Basic Real Analysis.
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Lecture 45
Riemann Integral and Riemann Integration – Part 3.

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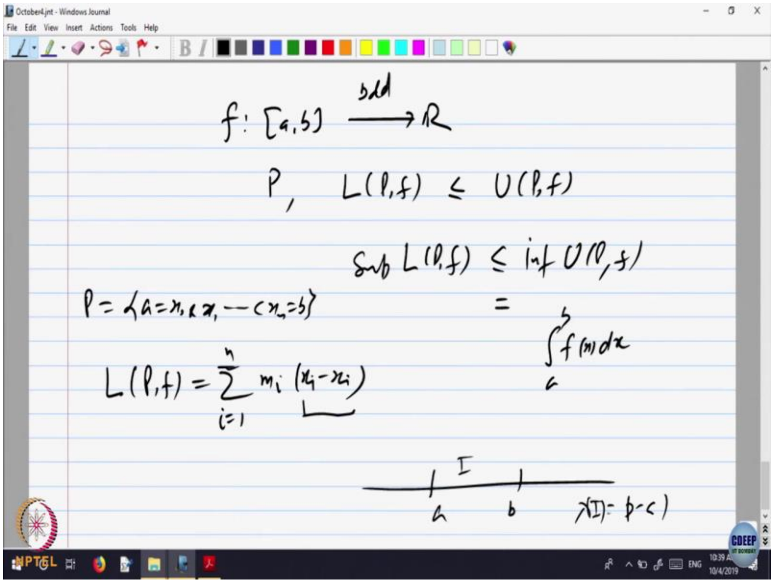

There is another way of extending the integral. So, let us look at that what is that way. So, Riemann Integral on a, b to R bounded. So, we define the with respect to a partition P the lower sum and the upper sums, they were all less than or equal to this and has the, you took the supremum. So, this was you look at the supremum of L P, f is less than or equal to infimum of U P, f and whenever these two were equal that was called the integral of a to b fx dx.

But, the important thing in this was the in the upper sum or the lower sum, say for example in the lower sum, if the partition P was x 0 less then x 1 less than xn equal to b, then you look at the lowest value you are approximating by looking at the lowest height of the function in the length of the interval. So, the idea was that you look at the length of that sub interval. At some time point somebody thought, so what we are doing is you have given a interval a to b on the line and you are trying to measure the length of that interval.

You measure the length by looking at saying that this interval b minus a. So, length is b minus a. So, that is a length of this interval this is I but I think probably that maybe a nice idea to introduce.

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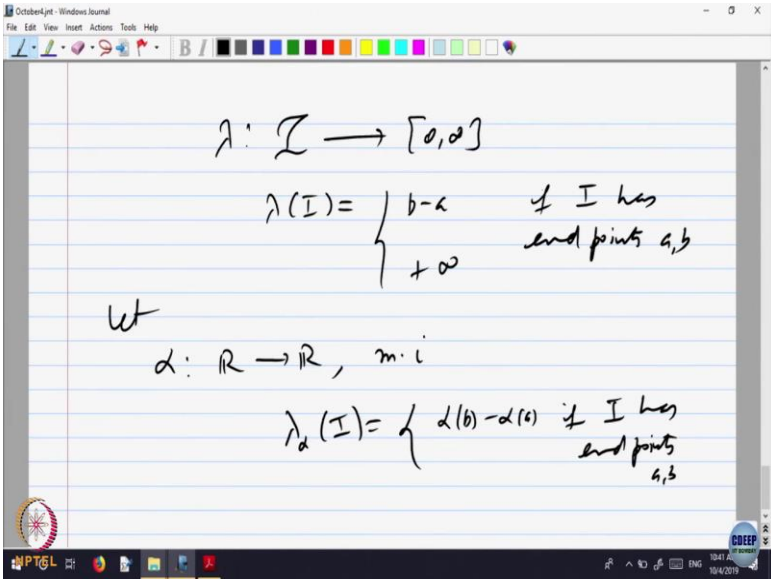

So, length on the class of all intervals taking value 0 to infinity. So, length of interval I is defined as b minus a, if I has end points a and b otherwise, we write as plus infinity if it is n bounded interval the length is defined as plus infinity. But this is something I can change its notion of length. So, what I am here, when the end point is a and b I am saying it is b minus a but let us take alpha a function on real line to real line which is monotonically increasing.

So, let us take a function then, what is wrong in saying I define the length of interval I to be equal to alpha b minus alpha a if I has end points a, b. What is wrong in saying that they have the length to be equal to alpha b minus alpha a. That means what? At every point I am giving the weightage different, different weightage to every point, where the earlier the length was b minus a, then now the length is alpha b minus and if alpha a is equal to identity function, then it original length if alpha is identity function then it is original length.

