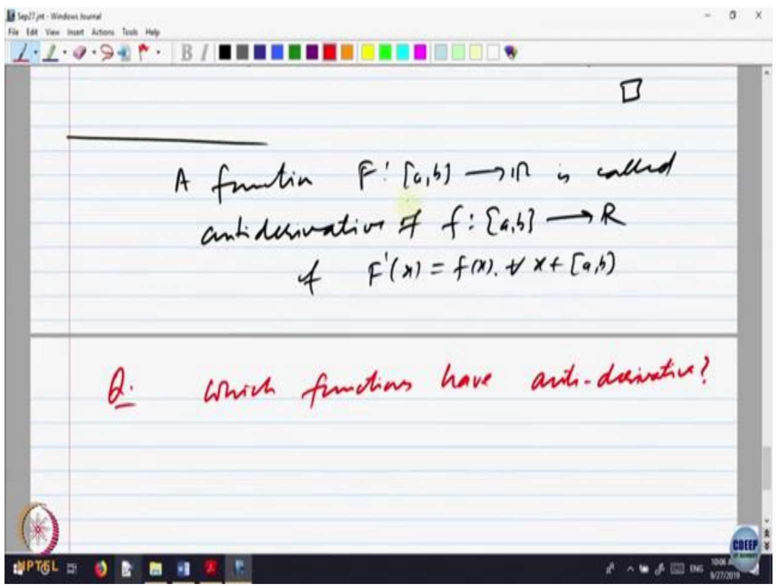
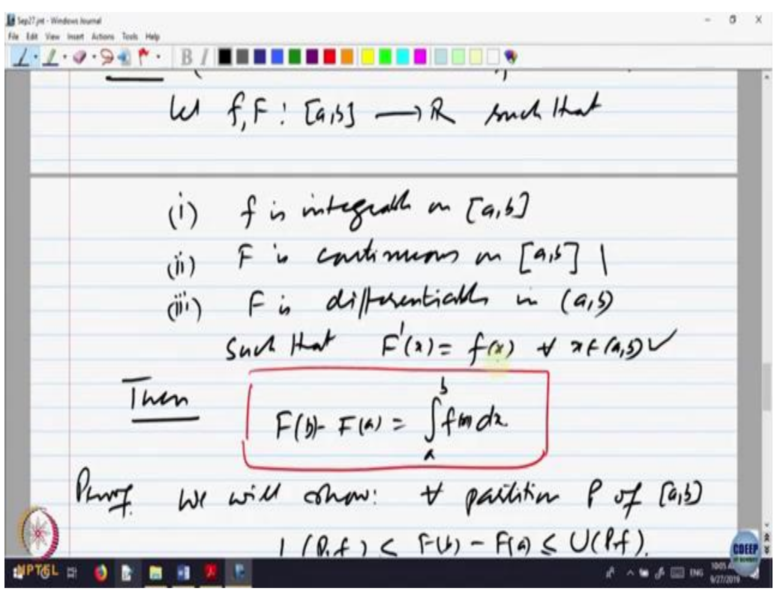
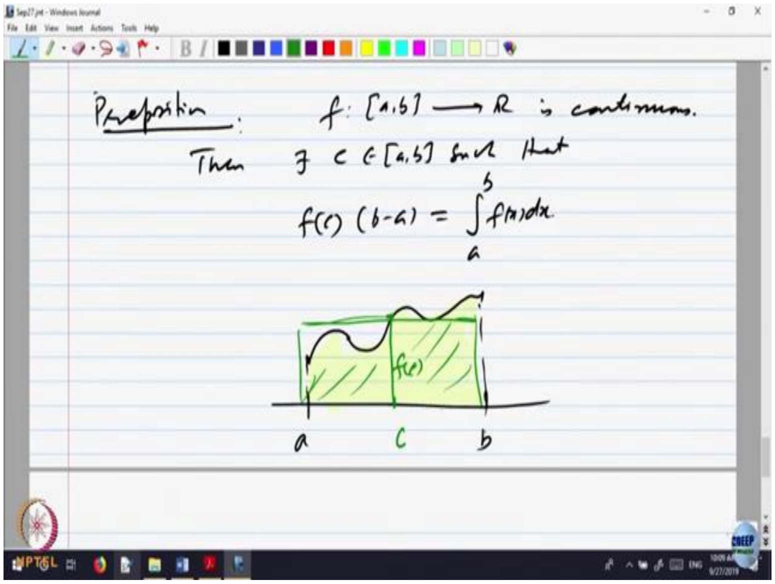
Basis Real Analysis
Professor Inder. K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 41
Riemann Integration Part V

