Basic Real Analysis
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Lecture 43
Riemann Integral and Riemann Integration Part I

So, let us just recall we had started looking at alternate way of describing integration. So, we define integral via upper sums and lower sums. So, here is an alternative, alternative way of doing that.

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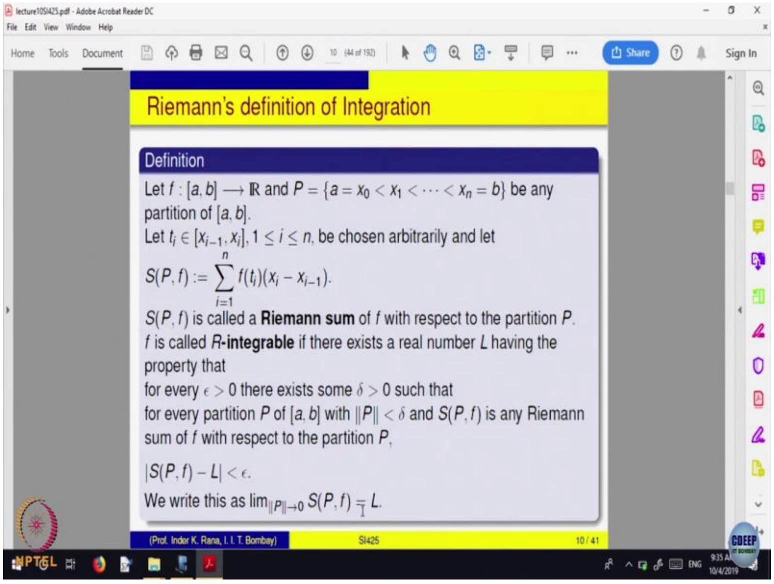

So, given a function on a b to R and a partition, one defines what is called sum SPf. So, in the sub interval x i minus 1 to x i choose any point ti normally called a tag but, so look at the value of the function at that point to the length of the interval, so that is a area of the rectangle with this height. So, that is approximate, approximation to the area of the variable of the curve.

So, this is called a Riemann sum of F with respect to the partition P and one says f is Riemann integral, integrable if the limit of this as a norm of the partition becomes 0 exists and that means that given epsilon should be bigger than 0, there is a delta such that whenever the norm of the partition is small less than delta, the difference between this sum and the number L is less than epsilon. So, one writes this as that limit of the sums is equal to the number L and that L is called Riemann integral of the function or we call this for the time being we call it R integral of f.

So, note that the function f is not assumed to be bounded in this to start with, because we were just looking at the value of the function at a point in between in the interval x i minus 1 to x i. Whereas in the upper and lower sums, you need to have the function to be bounded to start with.

So, what we want to prove is that if a function is Riemann integrable, R integrable in this sense, then it is also integrable, via the upper sums and the lower sums and the two integrals are equal. So, to prove that, we need to first show that if a function is integrable in this sense, then it is also a bounded function.

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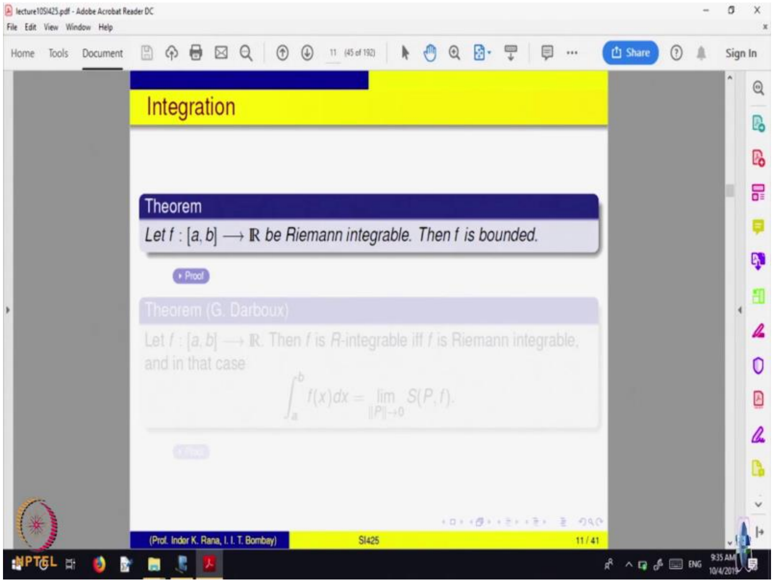

So, that was a theorem we were proving that if f is Riemann integrable, then f is bounded. So, we are almost completed the proof of the theorem except for the last step. So, let me just go through the proof again.

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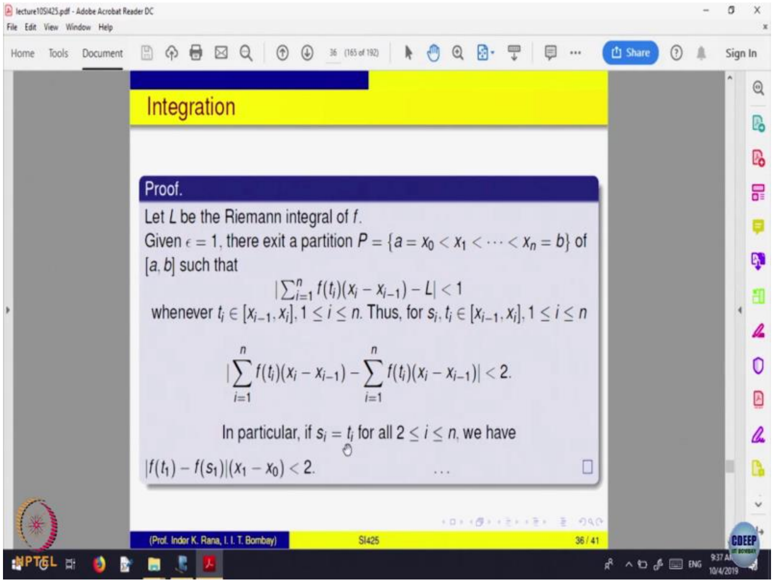

So, since f is Riemann integrable by the definition of integrability, given any number say epsilon equal to 1 there is a partition, such that the sum SPf minus the number L is less than epsilon, the limit of SPf is equal to L that exists. So, that means given any epsilon there is a partitions such that they are close.

