Basis Real Analysis
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Lecture 42
Riemann Integration Part VI

Then historically Riemann said, I do not like this idea. He did not say that. But essentially he said why we should assume that the function should be bounded.

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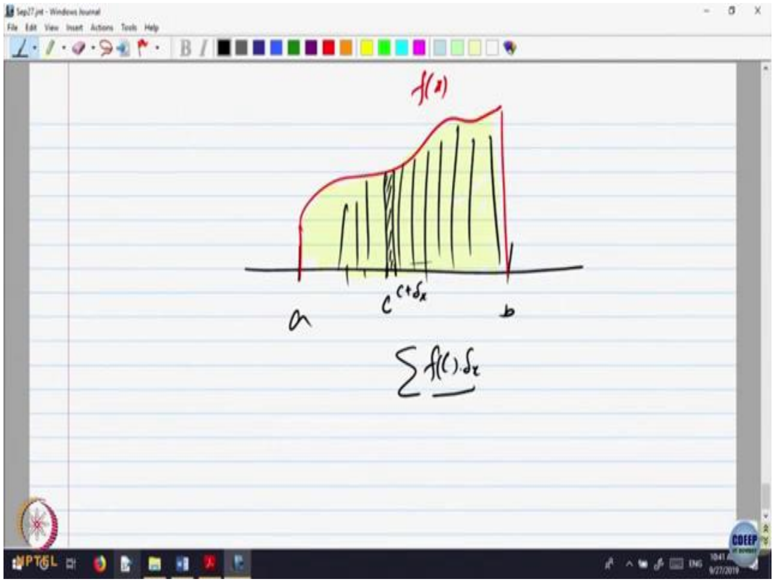

So, his idea was very nice one, he said, intuitively seems, so this is a and this is b. He said this is the graph of the function,f of x and what you want to do, you want to capture the area below the graph of the function. So, this is the area that you want to capture. So, to capture this area, what you are doing is you are trying to take upper sums and lower sums and trying to capture them in between.

But let me do this way, let me take any point in this, take this height and take a small nearby point C plus delta x. So, that will give me this. So, look at the area of this strip and try to imagine the whole area being made up of these thin steps. So, what is the area of this strip, you can take f of c into delta x and try to sum it up.

And the idea of summation is same, that you take partitions and make partition finer and finer. But his idea is, instead of taking the minimum and the maximum in the sub interval, take any value in that sub interval and take any rectangle of that height.

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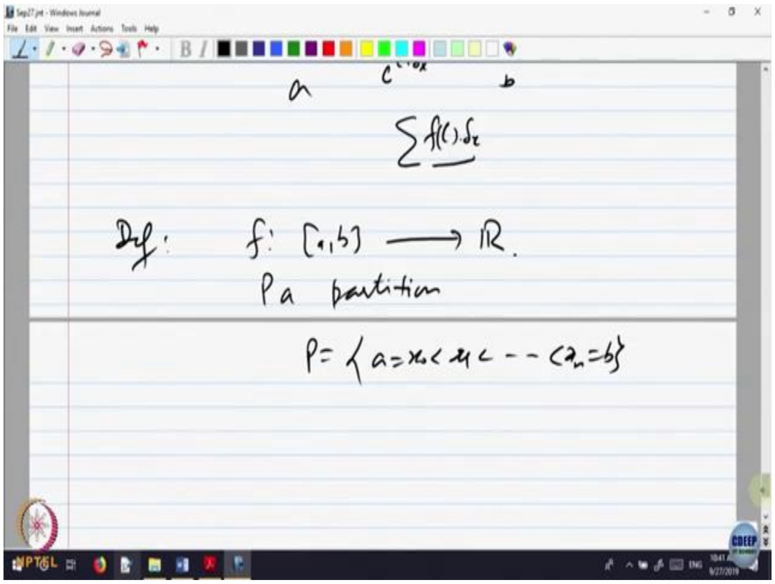