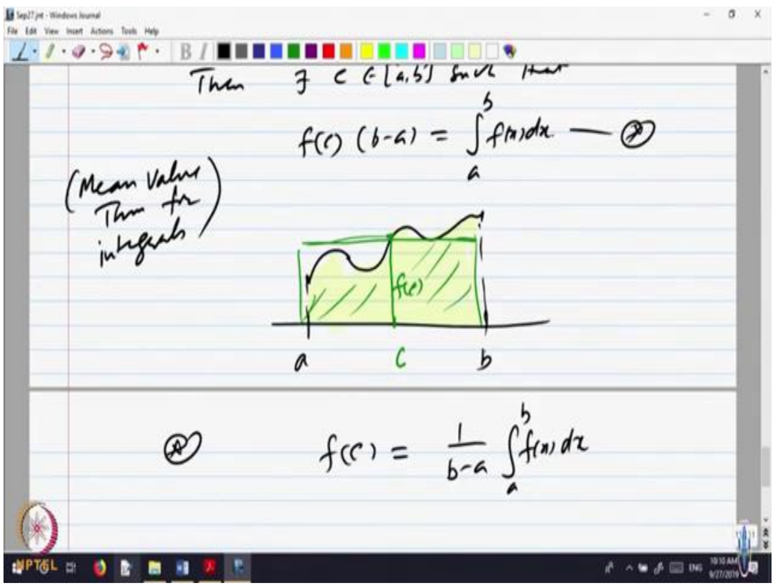
Now there is another question that comes. So basically what we are saying is, this is if you like you can call this as anti-derivative.

(Refer Slide Time: 00:32)









A function capital F a b to R is called anti derivative of a function, so one introduce is f, f dash of X is equal to f of x is equal to f of x for every x belonging to a, b. So, that relation that we are saying here, so one gives a name that capital F is anti derivative of small f. So, the question arises, so fundamental theorem of calculus is applicable whenever there is anti derivative for a function you can compute. So the question is what are the functions which have anti derivative, what is the class of functions or is there some, we can say that, if a function has this property than it will have anti derivative.

So, we want to answer that question, so question is which functions have anti derivatives? Keep in mind anti derivative is not unique. If we are just saying F dash would be equal to f, so if you take F plus a constant that derivative also will be same, so it is a class of functions, two anti derivative is differ by a constant. So, to answer that question let us first observe a small result, to answer this.

What should I call preposition. So, let us say f is a continuous function. Suppose it is a continuous function then we know it is integrable then the claim is there exist point c belonging to a b such that f of c multiplied by b minus a is equal to integral A to b fx dx. So, let us try to understand what is this theorem saying, because f is continuous the right hand side integral exist every continuous function, but what is the left hand side. This is f of c into b minus a, so this is the area of a rectangle with base as a b and height as f of c.

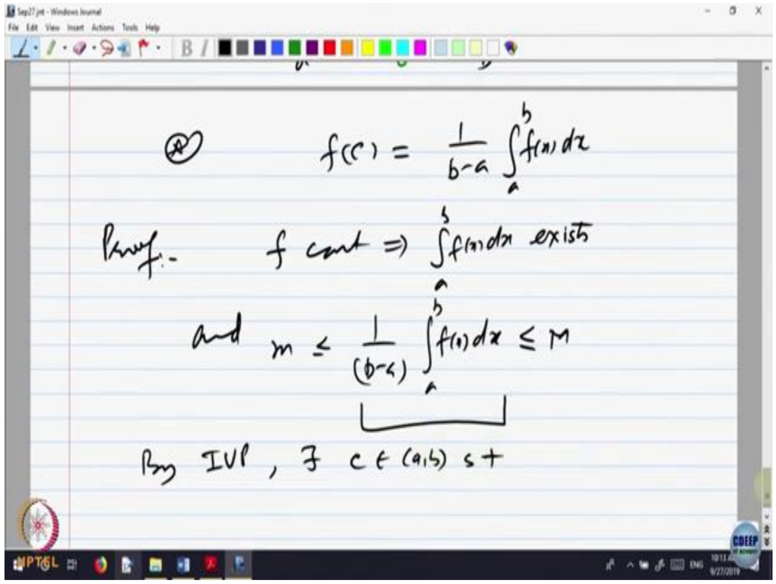
So, if you will interpret it geometrically it is very nice in the sense that if this is a and this is b, so this is the, your function right and you are looking at this area, area below the graph of the function that is your integral, so this is the right hand side. So, the right hand side is, so this is the right hand side, what is the left hand side?