So, we can take epsilon equal to 1, so that means this sum is less than 1. So, let us take two points Si and ti and look at th sum with respect t choice of Si and sum with respect to the choice of ti then this SPf with respect to the choice is less than, this is with respect to ti and there is a typo here, one should have a Si here, with respect to the choice Si then with respect to ti, each one of them is closer to L by 1. So, the difference between the two is less than between the two of them is less than 2.

So, in particular, if I choose Si to be equal to ti, then everything will be 0 except the first one. So, let us choose that that is what we had done last time. Choose Si equal to ti for all sub intervals except the first one. So, that all the remaining sums will be equal to 0 and you will get f of t1 minus f of S1 is less than 2 and from here, this is what every S1 and t1 in the first interval so we can fix t1.

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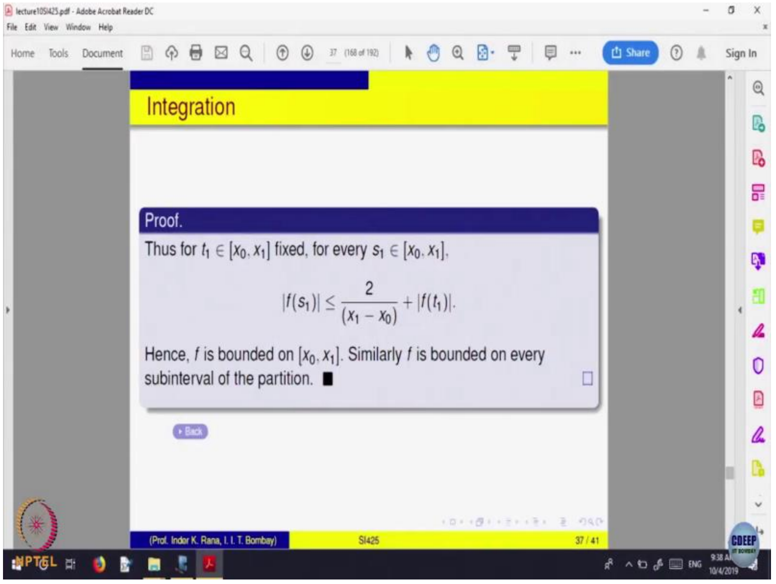

And that means f of S1 is less than 2 over this plus f of t 1. So, t1 is fixed, so the right hand side is a constant. So, that proves that up to here we had done it. So, just fixed now t1 and this gives you that for every S1 in the interval x 0 to x 1 this is bounded. So, it is bounded in the first interval and a same proof works for bounded in every other sub interval because I can choose Si equals ti in the second interval and so on. So, that proves that if a function is Riemann integral, Riemann integrable in the sense of what we have done just now. It is also bounded function.

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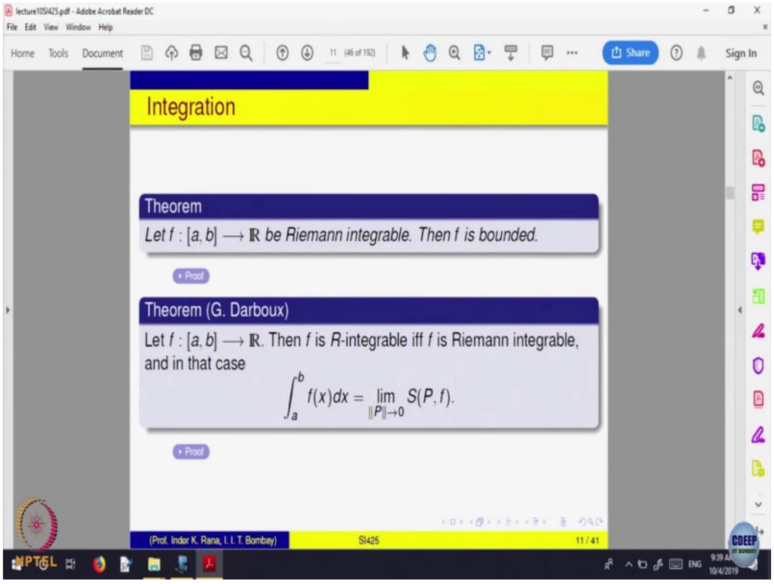

So, now, since it is bounded, one can define what is upper and lower sum so, the we want to prove this statement that if f is R integrable, then R is also Riemann integrable and the two integrals are same that means the integral where the upper and lower sums is same as the integral via this method and both the integrals are same. So, let us give write a proof of that.

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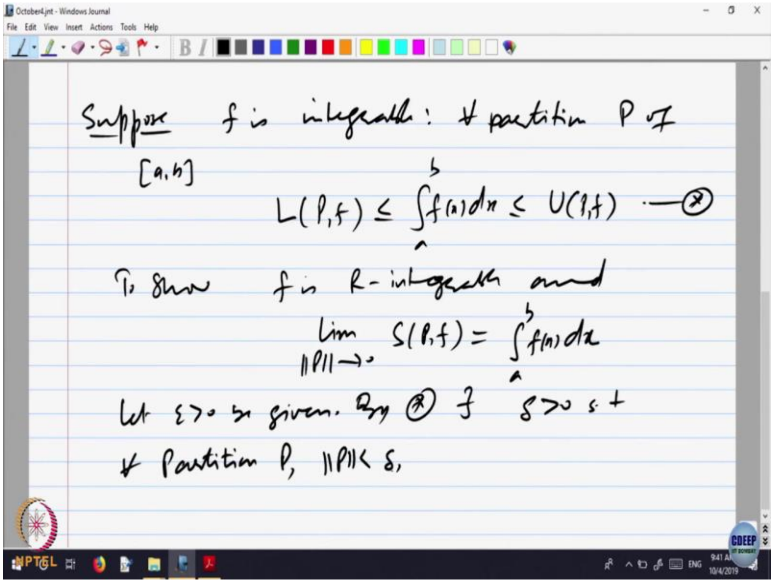