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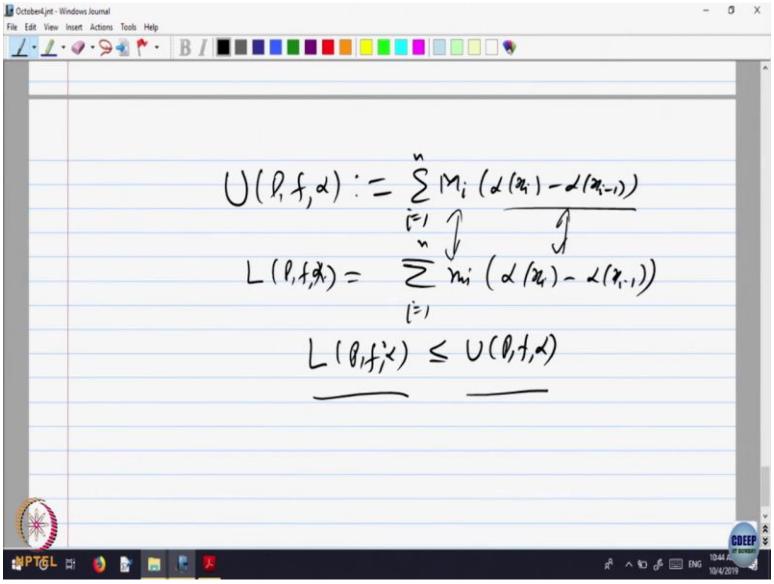

So, the idea came that when defining that upper sum and the lower sums U P, f, let us define the upper sum with respect to a new length function alpha. So, what is it going to be the definition? It is the maximum value of the function as it is, the height remains the same but the base length changes to alpha xi minus alpha of xi minus 1 and summation i equal to n. Change the length of intervals and similarly, the lower sum with respect to this alpha I can define it as sigma mi alpha xi minus alpha of xi minus 1.

Still the same property remains true L P, f alpha is still less than U P, f alpha because these two quantities are same the weightage of the length is same, there is small mi and capital Mi. So, that still remains the same. So, still I can ask can I what will happen if refine a partition? Because if you keep in mind the in the lower sums and the upper sums when you refine a partition, length does not play any part it is under supremum or the infimum that change.

So, same properties of lower sums and upper sums will remain true, whenever you measure the length in terms of a monotonically increasing right continues function monotonically increasing function. So, for this, so you can look at what is the supremum of the lower sum, what is the infimum of the upper sums, and whenever they are equal you will get a new integral which is defined with respect to a weighted measurement of length on the line.

So, this is possible and this is what is call Riemann Stieltjes Integral. So, let me so let me cause not many changes come accept for a few places, otherwise the whole theory goes as smoothly as the earlier one. So, that is what we I am yeah I think.

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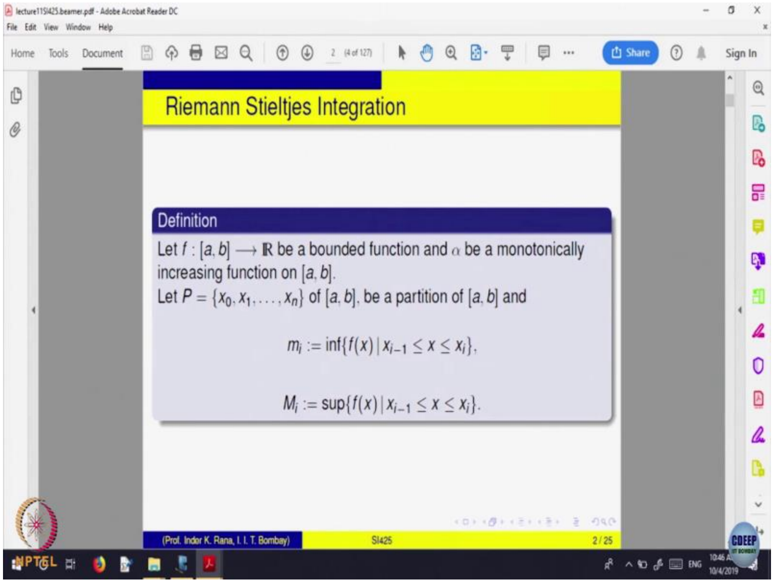

So, let us start with a function a b to R and alpha is a monotonically increasing function on a b. Given a partition, look at the infimum look at the supremum like we do it for upper sum and lower sums.