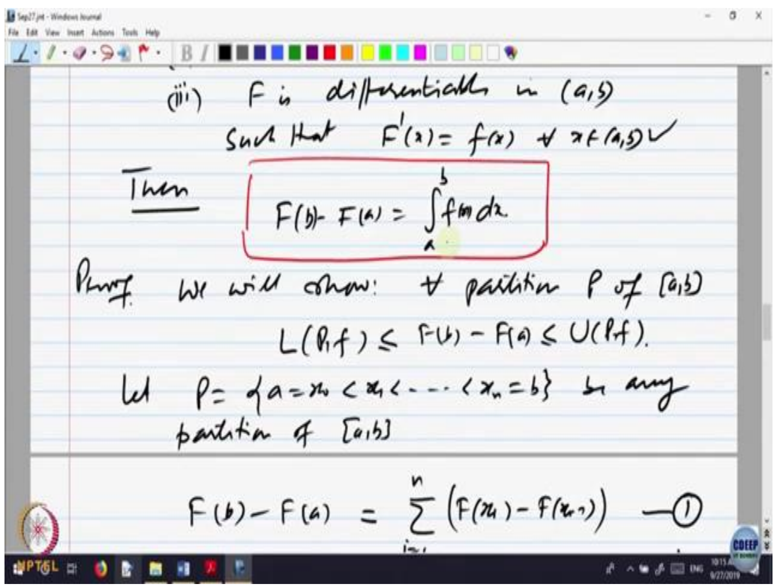
It says there is a point c in between so there is a point somewhere here c. So, is that if you look at this height, so that height is f of c. So, this is a rectangle so look at the rectangle, so area of that rectangle is same as the, that is the area of below the graph of the function. So, there is a point c in between, in the interval a b, such that if you look at that height and the area of the rectangle with that height that is equal to the integral of the function.

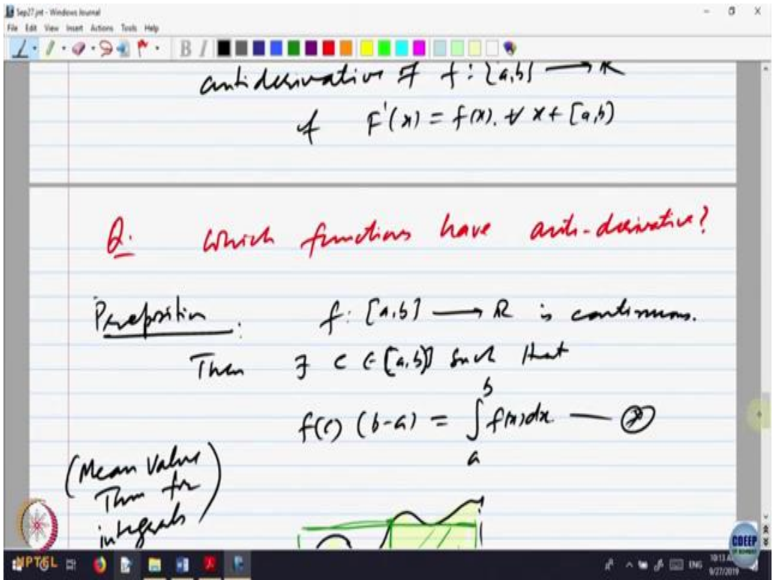
So, this is the reason why it is called, this is called mean value theorem for integrals, so this is called mean value theorem for integrals. By mean value you can also interpret this, so this the claim star, if you interpret it as f of c is equal to 1 over b minus a, integral a to b fx dx. Then what does geometrically right hand side represent. The thing that right hand side looks like an average of the function, f of x is a value at a point in a b sum of the values.

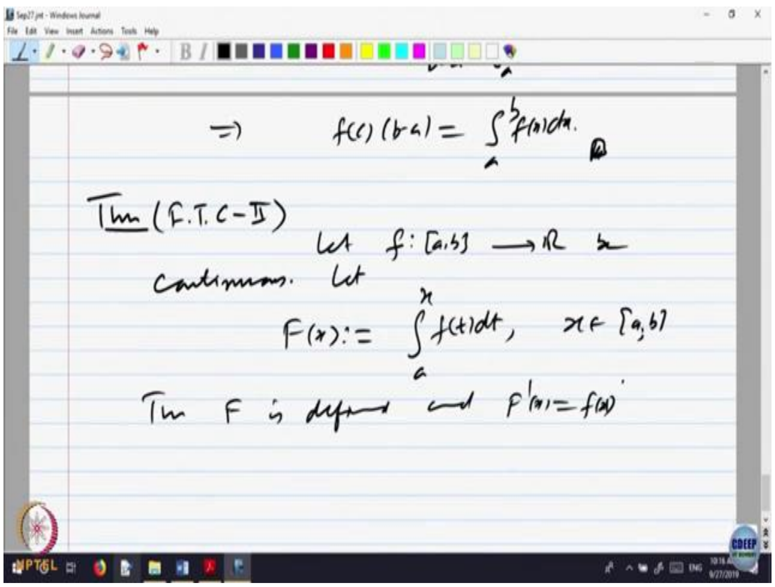
So, summation is integral a to b divided by the total number of values that is the length of the interval, so right hand side can be called as the average value of the function on the interval a b right hand side, can we called as the average value. So, the theorem says if f is continuous the average value is attained at some point in between the average value is attained at a point c in between a to b, so that is another way of saying, so this is you can call it as the average value.

