Basic Real Analysis
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Lecture 44
Riemann Integral and Riemann Integration Part II

So, this says that what is the advantage of this kind of a theorem?

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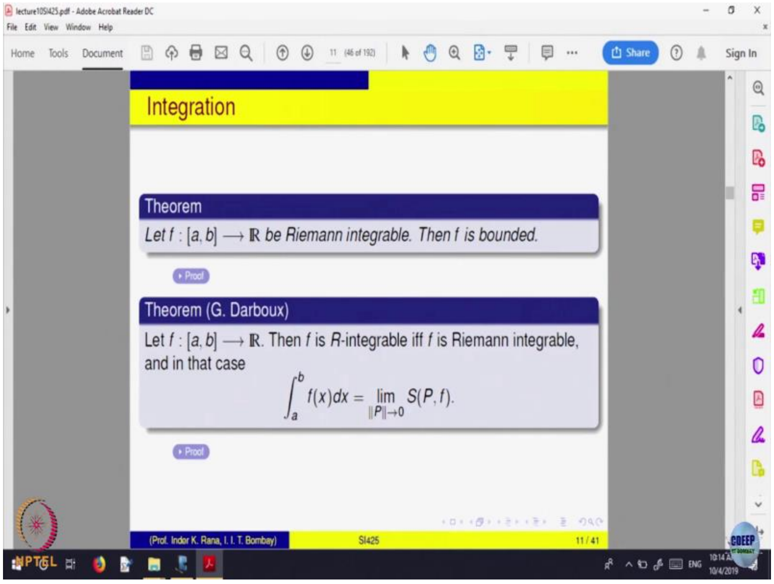

See this advantage of interpreting Riemann integral as limit is that. Now, if you want to look at the algebra of integrable functions, f is integrable g integrable, then we want to look at whether f plus g is integrable or not. If you want to go by upper and lower suns, then you have to relate the upper sums of f plus g with the upper sum of f and with that of, g that becomes a bit difficult one can do that.

But nowadays being the limit, if I take f plus g so, what will be integral of f plus g the limit of integrals limit SP f plus g, but the limits splits, limit of f plus g is equal to limit of f plus limit of g. So, that is the advantage that sometimes this way of proving that is integrable is useful to get the results.

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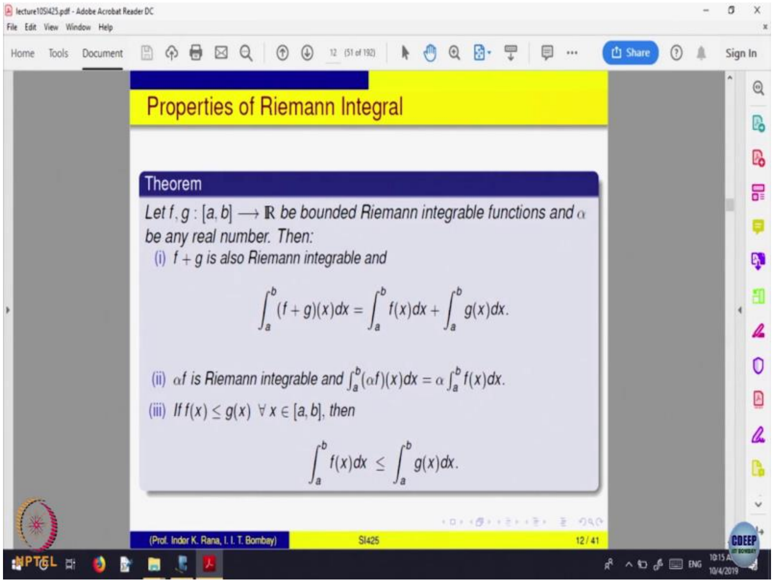

So, one proves the theorem like this if f and g are integrable than f plus g is integrable and integral of f plus g is integral f plus integral g. So, this is via the limit operations because the left hand side will be a limit of norm p going to 0 and that splits into limit of SPf respect to f, SPf, because what is Riemann sum? f plus g at some point ti into the length, so that is splits into two parts.