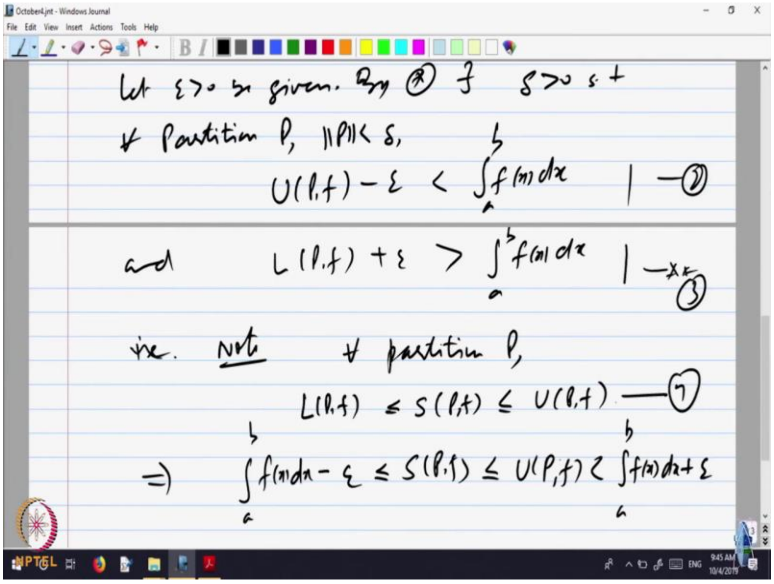

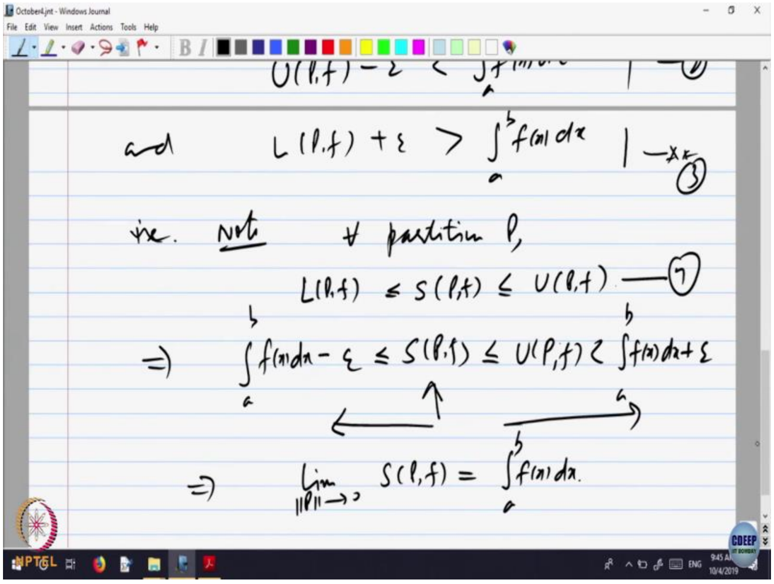

So let us first suppose f is integrable. So, what does that mean? That means that via upper and lower sums, okay. So, integrable, that means for every partition P of a b. If I look at the lower sums, that is less than or equal to the integral a to b f x dx is less than the upper sum and what do you want to show? So, we are given that is integrable with respect to this, to show that f is R integrable and limit norm P going to 0 of SPf is equal to this a to b fx dx.

So, to show this, so let us take let epsilon greater than 0 be given, we want to show that this limit is equal to this, so then by this integrability condition. So, by star there exists some delta bigger than 0 such that for every partition P with norm of P less than delta. See this integral is the least upper bound of LPF and greatest lower bound of the lower sums.

So, given epsilon we can find a partition such that upper sum minus epsilon this cannot be the right. So, there is a partition such that this happens and this is less than integral a to b fx dx is that okay, because this integral is the upper bound of U, UPfs. So this, we should say, this is the lower bound to this minus, so we should say, we are looking at this as, I just, integral of this is the greatest lower bound of UPf. So UPf minus epsilon cannot be, so this is okay.

And similarly, and LPF plus epsilon is bigger than integral a to b fx dx. So, that is now note one thing. So, let us note one thing that for every partition P if I look at SPf, what is SPf? That is the value of that sum at when you choose any point c i in that sub interval. So, this is always bigger than LPf and always less than UPf, because this is a value at a point in the interval x i minus 1 to x i.

LPf is the lowest value you choose and that is a upper you choose the largest value. So, now let us combine this with this, so implies SPf is bigger than or equal to LPf and LPf is bigger than integral minus epsilon from this, this star. So, this is SPf is bigger than LPf and LPf is bigger than integral a to b fx dx plus epsilon. So, we will call it 2, call it as 3 and call it as 4 from 4 and 3 this happens and if I will use second, then upper sum is less than this, this is minus epsilon, this is minus epsilon and upper sum will be less than this integral plus epsilon so UPF, which is less than integral a to b fx dx plus epsilon.

So, that means what that means SPf on the left, it is less than this on the right is less than this same quantity minus epsilon and plus epsilon. So, that implies that this is for a given epsilon we have found a partitions that means limit P going to 0 of SPf is equal to integral a to b fx dx is that clear.

So, what we are done is given, that the function is integrable, we have the integral exists a to b and that is the supremum of LPf and infimum of UPf and we want to show that this value this also the limit of SPf, the norm P goes to 0. So, to show that we had to show that given epsilon there is a delta such that whenever norm of P is less than delta this quantity SPf is close to this integral between and the most the epsilon distance between them.