(Refer Slide Time: 07:19)









So, and the prove is quite straight forward because, f continuous implies integral A to b fx dx exist and if I look at this number average value this is less than or equal to capital M times b minus a, because what is that is a you take the partition with n points a b only, m is the value. So, supremum capital M on the interval a b. So, m times are like there is upper sum with respect to the trivial partition and this is b minus a we are taking it down. So, let us do not write there and less than are equal to small m.

M times b minus a is less than the integral upper sum, lower sum is less than the integral less than the upper sum. The interesting thing is if I look at this value, this is a number which is caught between the smallest value and the largest value or the function, so what does intermediate value property say this must be attained at some point that’s all. So, the intermediate value property there exists a c belonging to a b such that f of c is equal 1 over b minus a, a to b fx dx. So, that implies whatever we want it to say that f of c times b minus a is equal to integral a to b fx dx.

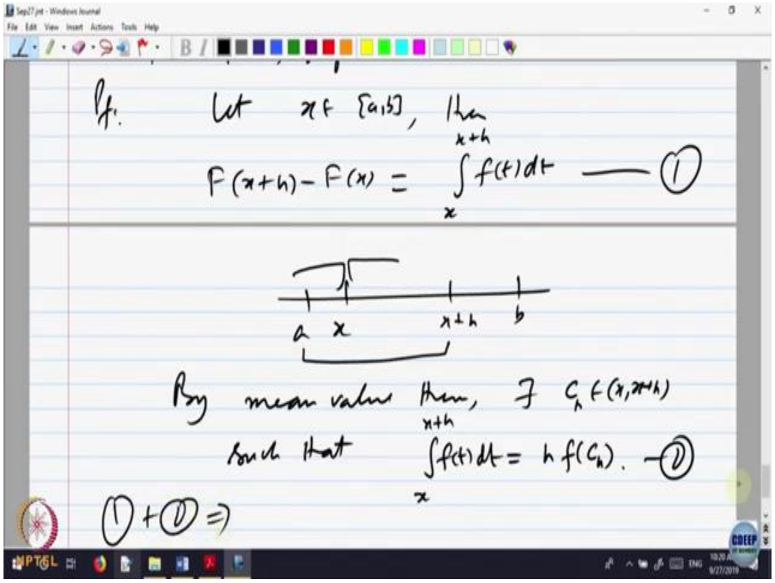
So, there is a point inside the interval in the open interval a b, while writing the theorem probably I did not yeah I should have said in the open interval a to b you can specify that is what intermediate value property says, there is a value alpha, there is a value beta and something in between and that must be attained at a point in between.

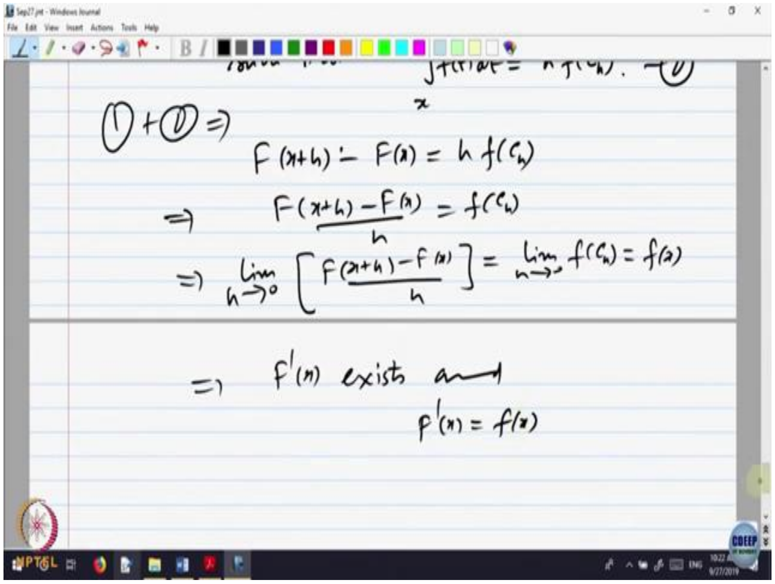
So, that is intermediate value main value theorem for integrals, we will an application for this in fundamental theorem of calculus point two, which says the following that let f, so R be continuous, let us af is a function which is continuous see actually, what we are trying to do is find out an anti derivative for the function. What could be an anti derivative for a continuous function?