So, limit of sum is equal to sum of the limits. So, using that one proves all these results as a consequence of the limiting operation that if f and g are integrable, then alpha times f is also integrable and alpha comes out, because in the limit, limit of alpha time something is alpha times the limit and similarly f and g. So, if you look at SPf with respect to f will be less than or equal to the Riemann sum with respect to g because f is less than g, the value at a point of ti of f will be less than. So, using that criteria of integrability one proves these things

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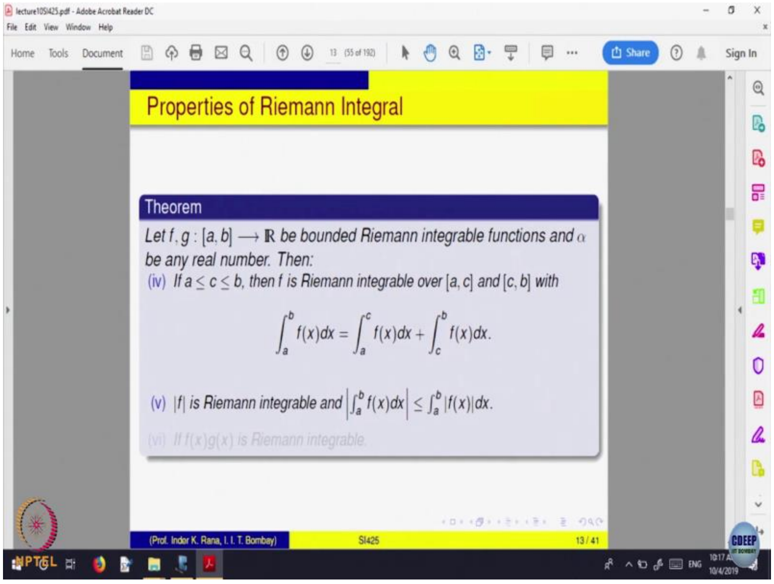

And this is also not difficult at all saying that if f is integrable a to b, then it is also integrable between a and c in between c is point in between plus the integral between c to b because a partition of the whole interval can be put it as a partition of a to c and c to b introducing the points c in between.

And this is because for in f which is integrable, integral may be negative. So, absolute value of the integral is less than or equal to integral of the absolute value. Again when you look at the limits, SPf, you look at the look at the absolute value of the Riemann sums. So, that would be less than or equal to mod f at into the length. So, this becomes obvious in that case.

So, using that the very useful result in proving integrability and not only that, that frees the notion of integrability from function being bounded, historically it is of great importance because that gave a lot of interest in looking at what is called Fourier series problem or in probability and statistics we will find what is a characteristic function of a distribution coming and looking at there, Fourier series problems. So, this is integration.

So, what we are done is we have given a function f on interval a b to R. We defined the notion of geometrically the notion of area below the graph of the function and that we interpret it as via lower sums upper sums or via the Riemann sums. There are some situations where you can extend this notion of integral see f on a b to R, f is bounded, the domain of the function is a bounded interval.

You can extend this notion when either the domain is not a bounded interval or the function is not bounded on the bounded interval. So, one can define the notion of integral. So, that is called the one way extension of Riemann integral and that is called the improper integration.

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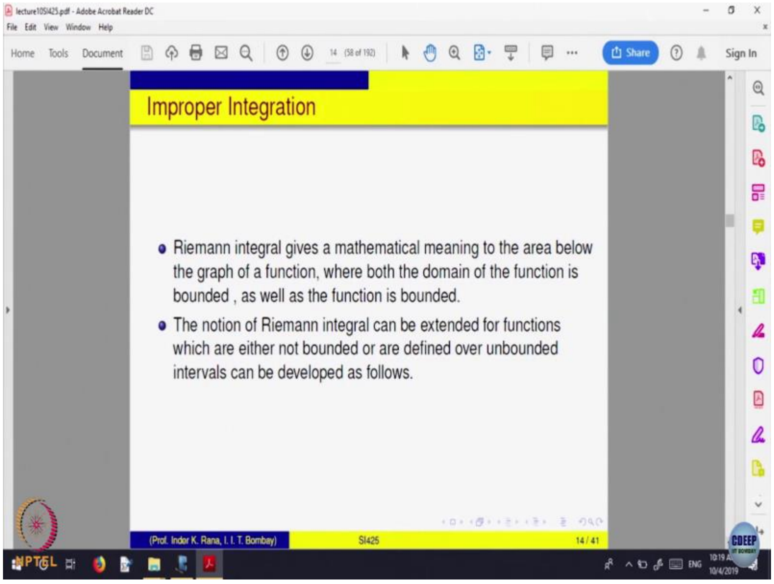

So, one looks at the function defined either on a interval which is not bounded right but the function is bounded or the function is defined on the interval a b which is bounded but the function is not bounded itself on that interval right. So, these two situations can be handled in some cases. So let us look at some example.