So, now we start with using from arbitrary we have go to the upper and lower by looking at the definitions, so given epsilon find a partition P such that the upper sum minus epsilon is less than this integral and similarly, the lower sum plus epsilon is bigger than that quantity. So, that is why the integrability by upper and lower sums.

And, now we observe that Riemann sums they are with you pick up points ti in a interval x i minus 1 to xi and LPf is with respect to the lowest value of the function in that interval, this is upper sum with respect to the maximum value. So, that is always true. So, now combine this to get the required thing. So, that says that if f is integrable, then it is also Riemann integrable and the two integrals are same. So, let us prove the converse.

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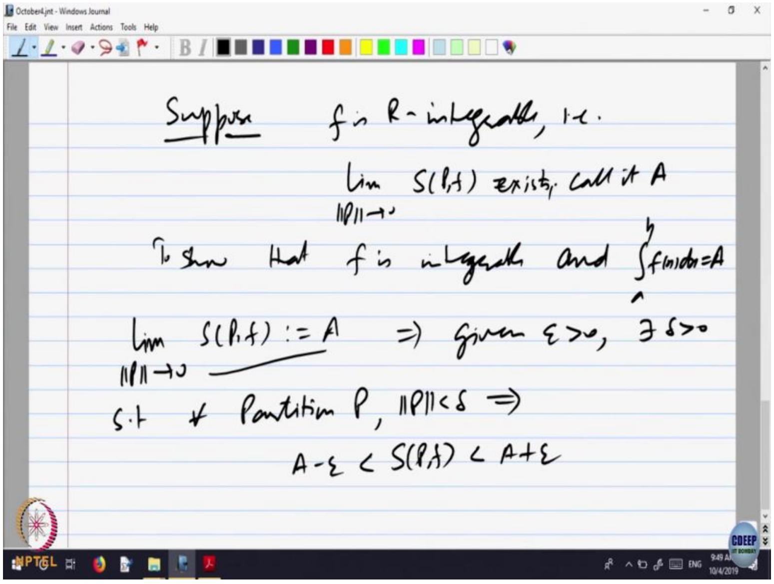