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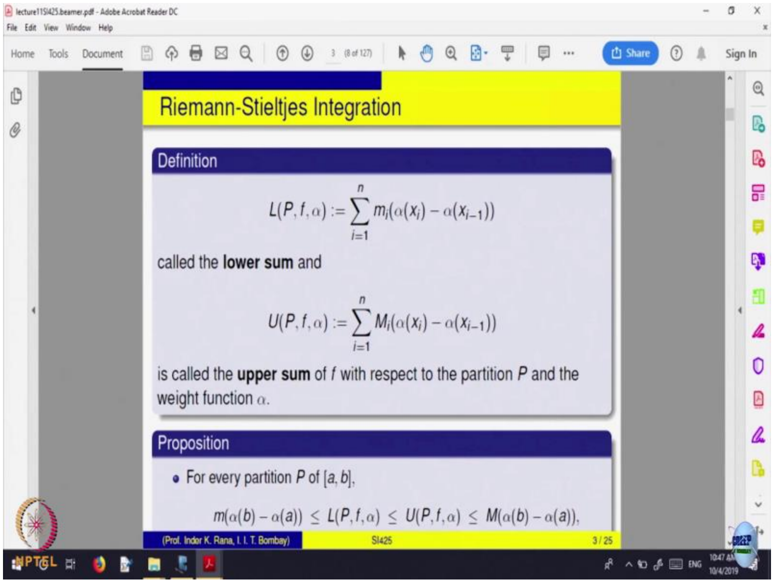

Now, how is the new upper sum defined, how is the new lower sum minimum value into alpha xi minus alpha xi minus 1. So, lower sum similarly the upper sum with respect to the, is a weight you are attaching to each. See why this is important in probability and statistics you assigned different weights to different points. So, you will probability distributions which come via such kind of functions monotonic functions and I expected value of the functions will be with respect to the weighted measurement of length of that interval.

So, that is why this is important from statistics point of view, and mathematics point of view it is a generalization of the Riemann integral. So, upper sum and lower sum so you prove those property that the upper, lower sum is always less than or equal to upper sum, that is not a difficult job there is a same thing, same proof continues.

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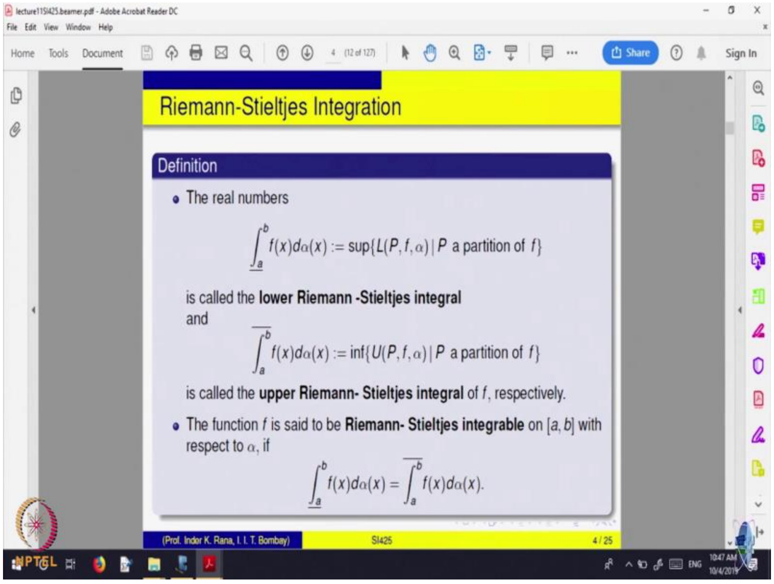

You can define upper Riemann Stieltjes integral lower C1 as a supremum and the infimum of the upper and you say the function is Riemann Stieltjes integral. If the supremum of the lower sums with respect to alpha is equal to the infimum of the upper sums with respect to alpha. All the everything goes on smoothly. So, you say that is the integral and common values is called the integral.

Probably all the theorems go as in Riemann Integral, except probably some points I will point out where the things go a bit.