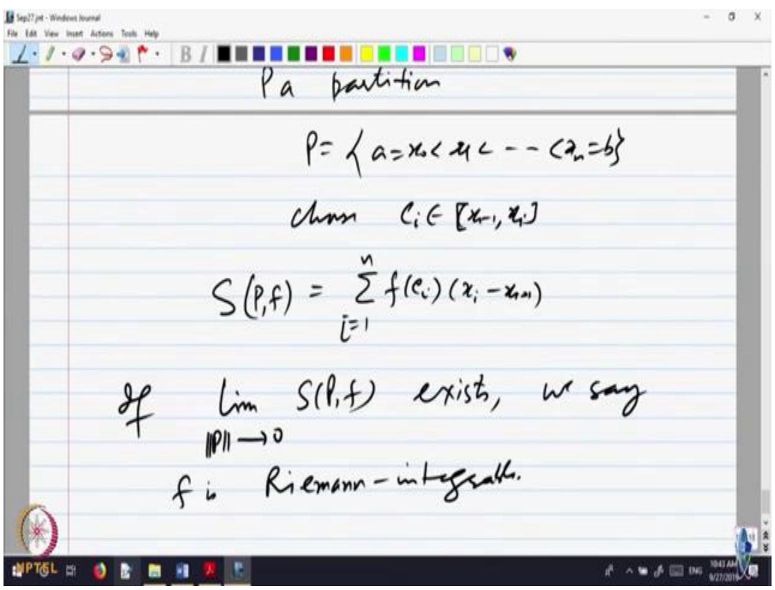

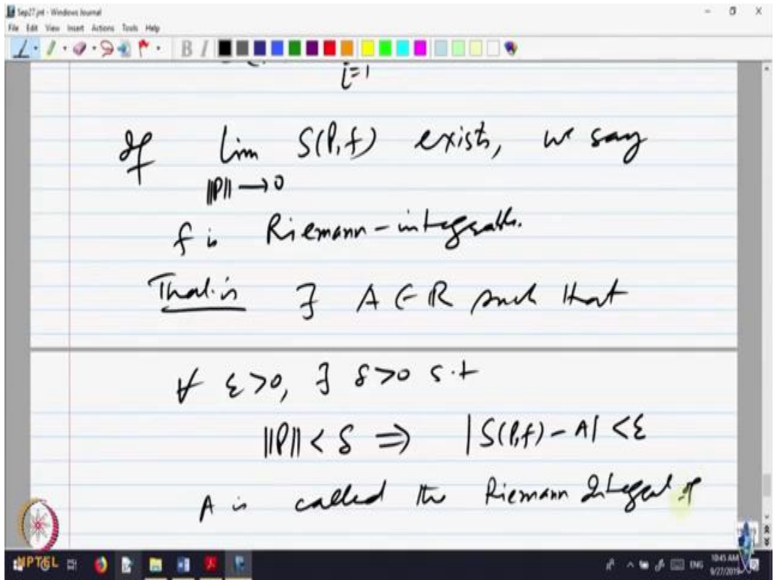

So, let me define that and so definition f is a function on a b to R. So, let P be a partition x1 less than xn equal to be take any partition and choose any point c i belonging to x i minus 1 to xi take any point in that close interval xi minus 1 to xi, look at the height f of ci into the length. So, that is that the strip, take the summation that gives you an approximate size of the area below the graph of the function. So, let us call it a S P, f.

Now, there are two things one should observe. To define this sum, we do not have to assume f is bounded. We do not have to assume, because we are looking at the value of the function at some point. So, we are taking the height of the function at that point into, so right hand side is meaningful. So, the sum is well defined, we do not have to assume. At the same time, note that the point ci is a arbitrarily chosen point in the interval xi minus 1 to xi, you can take the left hand point you can take the right hand point, you can take the midpoint or you can take any point does not matter.

So, look at this and look at the limit norm of P going to 0 make the partition finer and finer, S p f. So, if exist we say f is, now we will call it as Riemann integrable if this limit, so what is the meaning of this limit exist? Concept of limit again, but norm of P going to 0 what is the meaning of this saying that limit exists. So, let me explain that because that is for every epsilon bigger, limit exists, so first of all what is the limit one should say that there exist some number A belonging to R such that for every epsilon bigger than 0, there is a delta such that whenever the norm of the partition is less than delta that implies this sum S P f minus A is less than epsilon.