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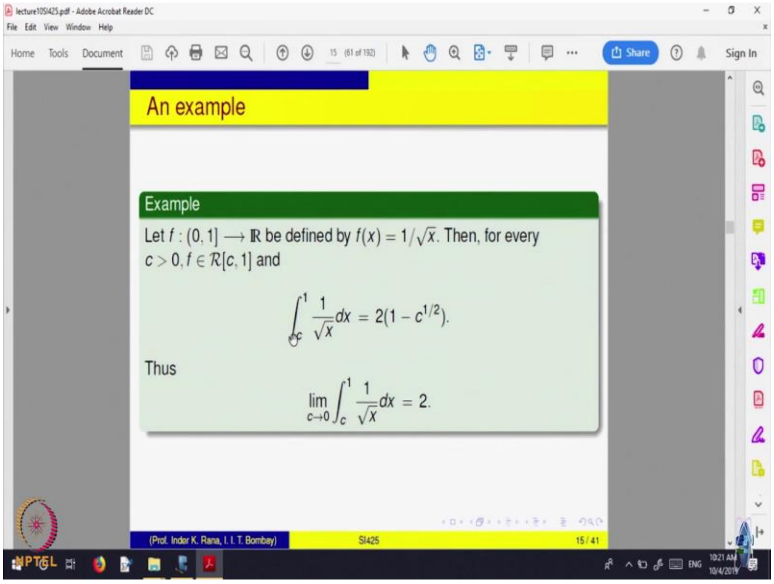

For example, look at the function defined on the bounded interval zero to one. Okay, open at 0 close at 1, f of 1 over square root x. Okay. So at every point that bigger than zero this function is defined, okay. And what is the integral, integral of 1 over square root x from any point c, if we take a point c bigger than 0, so that is equal to 1 over x raised to power minus 1 by 2 so, that is the integral.

Now, in this, the interesting happens if you let us c, go to 0, c is bigger than 0. If I let come c closer to 0, this integral has a limit. When c goes to 0, this goes to 2. So, what we are saying is, even though the function is becoming larger and larger as you come closer to 0, still, I can think of saying what is the area of this 1 over square root x on the interval 0 to 1, area below the graph of the function.

Though it is becoming very large near 0. So, this is a situation where we can define, so this limit will be called as the integral of 1 over square root x in the interval 0 to 1. So here, this is a situation where the function is becoming bigger and bigger in a bounded interval. Let us look at another example.

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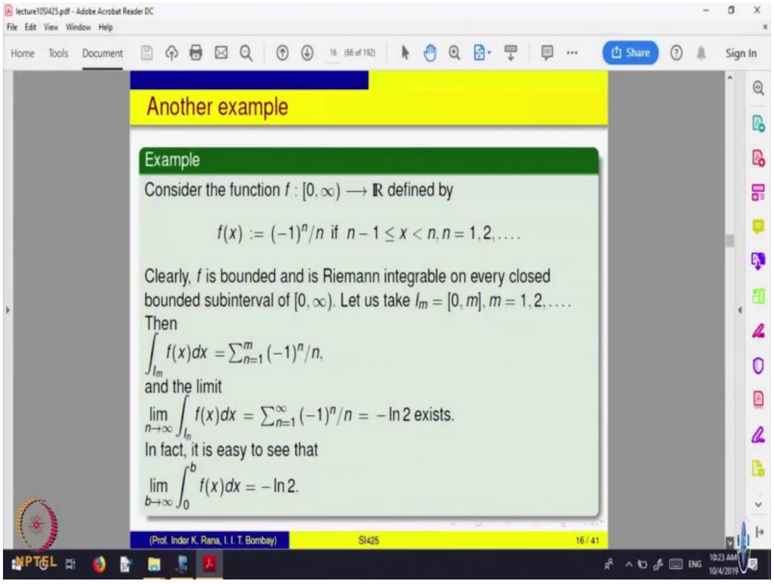

Look at this function, f of x is minus 1 to the power n divided by n and the function is defined on a in finite interval 0 to infinity, x is positive. If x lies between n minus 1, to n, then the function is defined this way. So, we would like to know, Can we say something like the function has some integral 0 to infinity? unbounded interval. So, what we do we have only define the notion of function integral when the function is bounded over a bounded domain.