Fundamental theorem calculus itself gives you an answer it says the relation between small f and capital F if I know small f I can define what is capital F from this. So, let us write that straight away that equation, so let F of x be defined as integral A to x of ft and dt, x belonging to a b. So, for every point x in a b, let us because f is given to be continuous. So, it is integrable and this function capital F is defined, then F is defined and F dash of x is equal to f of x.

So, this is the function, so this theorem says every continuous function will have an anti derivative, as soon as you are able to recognize the anti derivative you can get back the integral. So, class of all continuous functions have anti derivatives and for such functions computation of integral is straight forward by fundamental theorem of calculus.

(Refer Slide Time: 12:38)





So, let us just prove that this is the case we have prove. So, I have to find the derivatives so let us take a point x belonging to a b. How do I compute the derivative of capital F. what is the definition? f of x plus h minus f of x divided by h limit h going to 0. So, let us look at f of x plus h minus f of X, so what is that quantity equal to that will be equal to integral a to x plus h minus integral from a to x, I have not said it those properties of integral namely that the integral is additive over the intervals you right.

So, integral from a to c plus integral from c to b is same as integral from a to b. So, those I will state, so I should have said it that first what anyway, let us use that for the time being namely this is integral from x to x plus h ft dt is that, this is integral a to x plus h minus the integral from so this is x, x plus h and this is a. So, integral from a to this minus the integral from a to x, so what is left is integral from x to x plus h. So, that is what we are saying.

So, F of x plus h is integral a to x plus h minus F of x that is integral from a to x so what is left is integral from x to x plus h. So, let us continue with that, so F of x plus h minus F of x is equal to that quantity. Now look at this quantity, f is a continuous function, f is a continuous function, so it is continuous in the interval x to x plus h also and just now we proved the mean value theorem.

So, there must be a point in between x and x plus h say that this integral is equal to f at that point into the length, length is h. So, let us write before let us just by mean value theorem there is a point it depends on h belonging to x to x plus h, here I am taking h is positive just for illustration h could be positive or negative, x plus h could be on the left hand side or on the right side does not matter actually.

So, ch such that integral x to x plus h ft dt is equal to h times f at ch. So, put this value in this, so the call that as of 1, call this as 2, 1 and 2 imply F of x plus h minus F of x is equal to h times F of ch, so that implies that the ratio F of x divided by h is equal to F of ch and to compute the derivative we should take the limit as h goes to 0.

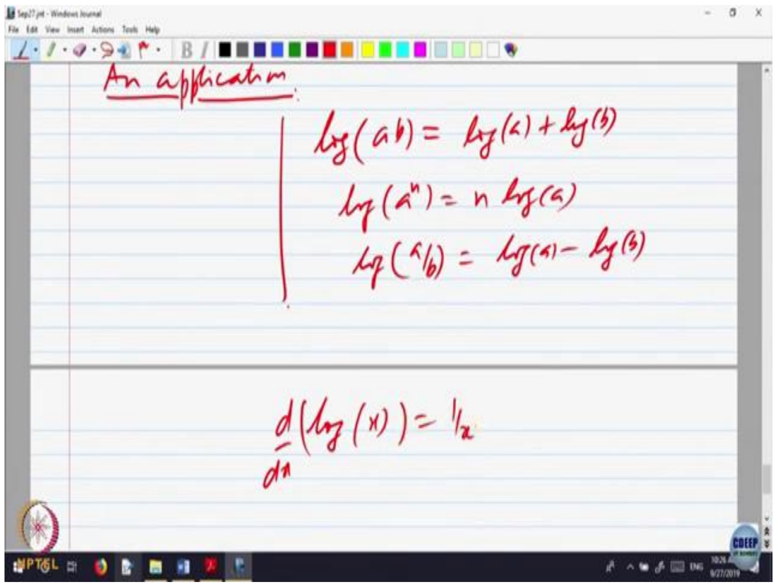
So, take their limit, so implies limit h going to 0 of F of x plus h minus F of x divided by h is equal to limit h going to 0, f of ch and what is that equal to, ch in a point in between the interval x to x plus h, h goes to 0, f is a continuous function. So, that limit, so c of h will go to x. So, f of c plus h must go to f of x by continuity of the function f. So, that is equal to f of x because of the continuity of the function f.