For every choice when P is a partition S p f depends upon the choice. So, this is for every choice of those points Ci whatever choice you choose, you construct S p f that should be close to the number A that is what the limit means. So, you say it is integrable and this number A is called the integral we should say Riemann integral for the time being because we have notion of integral coming from upper sums and lower sums called the Riemann integral of f.

So, now, once we give a another way of interpreting the area, we should say that this way of defining the area is same as the one we are done by upper and lower sums both are same, there is no difference between them. So, let us prove that, but before even proving that, important thing comes namely that if f is Riemann integrable and if we want to prove that it is this integral is same as the one which we obtained from upper and lower but that was defined only one bounded functions. So, there should be a theorem saying that if f is Riemann integrable implies, f is a bounded function.

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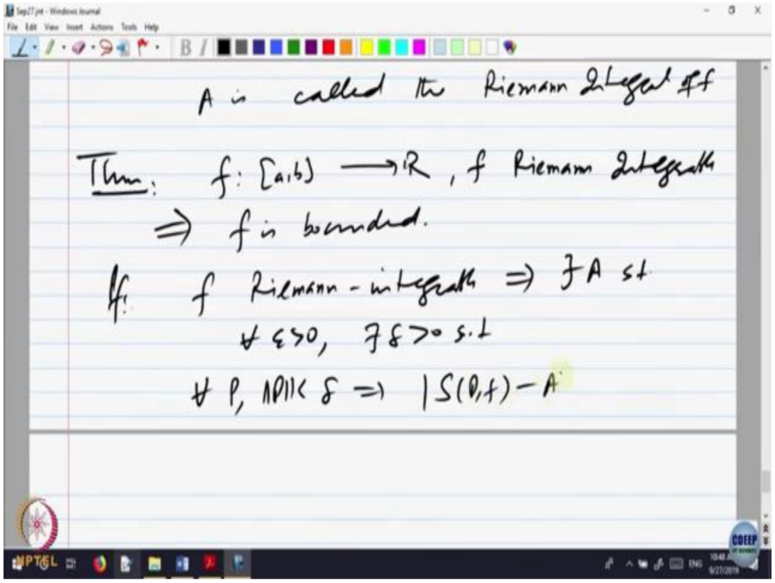