So, let us look at the integral of this from 0 to some point then it becomes a bounded interval. So, let us choose a point say Im which is 0 to m and look at the integral of this function in this interval. So, it is a continuous function piecewise continuous function in that interval. So, what will be the integral? This is defined as constant function in this interval.

So, what will be the integral? the value of the function into the length of the interval summation. Now this as m, m is the interval Im 0 to m. Now let us let m go to infinity, then what happens to this series minus 1 to the power n divided by n? Have you come across series in your courses, so this is a convergent series, this is a alternating series, actually. So, this is a convergent series, we will do it again also later on, and its sum is equal to minus ln of 2, ln is a log function.

So, though the function is defined over the whole interval 0 to infinity which is unbounded, but in every bounded part it is integral is defined as we stretch that interval to infinity, the limit exist in the earlier case, it was c 0 to 1 and c was being pulled to 0. So in all these cases, they are the function was unbounded. Here the function is bounded. But the domain is unbounded interval. So such things are called improper integrals.

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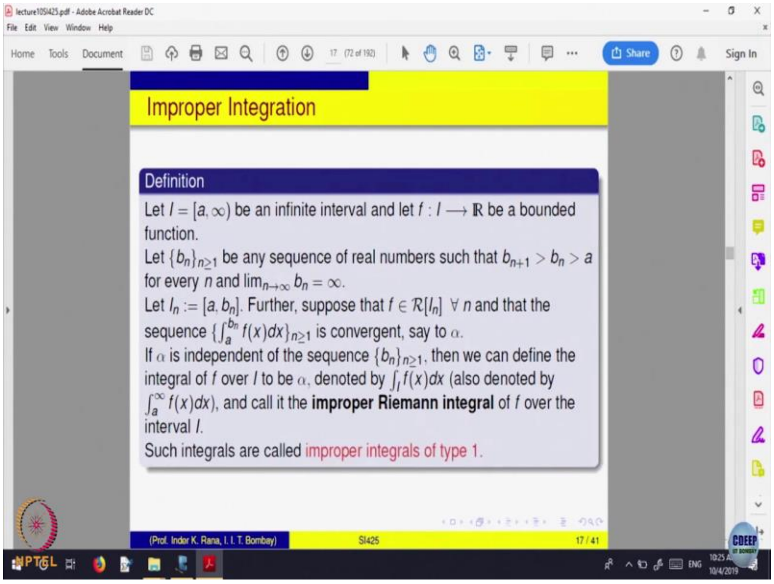

So, well let us make a definition let us say I is interval a to infinity and f is a bounded function on that, the interval is 1 to infinity, but the function is bounded. So, if I take part of a to something, it will be a bounded function on a bounded interval and suppose that integral exists and as we take the limit of that point going to infinity that also exists, then we can say f is integrable on the interval a to infinity.

So, let us define that, so take a sequence bn of numbers which is increasing to infinity and In is a to bn and suppose f is integrable R is the symbol Riemann integrable on the interval In that is a to bn. If that exists take the limit of this and suppose that limit is equal to some number is convergent to alpha and that number alpha is independent of the way you go to infinity.

It should not depend upon that, then you say that the function has a integral and that integral is called improper integral of f over the interval I. These are called here the domain of the function is unbounded. So, these are called improper integrals of type 1 where the domain is unbounded but the function is bounded, the other one which we saw earlier where the domain is bounded but the function became unbounded and still the integral existed.

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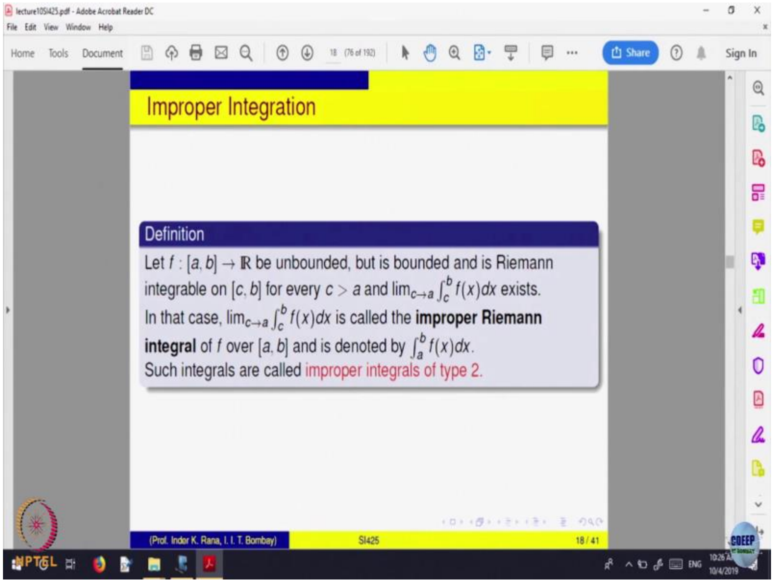