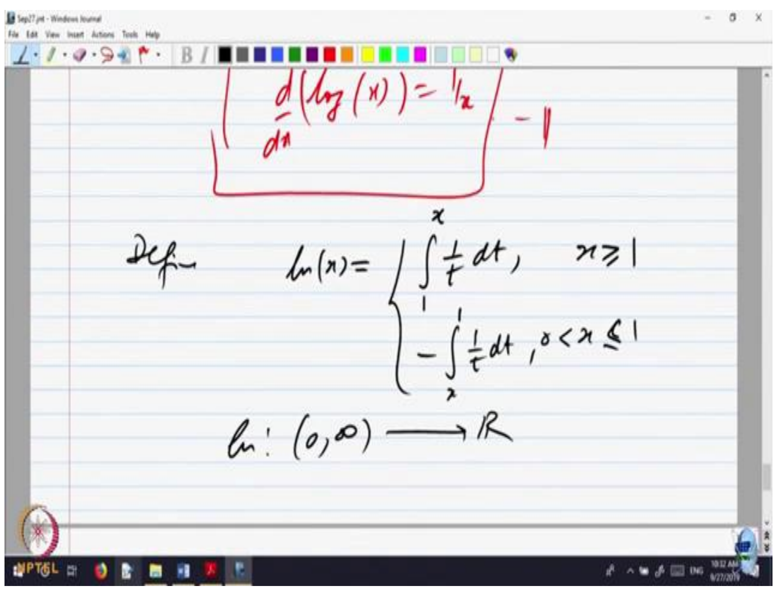
So, that proves so implies f dash of x exists and f dash of x is equal to f of x. I just said this h could be positive or negative right if x is the interior point, if x is the interior point, if X is end point end of point then what will be happen it you will only have right hand side, so only have the right hand derivative. So, taking care of that, these are minor things, so you can x in the open interval than plus minus does not matter and at the end points you can just have one sided limits.

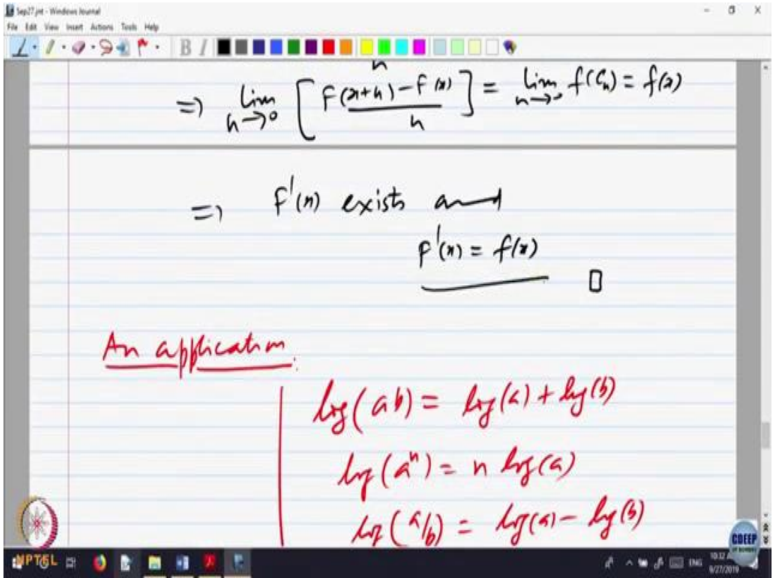
So, you can prove for everything, so derivative at the left hand point means it is the right hand derivative and derivative at the right hand point means it is only the left hand derivative possible.

So, every continuous function has an anti derivative and for every function which has anti derivative integral can be computed, so that is what is the importance of fundamental theorem of calculus.

(Refer Slide Time: 19:14)







So, let me give you one application of this fundamental theorem of calculus, which is quite useful or of importance rather. So, let me guess an application of course it is of a great importance this theorem it allows you to compute integrals without going into the limit operations and it has implications in historical in Fourier series and so on. But let us look at a simple application which we will all without knowing Fourier series you will understand.

I think we start using a function called the log function from our school times. I think probably standards 8, 9 or somewhere a log function. What is the definition of log function? we do not know how is log defined? For us the log is defined by the log tables or another example is what is a trigonometric function, sin theta, we defined sin theta as a height over base.

But if you want to define sin function as height over base and you want to prove that function is continuous, how do you prove it is continuous, how do you prove that function is differentiable because we start using this facts that sin x, cos x all are defined functions, well defined, periodic and so on, sin is continuous infact differentiable derivative of sin is cos so if we take the definition of those geometric definitions, you do not get those properties easily you have to assume those properties.

Also the log functions, log function is a function which has some nice properties what are the properties of the log function normally we have log of what is log of ab?