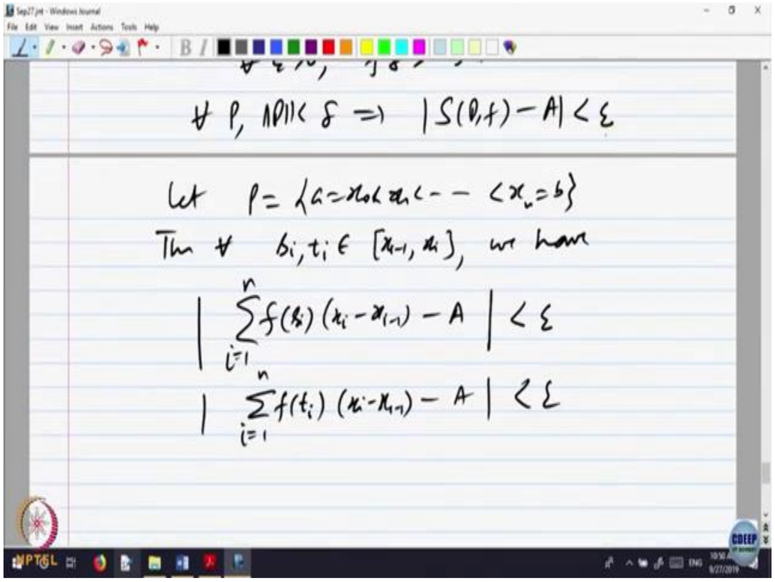

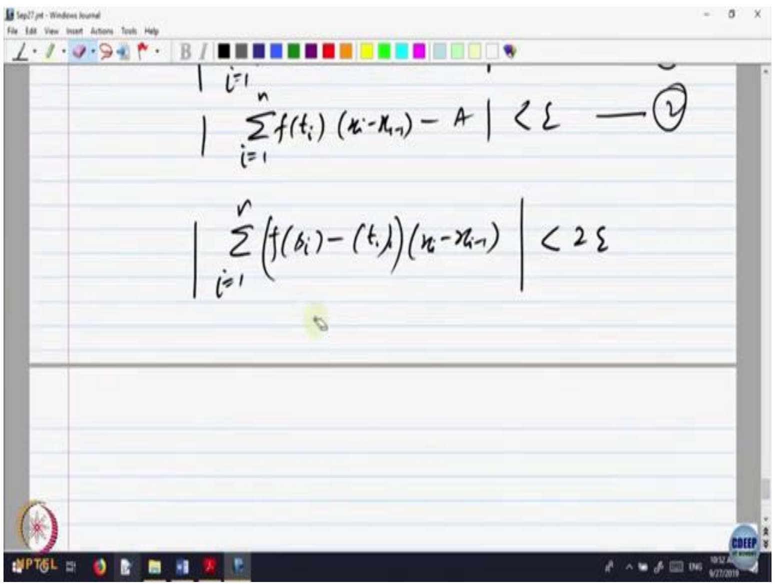

So, let us prove that first that if f is, so theorem first f a b to R, f Riemann integrable implies f is bounded intuitively it seems quite, because it was a function is unbounded those strips areas you can keep on increasing, that is one way of looking. But let us know look at another way of proving this. So, f Riemann integrable implies there is a number A such that for every epsilon bigger than 0 there is a delta such that for every P norm P less than delta implies S P f minus A less than epsilon.

See boundedness means what? We should be able to show that for every point x mod of f of x is less than or equal to some constant that is what we want to show. So, let us say this P is a partition. So, something say is A equal to x is 0, x 1 less than x equal to b, then for every point Si and ti, see the S P f requires choice of a point in between the interval xi minus 1 to xi. So, let me choose some points xi, ti between xi minus 1 and xi.

Then for every say choice we have. So, let me write corresponding S pf, f of Si, xi minus to minus xi minus 1 sigma i equal to 1 to n minus A less than epsilon that is a sum corresponding to the choice of Si. But let me also choose the corresponding to some other choice f of ti, xi minus to xi minus 1 minus A less than epsilon, for two different choices let us so, that is what it says.

But, we only want at a point. So, let me make a choice. So, let us choose Si equal to ti. No let me just let us try to take from these two equations, let us subtract them and see what we get sigma of f of Si minus f of ti equal to 1 to n, if we subtract, I want to look at this. So, that means if I what will I get less than. sorry, into that length of that interval. So, let me minus f of ti into xi minus xi minus 1. I want to estimate this quantity.

So, I can add and subtract A. See this is summation this the first term minus mod A, this minus mod A. So, summation f of Si minus capital A times this plus minus of that, so I am saying this is less than 2 epsilon is that okay. Because what is the first term, summation f of Si times this length, so I can add minus capital A. So, I will get one epsilon by using triangle inequality and that other one, I will combine it with the other one. So, triangle inequality so add and subtract capital A in this thing and use triangle inequality and use this 1 and 2.

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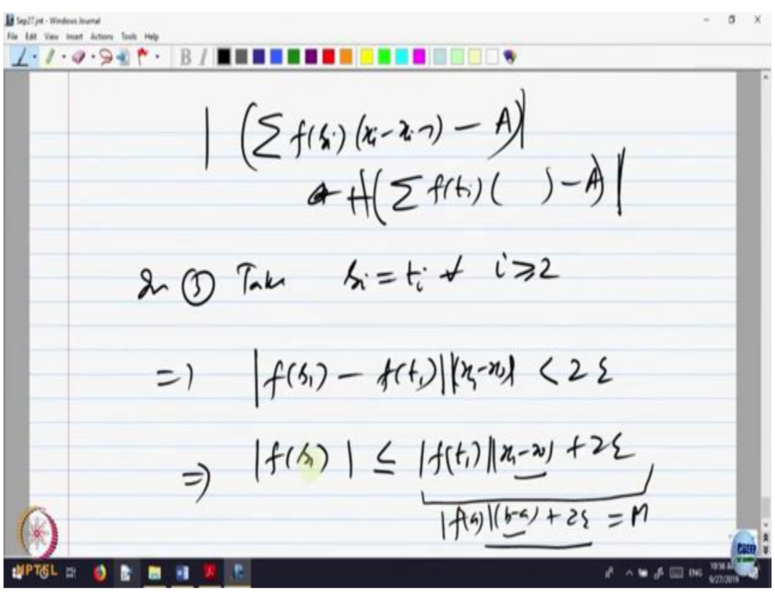