So, that was the situation a to b from c to b the function is integrable and limit c going to a exists then we say this is improper integrable function of type 2. So, domain is bounded function is unbounded in the other one function is bounded but the domain is unbounded. These kind of situations arise for functions which are both important in mathematics probability and statistics. Okay, the improper integrals. So, we will give some more examples of this.

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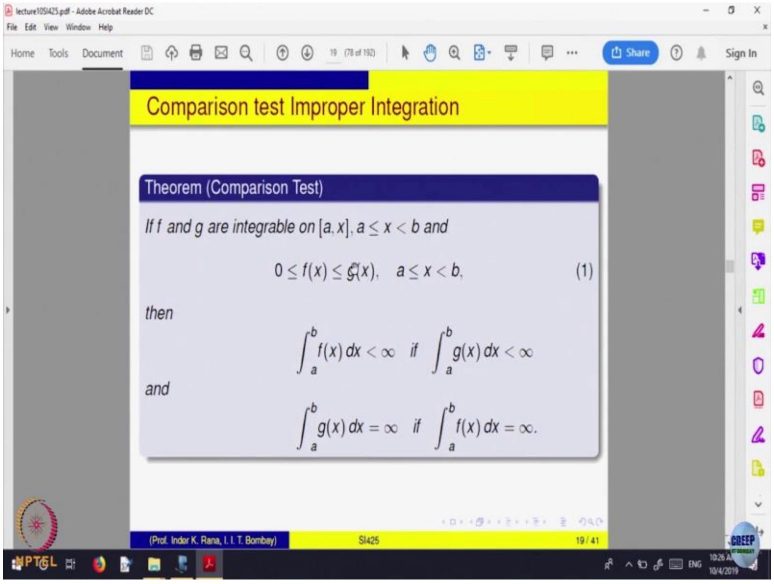

One way of checking is called comparative comparison test, which says if how to check whether some integral will exist or not. So, if f is less than or equal to g, and if so saying that the integral a to b is finite. That is saying that the improper integral exists then. So, if integral of the dominated function improper integral exists then the of the function which is being dominated that also exist and if this is infinity.

If this is infinity then g being bigger that will be also that means, when the improper integral exists you say it is less than infinity, when it does not exist you say it is equal to infinity or one uses the word convergent and divergent improper integral is convergent, that meaning that integral limit exists divergent is other way of saying. Here because which zero so, that is another way of writing.

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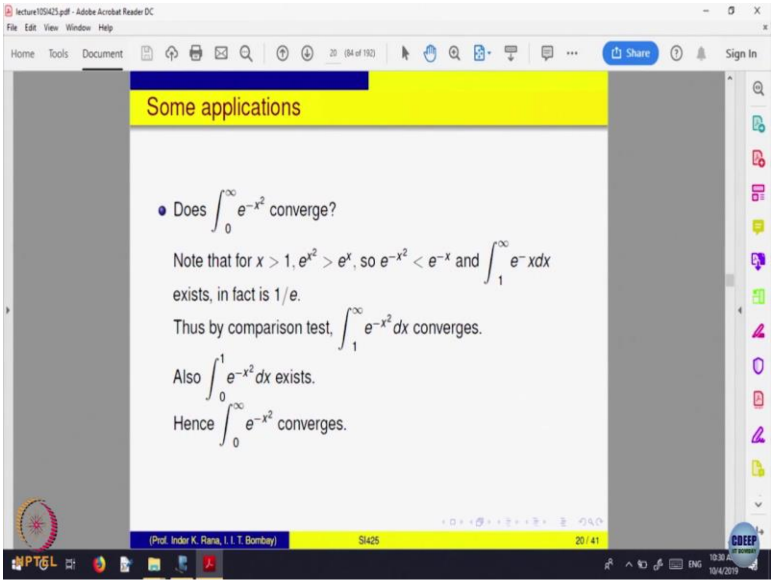

For example, look at 0 to infinity e raised power minus x square. Here the function e raised power minus x square, the interval 0 to infinity that is unbounded. So, let us try to analyze this in two parts, we will see e raised power minus x square behaves differently between 0 and infinity. So, let us look at if x is bigger than 1 in this, then what is the inequality e raised power x square is bigger than e raised power x and so, e raise power minus x square because the function is e raise power minus x square is less than e raise power minus x.

