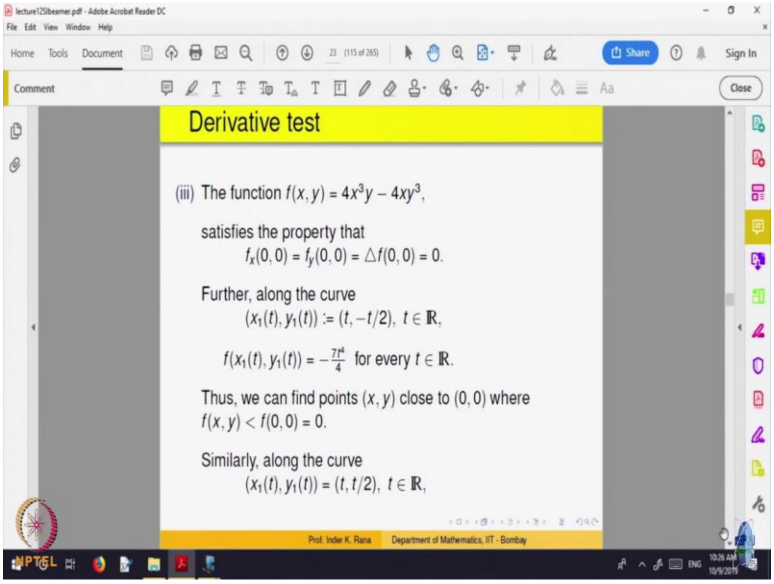


So, let us look at this is another simple example for, example if you look at minus x4 y4 we know that what does a graph look like x4 y4 so it is always positive, it will be like a cup, and the point 0, 0 will be a point of local it is a negative thing. So, point 0, 0 will be a point of absolute maximum actually not only, at every other point the value is negative but if you look at the discriminant at that point 0, 0, what is the discriminant?

First derivative will give you 4x cube, and you compute that second derivative also and compute the discriminant that comes out to be 0. So, test fails in this case the test fails discriminant is 0 and the function has local maximum at the point 0, 0. So, if a test fail does not mean neither is possible, anything is possible. Test fail means you cannot conclude anything while looking at the discriminant if it is 0.

So, this is an example of a function where the test fails but the function has local maximum at the point 0, 0. And if I invert that means if I take x4 y4 without the negative sign, then again the discriminant will still be 0. But the function will have a local minimum at that point. So, test may fail but anything is possible, so that is what this example is showing. Look at this example.

(Refer Slide Time: 16:50)



So, in this example again if you compute discriminant that will turn out to be equal to 0. So, as such test does not help you to do anything, it does not help you to compute a conclude anything for this you can try to maneuver, for example if I look at it is if I look at this line. So, y cube is there, if I look at the line x is equal to minus y, what will happen? I can analyze this function along a curve passing through 0, 0. So, for example I can look at the line where y is equal to minus x and along that line.

So, if you look at that, then what is the function look like, is negative y equal to minus, or look at say look at the value of the function x1 t y1 t along the curve t minus t, along that the function takes the value negative along some other curve passing through it takes the value positive y equal to x. So, that means along some curve it is a minimum along some other curve it is the maximum that is good enough to say that is the point of saddle point.

So, this is by analyzing the behavior of the function at points closed to the point where we are looking at along some curves but the curve should pass through that point that is important. We should not look at arbitrary curve, because you want look at every neighborhood of that point, should have a point where the value is bigger and some other point where the value is smaller. So, you should have a curve passing through that. So, these are the way you analyze that 0, 0 is a saddle point.