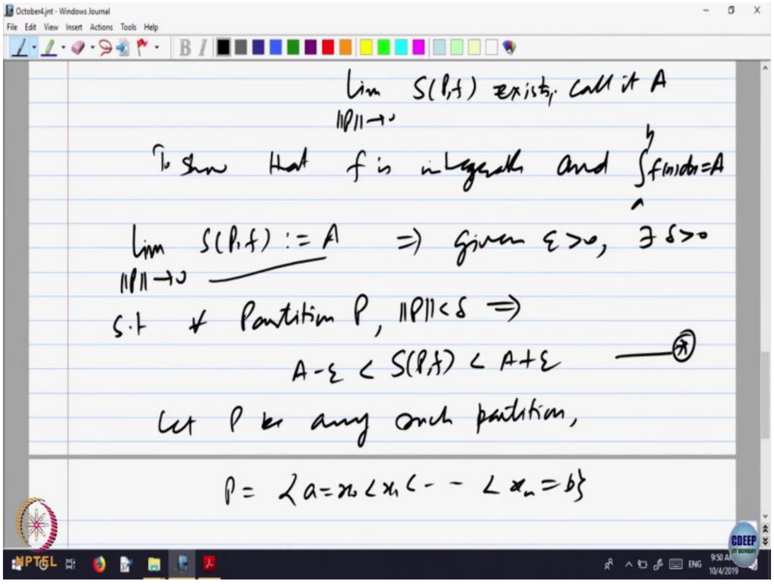

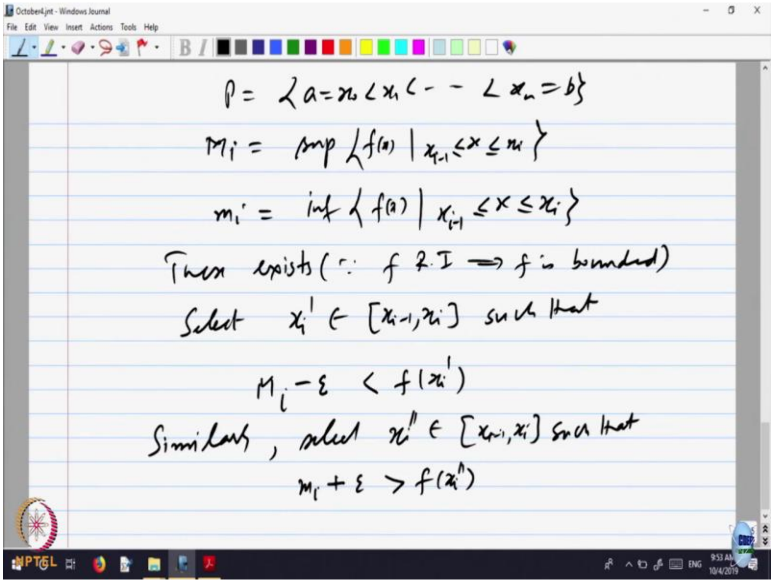

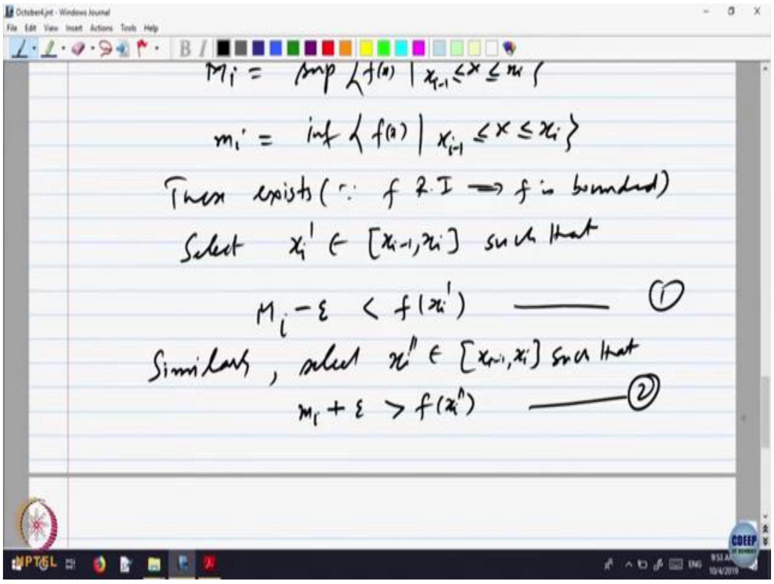

So, what will the converse be suppose f is R integrable, Riemann integrable, so that means, limit of norm P going to 0 of SPf every Riemann sum exists. Let us say call it a number A, so this limit exists and we will let us call it as A to show that f is integrable, f is integrable and the integral a to b fx dx is equal to this number A. So, when you want to show f is integrable what does that mean? That means, we should show that this number A lies between upper and the lower sums for every partition.

So, to do that, so let us start with because this is given to exist. So, limit norm P going to 0 SPf that quantity we are called is A, implies given epsilon greater than 0 there is a delta such that for every partition P norm of P less than delta implies the sum SPf is close to this limit is equal to A. So, it is close to A, so let us say it is less than A minus epsilon and A plus epsilon.

So, that is the meaning of the limit is equal to A. Now from this so let us call this star we had to go to upper and lower sums, to show that the function is integrable in the other sense.

So, let us side this partition, so let P be any such partition, for P is well as give some name a equal to x 0 less than x 1 x and x n equal to b. As before let us call capital M i equal to supremum of fx x in the interval x i minus 1 to x i and small mi equal to the infimum in the same interval, we have to go to upper and lower sums, so we have to write down all this quantities and these exists why do they exist?

Because function is given to be Riemann integrable. So, this limit exists, so by the earlier theorem it is a bounded function because f Riemann integrable implies f is bounded just now, we proved that so, it is a bounded functions of these quantities will exist.

And f is a function on the interval a b to R. So f is a b to R this is a partition. Now, we are not given f is continuous we do not know that, otherwise things would have been very easy because this Mi would have been attained at some point, we are not given to be continuous. So, we can choose select given that Mi is the supremum select some point x i dash belonging to x i minus 1 to x i such that this capital M i minus epsilon cannot be that supremum, Mi is a supremum.

So, something smaller cannot be the supremum that means, there must exist some point x i dash, such that this is less than f of x i dash, this cannot be, so there must be point on the right hand side and similarly, select some point x i double dash belonging to the same interval x i minus 1, to xi such that we have to go to upper and lower.

So, that is why we are trying to manually maneuver all these things, such that small m i plus epsilon that cannot be that right on the, so mi is the infimum. So, this plus something cannot be the infimum. So, that will this must be bigger than f of x i double dash. Now, let us form Riemann sums with respect to this choice xi dash and xi double dash so, let's call this as 1 call this as 2.

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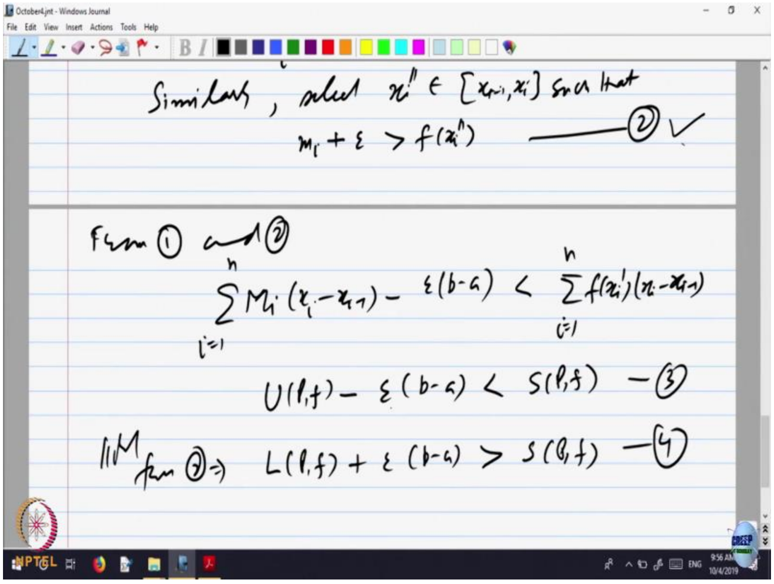