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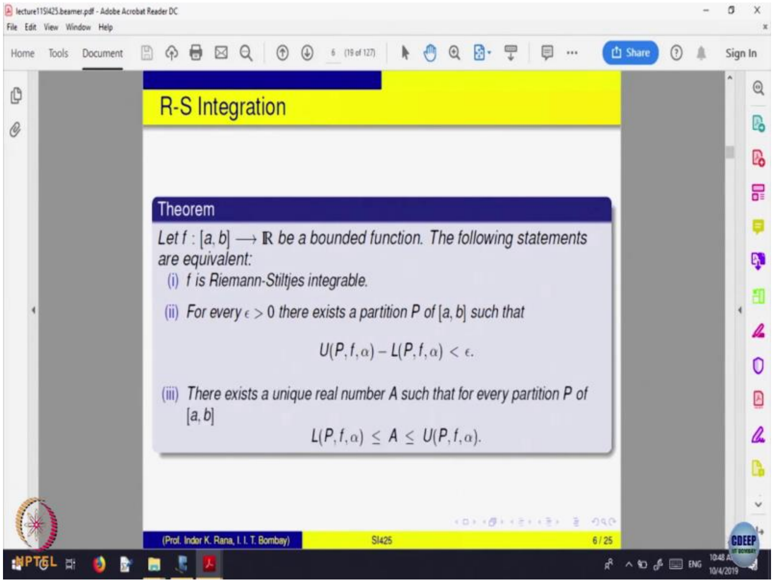

So, for example upper is always lower than the bigger than lower and the function is integrable whenever the difference can be made small, because that is only thing that is make a difference alpha length of does not contribute anything actually. So, all those theorems, same proof go over only with the difference in the proofs will be that instead of writing xi minus xi minus 1 you will be writing alpha xi minus alpha xi minus alpha xi minus 1, that is all nothing more, so all this proofs go.

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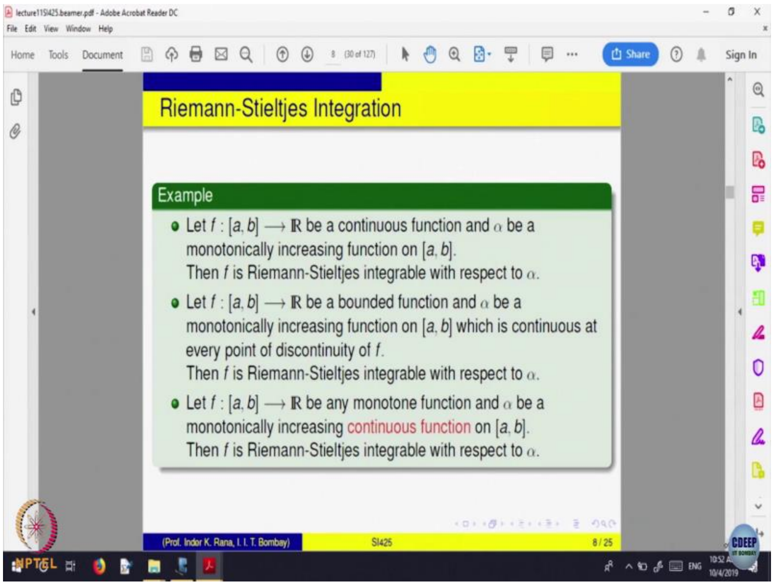

So, here is something if every continuous function is Riemann Stieltjes integral like in Riemann Integral, we proved that if f is continues on a b then it is integrable. So, what was the proof basically? The proof basically was that, because f is defined on a close boundary interval it is continues or it is uniformly continues. So, given epsilon there is a delta such that whenever two points are closed by distance delta their distance f of xi the values are also closed.

So, now what are the upper sum and lower sums? For a partition you look at the points upper sum minus the lower sum, capital Mi minus small mi, what is that? If a function is continuous capital Mi is attend at some points small mi is attend at some point, and if that means in the interval xi minus 1 to xi capital Mi is f of something, in that interval small mi is f of something.

So, if your length of the partition is less than delta, then their values will be less than epsilon because of uniform continuity. So, given f is uniformly f is continues by uniform continuity given epsilon choose a delta choose a partition whose is norm is less than delta. So, you will get capital Mi minus small mi will be less than epsilon into the length, so that will be small. So, that was a proof.

So, in the same proof, if f is continues uniform continuity everything makes sense. So, what we will be doing instead of multiplying by xi minus 1 to xi will be multiplied by alpha times xi minus alpha time xi minus 1, same proof no change will come. So, every continues function will also be Riemann Stieltjes integrable with respect to alpha for the same proof if f is continues. No change comes.

The change comes supposing f is not continues it is only a bounded function, and alpha is monotonically increasing we know that for a bounded function Riemann Integral will not exist. So, the modified theorem for Riemann Stieltjes Integral is, if that alpha function, alpha which is monotonically increasing is continuous every discontinuity point of f, then it becomes Riemann Stieltjes integrable.

So, the discontinuity of f is taken care by the continuity of the point, continuity of the monotonically increasing function (())(13:44) monotonically increasing, so it is discontinues at the most at countably many points, we know that. So, the only change comes is, it is continues at every point of discontinuity of f then. So, that is where any bounded for Riemann integrable function, we said that every monotonic function is Riemann Integrable.