So, this is less than right and look at the integral 1 to infinity of e raised power minus x what is that integral? 1 to infinity that will look at the integral 1 to some finite quantity of e raised power minus x and limit of that, as that point goes to infinity. So, what will be that integral right? So, that goes to infinity that other part will go to 0. So, this limit is equal to 1 over e, integral of e raised power minus x from 1 to some point c, exponential derivatives itself integral is itself with a negative sign so negative.

So, this integral exists. So that means e raise power minus x square from 1 to infinity will also exist by comparison theorem, because this integral is finite. So, by comparison test, the integral e raised power minus x square will also exist and 0 to 1 e raised power minus x square. Does that exists? domain is bounded, 0 to 1 e raised power minus x square what is happening to the function it is continuous function. So, integral exists we know that integral that is the ordinary integral.

So, the integral 0 to infinity will exist and this is something similar to what is called normal distribution, normal density function that will come in statistics. So, that is improper integrals. So, that is one of the uses of improper integrals.

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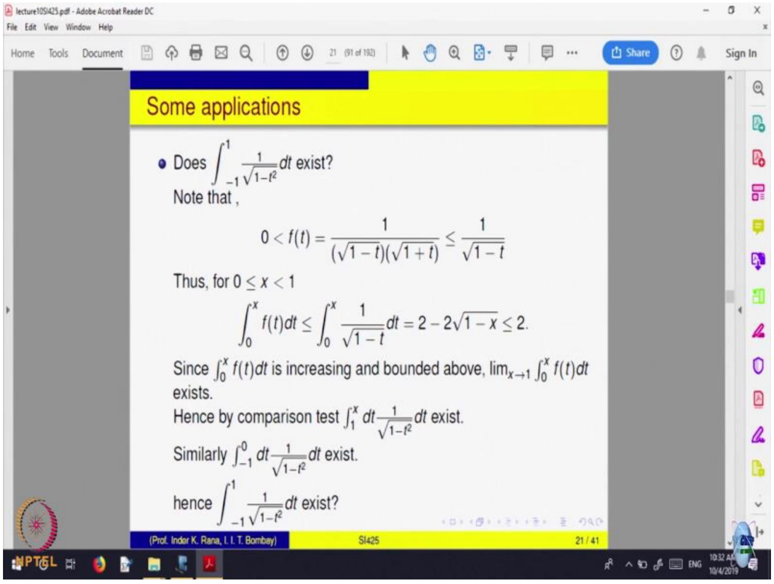

Here is another application which we will not go in much into. I want to look at the integral I think I pointed out earlier, look at the integral of 1 over 1 minus t square dt minus 1 to 1. So, let us try to split 1 minus t square factorize that is 1 minute t and 1 plus t and the 1 minus t into 1 plus t, t is between minus 1 to 1.

So, what happens to 1 plus t, there is always 1 over 1 minus 1 plus t that is square root. So that will be, so this quantity is less than 1 over 1 minus t. So, if between 0 and 1, look at the integral, integral 0 to x f t, because this is less than this quantity, so it is less than or equal to this quantity 0 to x.

So, I am trying to look at the integral from 0 to x of 1 over 1 minus, I am not going to 1, because if I try to go to 1, the function becomes unbounded 1 over 1 minus t, 1 over 1 minus t square root as t goes to 1 it is becoming unbounded function near 1. So, I should avoid 1. So, to look at 0 to x from 0 to 1 only that integral exists and that always remains less than or equal to 2, because 2 minus something.

So, that means what 0 to x f is non negative. So, this is a increasing function bounded above. So, limit of this 0 to x, x goes to 1 will exist, are you following? Because 0 to x non negative quantity the interval is increasing as the non negative function integral will be non negative.