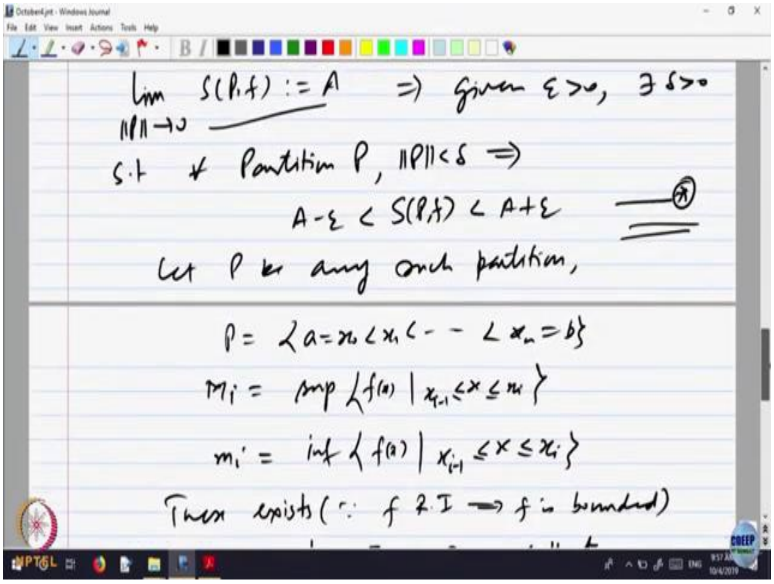

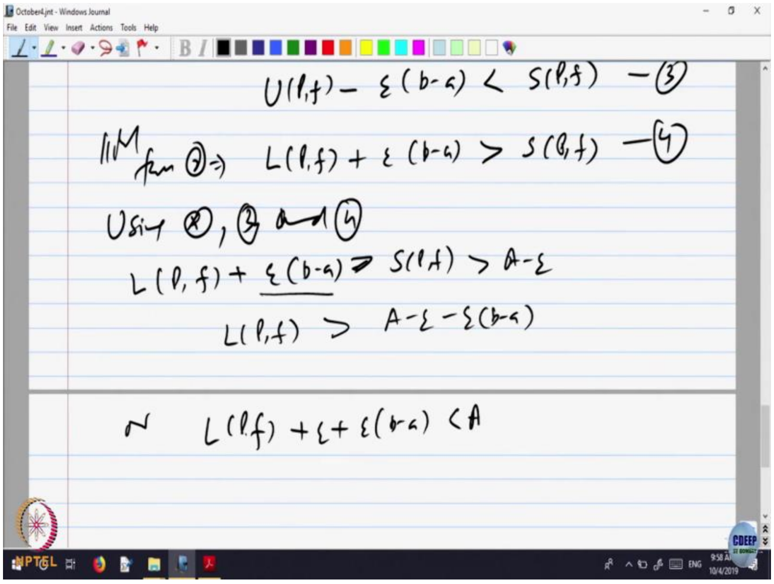

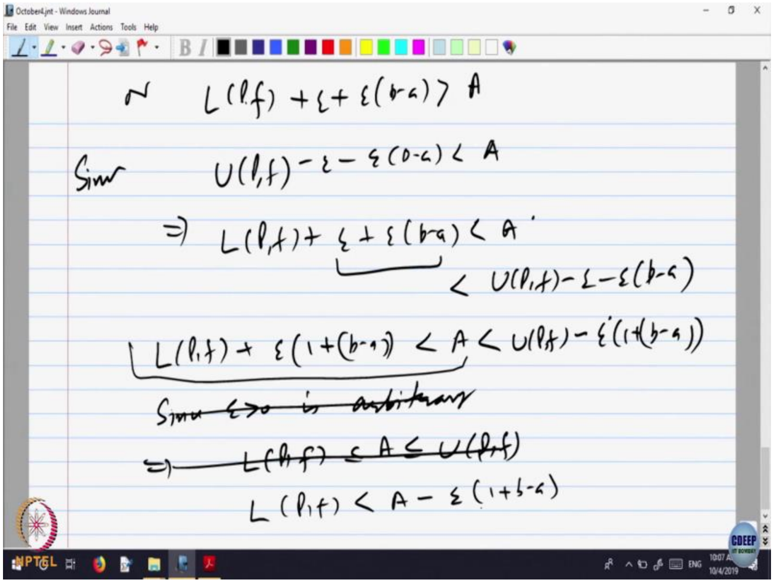

So, from 1 and 2, how do I go to upper sum, so, I want mi times the length of the interval, so that is x i minus xi minus 1 summation i equal to 1 to and that will be the upper sum. So, here I multiply both sides by this number and try to add.

So, what will get? I will get minus epsilon into b minus a is less than so, sigma f of xi dash into xi minus xi minus 1 i equal to 1 to n. I have multiplied equation 1 both sides by xi minus 1 xi and taken the summation, here when you multiply things will cancel out and you will only get x epsilon times b minus a, this is a upper sum. So, UPf this is a upper sum Mi times minus epsilon b minus a is less than SPf.

These are even sum with the selection of the point xi dash in that interval similarly, from 2 you will get if I multiply this equation 2 by xi minus 1 to xi both sides and add what will that equation give me, that will give me the lower sum with respect to the partition P plus epsilon times b minus a is bigger than SPf, multiply by the length of the sub intervals in equation 2 and sum it up.

So, similarly from 2, we will imply this thing, so let us call it as 3 and call it as 4. So, now, let us look, upper sum minus some quantity is less than SPf lower sum is that property and here is SPF lies between this and this. So, let us combine these star 3 and 4. So, using star 3 and 4, what will I get? So, 4 says LPf plus epsilon times b minus a is bigger than, so let us, is plus is bigger than SPf and what was SPf there in star SPF was bigger than A minus epsilon, was bigger than A minus epsilon. So, using star, that means LPF is bigger than A minus epsilon minus epsilon time b minus a.

So, this term I take it on the other side is less than or if you like even just if you want LPF on this side LP f plus epsilon, plus epsilon b minus a is less than A. So, rewrite take everything on one other side that is a better one. So, this was using star and using 3 so, if I use, 4 if I use 3 what I will get!? So using 3 now. So, it will be similarly UPf, where is UPf minus epsilon. So, if I take it on what is SPf? SPf is less than A plus epsilon, so here is this will be less than this quantity will be less than A plus epsilon.

So, that epsilon I can bring it on this side So, that will be bigger than UPf minus. So, UPf minus epsilon minus epsilon times b minus a will be less than less than, so this will be bigger than. So, what are we getting? SPf Is it everything okay?

Student: (())(26:16)

Professor: This was LPf plus epsilon is bigger than SPf is bigger than a minus action. So, everything on this side, so it is bigger than, I should have written bigger, I wrote it wrong here, this was bigger and UPf. So, that will be less than this quantity less than A, this I just rewrote. So as it was this, so, what does that mean? So, implies we are got the require inequality that LPf plus epsilon b minus a is between A is less than UPf minus epsilon minus epsilon times b minus a.

So how do I adjust all these quantities now, so I have got a number a between LPF and UPf plus some small quantities. So, that small quantities are arbitrary. So we can make it, how many of you are worried about this epsilon plus, I can readjust those things back. That is what we are do in our analysis I can make it epsilon by 2.