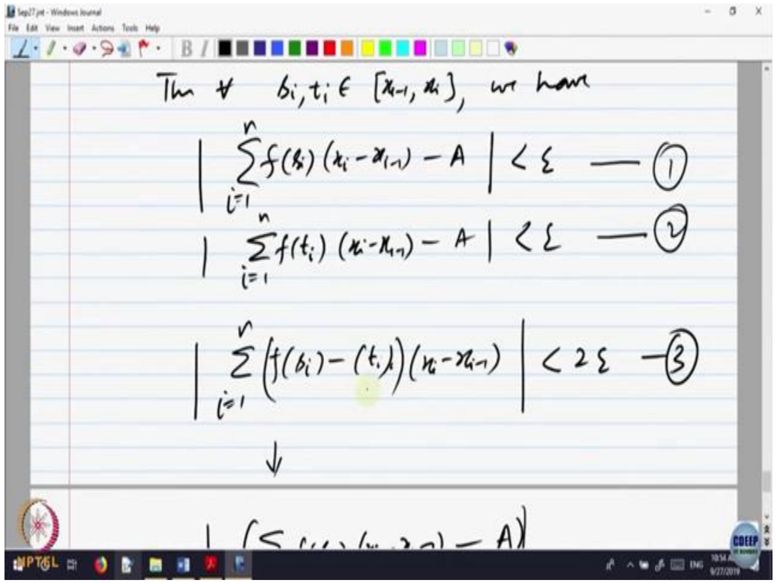

Is that okay or shall I write that step? So, let me probably in case you feel uncomfortable that we write. So, the reason is because this quantity is this summation f of Si into xi minus xi minus 1 minus A plus the other term minus summation f of t i that thing minus A, add and subtract and now use triangle inequality.

So, this will less than or equal to absolute value of this plus absolute value of that and use 1 and 2 to get the required thing. So, this is less than 2 epsilon, so let me call this as three. So, in this three, let us put take Si equal to t i for every i bigger than or equal to 2 then what will happen? That terms will be 0 those terms will be 0. So, what is left is the first term f of S1 minus.

So, that will give me implies f of S1 minus f of t 1 absolute value into the length x1 minus x0 is less than x1 that is positive anyway less than 2 epsilon. In equation 3, I am putting S2 equal to t2, S3 equal to t3 and so on all remaining terms will be 0 except the first one. So, first one will be f of S1 minus f of t1 into the length. So, that is x1 minus x0. So that is the case, so what is mod f of S1 is less than or equal to, what is mod of f?

So, mod of f t1 into x1 minus x0 plus 2 epsilon. For where this implies this mod of A minus B less than something. So, what is mode A less than triangle inequality again you can add and subtract if you like and now, now look at this quantity f of t1is a value at some point t1. Length of the interval, so this is less than or equal to some number, f of t1 the value of the function at a point t1 of your choice you can take f of T1 to be a left hand point A1, A

So, for f of A into the length. So, you can put maximum b minus a, this are the length of sub interval you can bind it by the total interval plus 2 epsilon, f of t1, t1 was a arbitrary point in the interval xi minus 1 to xi. So, let me take the arbitrary point as the left hand point A. So, T 1 is equal to A, x1 minus x0 that is the length of the sub interval.

So, that is less than the length of the full interval there is b minus a two epsilon. So call this as a some constant M. So what is S1, S1 is an arbitrary point in the first part, so what we are shown is the function is bounded in the interval x0 to x1, S is arbitrary.