So, these quantities are non negative increasing bounded by 2, so limit of this will exist so, this limit exists.

Because this limit exists, so similarly with minus 1 to 1 also the limit exist. So, you can say that this integral exists now, the integral is unbounded does a function is unbounded near the value 1.

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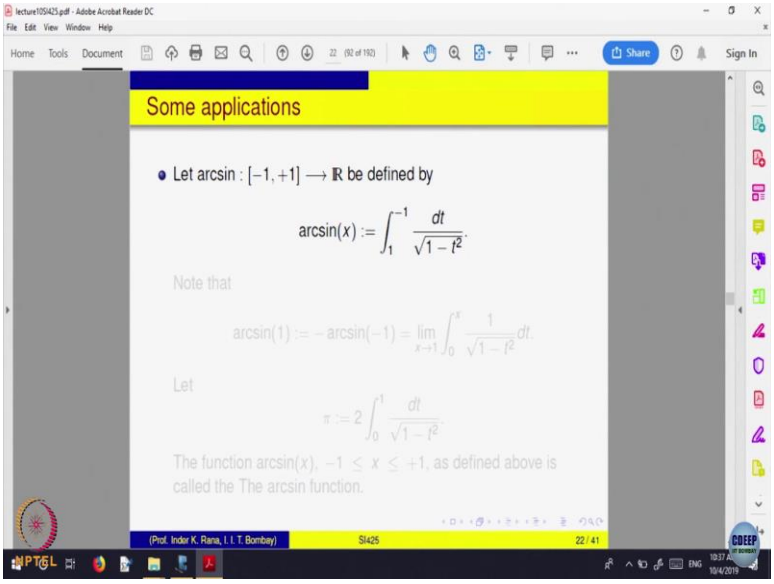

Now, because of this here is one application of this, look at the integral, this integral just know we said exists minus 1 to 1 of 1 over 1 minus t square look at 1 over 1 minus t square that is a derivative of what? 1 over 1 minus x square, square root, it is a derivative of sine inverse function.

So, if I integrate the derivative, if I able to integrate I should get back the function, but here the integration is coming via improper integral, when you are away from minus 1 or 1 you are applying fundamental theorem of calculus and getting value arcsine x and at the end point is the limit. So, that is also giving you the continuity of the sine universe function because the way it is defined.

So, one I will not will not go into this just for the sake of exposure saying that the improper integral can be used. In fact, here we are saying that we define sine inverse as improper integral of its derivative and once sine inverse is defined, this is defined from minus 1 to 1. So, what will be the value of sine inverse at the point minus 1? What do you think should be the value? that is

Student: (())(20:47)

Professor: Right, now as such may not defined Pi by Pi or anything. So, we are defining a sine inverse function between minus 1 to 1 taking values in R via integration, one proves it is continuous differentiable and all those properties it is a continuous function, it is a one-one function it is because what will be a derivative of this 1 over 1 minus t square positive? It will be monotonically increasing function derivative is nowhere 0 it will be 1 to 1 on to function between minus 1 to 1 to R. what will be the range?

It is a continuous function on minus 1 to 1 a interval range also should be a closed bounded interval and so, it should have a left hand pointer, it should have a right hand point that left hand point is called minus Pi by 2 and the right hand point will be plus Pi by 2, because you can see from here it is what is a odd function.

So, this is the way of defining sine inverse getting what is minus Pi by 2 what is Pi by 2 and then you inward that function you get sine function between minus Pi by 2 to Pi by 2 to minus 1 to 1 and if you look at the graph of the sine function, because sine inverse is 1-1 on to right that also is a 1-1 on to function between minus Pi by 2 to pPi by 2 and then you extend it periodically everywhere.

So that is a way of defining trigonometric functions and also in between you defined what is Pi. We defined Pi also via sequences. Look at the area of the circle. So, that was the beginning of our story of saying that a monotonically increasing sequence which is bounded above must converge. So, that the area of the inscribed and (())(23:06) was monotonically increasing and bounded and similarly outside was decreasing.

So, this is a way of defining, way of extending Riemann integral when either that domain is not bounded or the function is not bounded. There as many other functions with in applied mathematics also it comes from the something called gamma functions, if you have heard about those things there all improper integrals, so I will not go much into it, because we just want to give you an exposure of something called improper integrals as and when it comes to one or some of the courses will study more of them. So, this is one way of extending your integral.