Student: (())(21:11)

Professor: Log of a plus log of b and then we have log of say a raise to power n is equal to n times log a, log of a by b whenever defined, we say it is log of a minus log of b and so on. Is there any such function with these property, does such a function exist at all or not.

Not only that when we come to slightly higher classes, we say a log function is differentiable and log derivative is 1 over x. We start using that also, without any proof the reason is that these function are not easy to define, these function. For example, you will look at the polynomial functions, polynomial function, constant function is a constant polynomial. Function fx equal to x, x square, x cube their sums linear combinations all are polynomial functions and you see some sought of feel comfortable with them.

You can define them regressively, you can define a rational function p by q where p is a polynomial, q is a polynomial. But there are functions which cannot be obtained from polynomial functions in any way by any algebra you have to go the concept of limits that is a reason these cannot be defined.

For example, you may say that I can define sin x, where the series kind of a thing, but that is a infinite operation it will does not somewhere. There is a limit involved in it, so we will come to see series later on. So, let me just try to illustrate how fundamental theorem of calculus can be use to define what is called the log function, we will not going to all the details of this.

But we will at least initiate so that you can try to prove it yourself. Because it is quite reasonably easy. The clue lies in this thing, the derivative of log is 1 over x. So, if I integrate 1 over x I should be able to get back the log function by fundamental theorem of calculus. So, one defines, so define ln of x to be equal to integral, now what is the value of log of 1 that is like having a 0.

So, let us start from 0 and go to x, 1 over t dt for x bigger than 0 and if it is less than 0 what should I do , I can only write x to 0, so x to 0, but with a negative sign. Log of 1 is 0 what is log of a negative quantity, less than 1. What do you think the log of half, 2 minus log of 2. So, that is why this negative sign is coming, x to 0, 1 over t dt and this automatically says log of 1 is 0, log of 1 is 0.

So, I will not go into the details, but let me just say what is obvious from this?

Student: (())(25:35))

Professor: Oh sorry, you are right, bigger than or equal to 1 less than or equal to 1, sure of my oversight. Log of 1 is 0, for negative quantity log is not defined, it is only defined when x is bigger than. So, this is a function defined to ln is a function defined in 0 to infinity taking values in R. It is a function defined only for positive numbers, log is defined only for positive.

Student: (())(26:19)

Professor: Greater than, less than 1, x is, x is less than 1, what corrections she has said I should have applied it here also 1 and 1. Log of 1 you want 0. So 1 to x integral that is when x is bigger than 0, when is less than 0. You can write 1 to X, but normally you write lower limit to upper limits so with the negative sign.

So, that is what, so there is a function defined for all positive real numbers and what is obvious for this. 1 over t is a continuous function, 1 over t is a continuous function. So, ln of x is defined continuous function is integrable. So, ln x is defined one consequence immediately it is defined and what is the derivative, what is the derivative of ln x fundamental theorem of calculus again derivative of ln x is 1 over x.

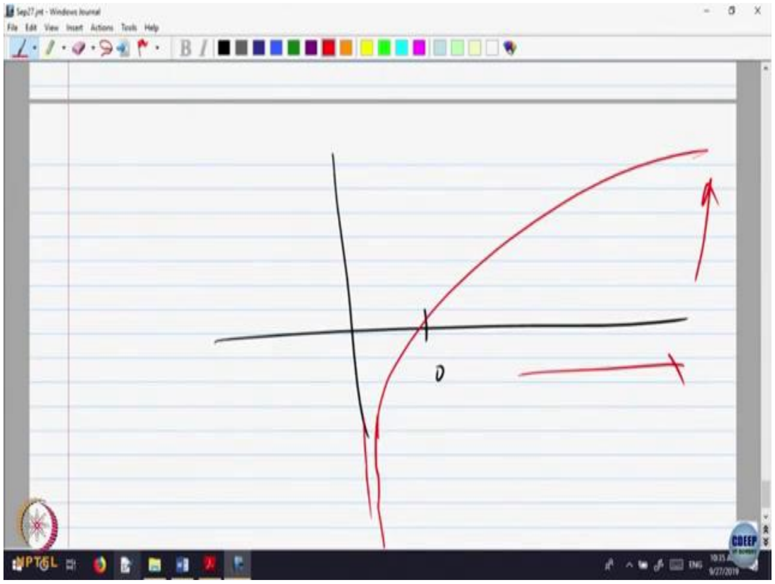
That gives you the derivative also it is differentiable and derivative is 1 over x is that okay? 1 over x 1 2 x derivative is that is fundamental anti derivative just now, we have said that, f of x was the integral a to x, so it is 1 to x. So, derivative is 1 over x, if the derivative of so it is a differentiable function with derivative is 1 over x. x is bigger than 0, so derivative is everywhere positive, 1 over x derivative, x bigger than 0.