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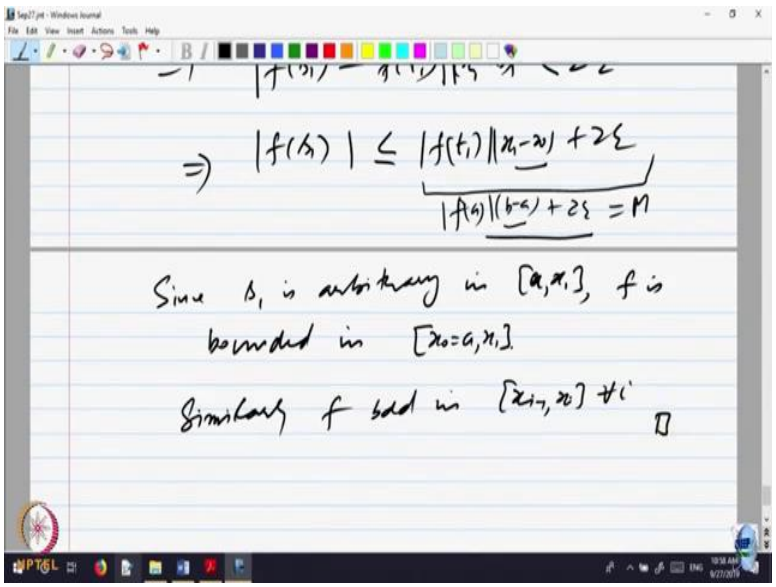

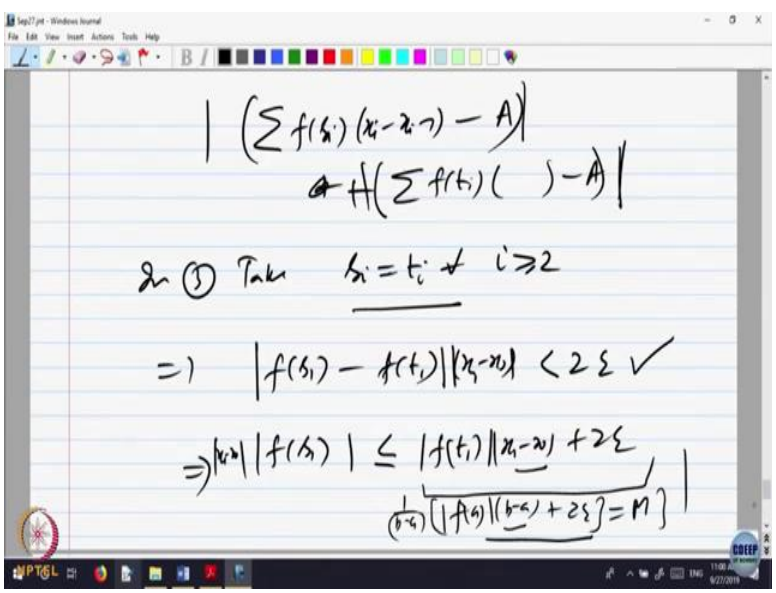

So, since S is arbitrary S1 is arbitrary in the first part that is a to x1, f is bounded in x0 that is a to x1. So, what we are showing is we found a partition now, so how did I get the first one by choosing Si equal to Ti, I can do it with other ones now, Si equal to ti everywhere for all i except 2. So, I will get boundedness in the second interval and so on. So, similarly, other sub intervals so let me write.

So, similarly f bonded in each one of xi minus 1 to xi for every i and that proves it. You can divide by that also. So, to be very precise, it should have been multiplied by this. So, 1 over of b minus a put that also if you like, take it on the other side, the constant multiplying that what your question is, this multiplied by x1 minus x0.

If I divide so it will be 1 over of this quantity, so 1 over of I should not write 1 b minus a, I will should write as something else. Yeah I think f of S1 minus that makes it smaller. Yeah there is minor thing I think from here if you want to go then you should, that is a small h I should because this will not give you directly this you have to divide by that. x1 minus x0, so this is not correct, we should have x1 minus x0 multiplied by this is less than or equal to this quantity. So, you will have 1 over of x1 minus x0. Can I bind that with something? Yeah, I think