In the beginning, epsilon by 2 times b minus a or simply say that L, one simply says it is epsilon times 1 plus b minus a anyway, that does not matter is less than A is less than UPf plus epsilon minus epsilon times one does not have to do all those cosmetic things one can just write this as 1 plus b minus a. So, one simply says since epsilon greater than zero is arbitrary, this implies that LPf is always less than or equal to A is less than UPf. So, arbitrary means I can let epsilon go to 0 if you want.

Student: (())(28:35)

Professor: Which sign?

Student: (())(28:40)

Professor: LPf is greater than A. LPf did I make a mistake somewhere LPf is this plus is bigger than A. So, this was LPf is bigger than A is bigger than that seems a bit strange should not happen. That means either I have made a, I want LPf to be, let us go through the steps this is okay. This step is okay because SPf is the limit, so given epsilon it is close to a so mod of SPF minus A less than epsilon so A minus that is okay.

So this star is okay and now the Mi is the supremum, so Mi minus epsilon cannot be the supremum that is okay. So, there is a point with this property, so 1 and 2 are also okay. So, so UPf is less than UPf that is this so multiply this equation by xi minus 1 to x i plus b minus a that is small mi, so this is ok.

So, in 1 if we multiply this is less than okay. So, UPf minus epsilon times b minus a is less than SPf that seems. So, this part is okay this part is okay third, so that means UPf. So, LPF is greater than this is ok seems okay. So, let us see from here what are we getting LPf this part gives me LPF is less than A minus, I think even that I think I should not write this way. Yeah, okay. I think probably the last line I should change, yeah I think this only we should change.

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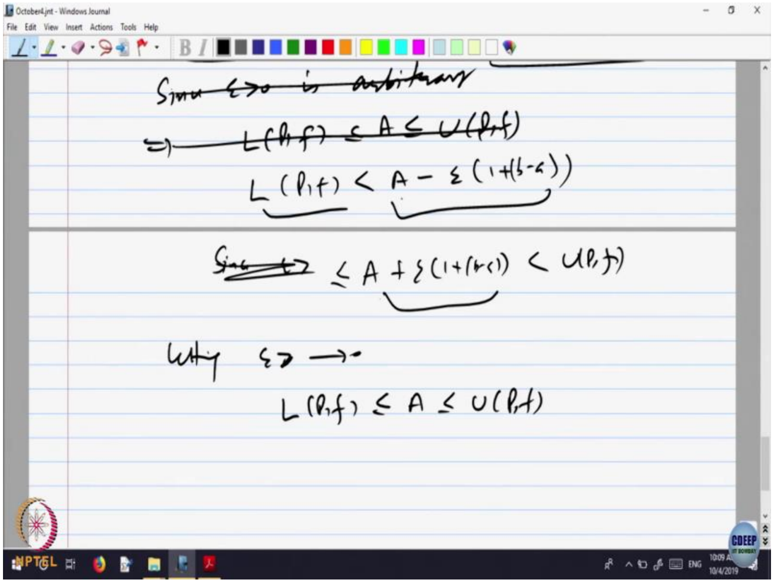

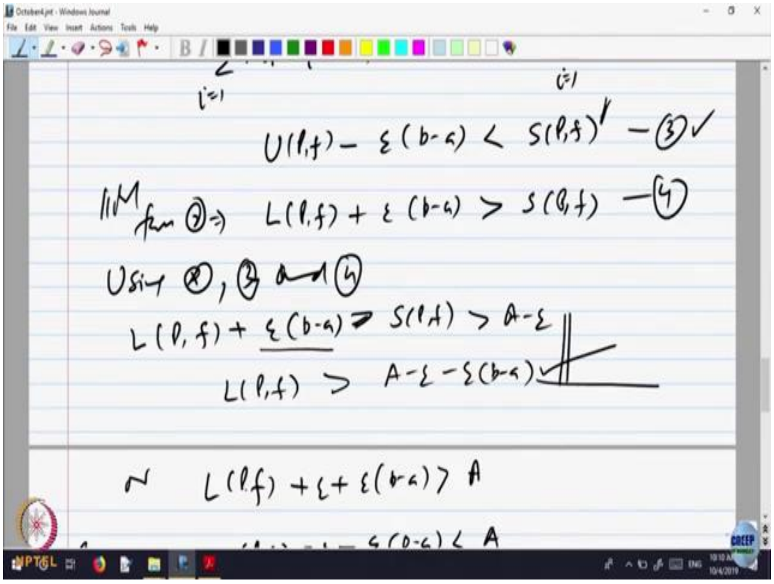

So let us rewrite to do this. So this says LPF is less than A minus epsilon 1 plus b minus a. This part LPf, this quantity I have taken it on the other side, b minus a. So, I think this is a place where we should say in this since epsilon greater than, we can write this and similarly other one will be A plus epsilon times 1 plus b minus a will be less than UPf.

This part will give me this. Now, just compare this quantity and this quantity can I say LPf is less than this quantity and that is less than equal to this quantity because this is A minus something and this is A plus something same quantity and that is a non negative quantity epsilon times 1 into b minus a. So, that, so if I do that, then I will get LPf is less than A times epsilon into this less than A plus the same quantity less than UPf. Now here if I let epsilon go to 0, so letting epsilon converge to 0, I get LPf is less than or equal to A is less than equal to UPf. That is okay.

Student: (())(35:28)

Professor: Which one?

Student: (())(35:32)

Professor: No, no I am using only this part.

Student: (())(35:35)

Professor: Which part?

Student: (())(35:38)

Professor: is this? Okay? LPf plus epsilon times b minus a is bigger than A minus epsilon. So is this part okay? Yes, I think okay, let me only I think this is the time to show you probably the better way of writing this in case.