For Riemann Stieltje you will require f to be, alpha to be a continues, not only monotonically increasing you require also it to be a continuous function. We will not prove all these theorems but I am just pointing out the differences is between the two statements of the theorems. So, that is properties of Riemann Stieltjes.

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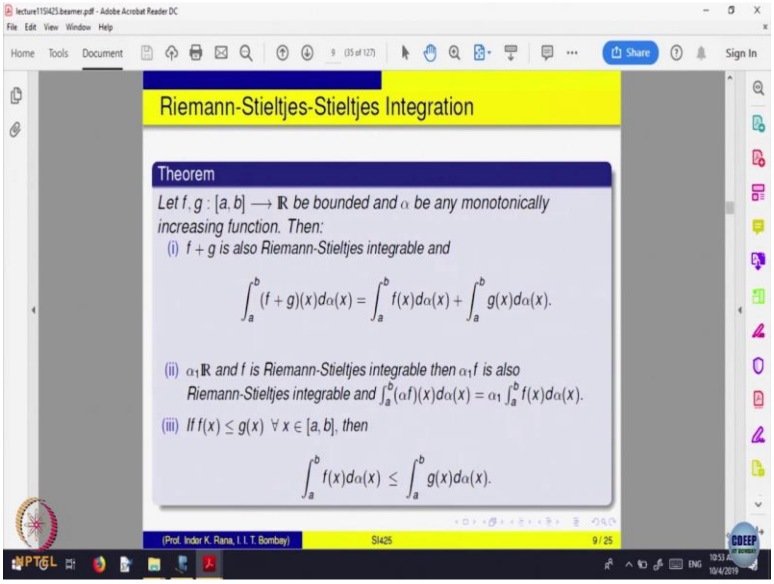

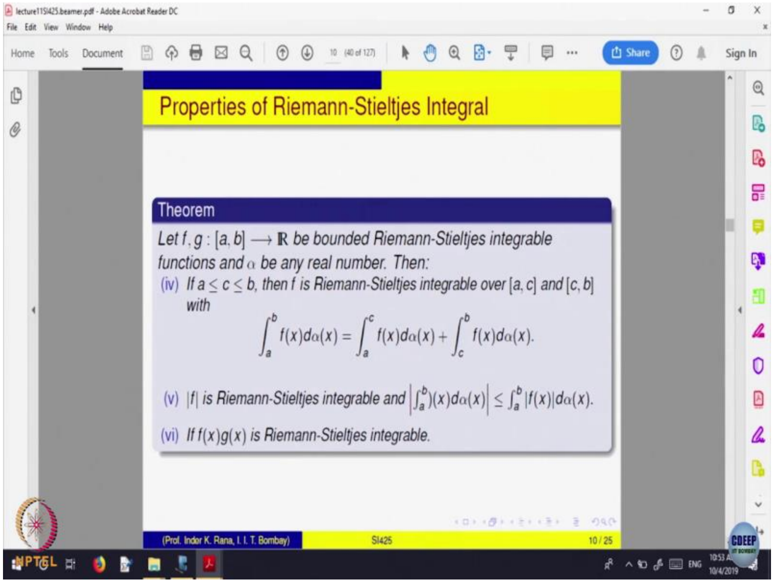

Other facts they all continue to hold if f and g are even Stieltjes integrable f plus g is even Stieltjes integrable, because the sums will split anyway into two parts. So, length does not change those property. So, Riemann Integral of f plus g Riemann integral of f plus Riemann integral of g same property holds. Scalar multiple integrable comes out f less than g than same property holds.

So, similar properties hold not much change comes. So, that is why not proofs are not very interesting for all this. Suppose we will not any way go into the proofs of these things. So, the basic fact is that, you can extend Riemann Integral in two different ways, one look at function being unbounded on a bounded domain or function being bounded on a unbounded interval that gives you one set which is called improper integration or you can keep the function values as it is, but change the measurement of length by some monotonically increasing function.

That gives you Riemann Stieltjes Integration and, the two have different that so the two branches which are, I think I do not, probably I should just say that this Riemann Stieltjes Integral plays a important part in further topics like measure theory and probability and statistics. It will come again and again there, and of course improper integral comes in a disguised form of gamma functions and gamma function and Cauchy distribution and so on.

Or normal distributions and so on. So, it will require those thing to exist. So, that is integration on of one variable. So, next time we will look at integration of function of two variables that is also important from many subjects point of view. So, we will look at integration functions of several variables next time.