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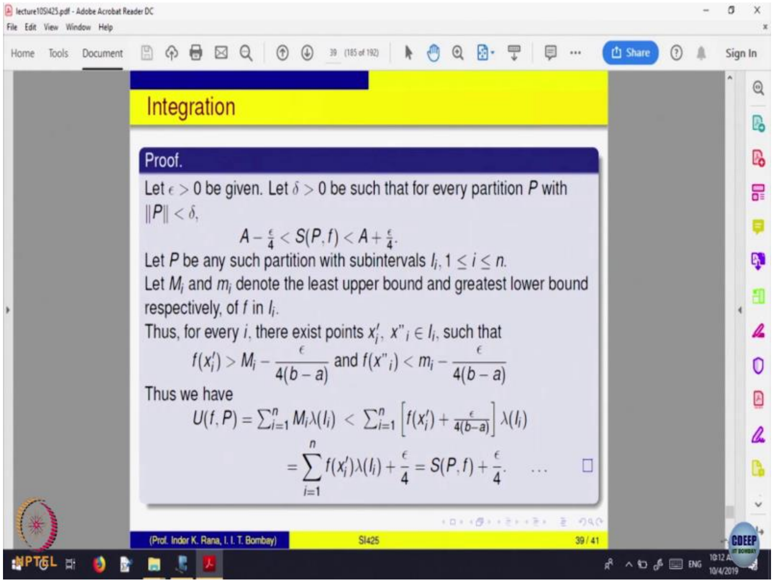

Let us look at, see SPf for a given epsilon greater than 0 you can find the delta, such that this quantity happens, there is at number A is close to it, where P is a partition. So, it has sub intervals Ii xi minus 1 to xi. Now, capital Mi and small mi are the supremum and the infimum. So, that means what I can find a point xi dash. So, that xi dash is bigger than mi minus that quantity. Because this quantity mi minus cannot be the supremum.

So, there must be a point and similarly for the other one, for the infimum small mi. So, now if I multiply both sides by the length, so multiply both these equations by the length of the intervals, so I will get these two okay and what is this quantity? So, what is this quantity f f xi multiply by lambda of xi that is SPf.

And that quantity that instead of epsilon in, the written proof it is epsilon divided by 4 times b minus a that is so, that is okay. So, you get upper sum is less than SPf plus, epsilon by 4, instead of that epsilon into 1 plus epsilon what we wrote and now this combined with this first one SPf is less than A plus epsilon by 4. So, this and this combined together you get upper sum is less than.

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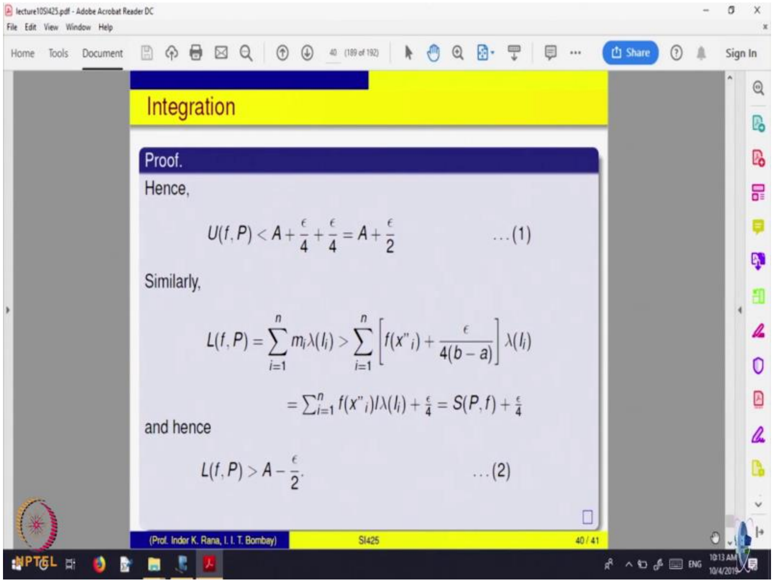

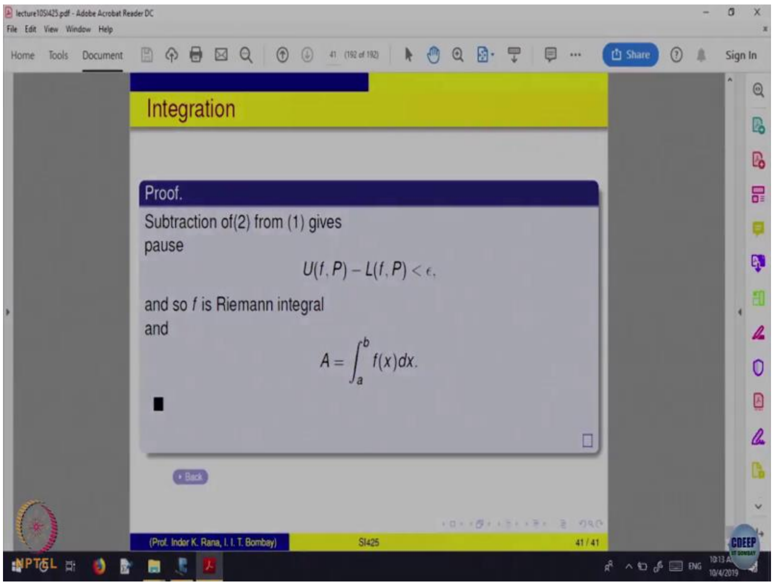

So, you get upper sum is less than A plus epsilon by 2 and similarly you will get the lower sum is bigger than A minus epsilon by 2 and now combine 1 and 2 what does 1 and 2 give you? The lower sum the difference between the two upper and lower is less than epsilon.

So, go through this proof from the slides again. So, basically the idea is that Riemann Integrability gives you tags that points ti arbitrarily, you can connect with the upper and lower sums by looking at the definition of Mi to be the supremum. So, that gives you that this number upper and lower is less than epsilon. So, it is integrable and a must be that sum.