So, ln x is a function with positive derivative go back I know something about the derivative what does it say about the function, if the derivative is strictly bigger than 0 the function should be monotonically increasing. So ln of x, defined this way, is a strictly monotonically increasing function in 0 to infinity and derivative is 1 over x.

So, what is the second derivative if you like 1 over t, derivative 1 over minus what is the derivative minus 1 over x square, x is bigger than 0 anyway, x square derivative is everywhere negative, so it is a concave down function, see all of your calculus, all those results that we prove derivative helps you. So, it is a continuous function, it is a differentiable function, it is concave down at 1 it crosses the x axis, and what happens as you go on increasing, what happens to ln of x, as x goes to infinity.

One can use this itself 1 over t to show that as x increases the area will keeps on increasing, it goes to infinity. So, it keeps on increasing and one can also prove those other properties, that ln of 1 over x is very nice actually, one should try to read the proof that ln of even all those properties, ln of a b is equal to all these properties can very easily, we proved for this definition and give beautiful applications of calculus basically.

(Refer Slide Time: 30:27)



So, so one observes that the graph of this function, ln of 1 is 0, so the graph should look like this. It never meets the horizontal and never there is no asymptotes of, this is asymptote but there is no horizontal asymptote it keeps on increasing. So, on this side as you go this side the function keeps on going up.

So, as a consequence of this ln x is a bi injective function from 0 to infinity to R which is differentiable. So, it should have an inverse function and what is that inverse function it is a familiar exponential function E is to power x. So, that is the precise definition of what is exponential function it comes from the definition of ln of x.

So, exponential is the inverse of the bijective function ln of x and one can use to show that because this function is ln x is differentiable function, universe function also is differentiable and derivative is 1 over the derivative of the original function. So, you get exponential is a function which is differentiable with derivative itself.

Derivative of E is to power x is, E raise to power x all this come because of this Fundamental Theorem of Calculus, so you call this as ln of natural base E, then you can define ln of other bases and so on. So that is definition of what is called transcendental functions and one of the transcendental functions is log and the exponential function.

Same way you can try to define what is trigonometric function sin we can try to apply the same kind of trick. What is the derivative of sine? Cos, but I do not know, cos, derivative of cos is sin so I am stuck. But what you do is to define sin it is good enough to define sign inverse. I can define sin inverse in a appropriate interval, say minus Pi by 2 to Pi by 2. There sin is, sin of minus, it goes the value, it is a one-one function.

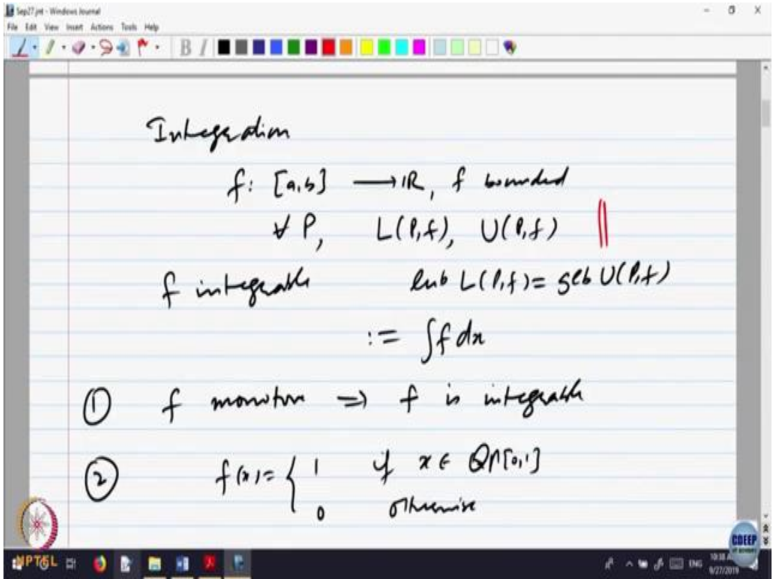
So, if I can define sin inverse, I can define what is sin as a inverse of, like here we are doing exponential is a inverse of log. So, sin inverse what is the derivative of sin inverse.

Student: (())(33:15)

1 over 1 minus x square, so that probably we can integrate. So, one tries to integrate that to get sin inverse and show it as all the nice properties and then inverse of that is taken as the sign derivative of sin is taken as cos and all those properties are proved. So, that is a definition way one uses rigorously fundamental theorem of calculus to define functions which cannot be defined algebraically.

So, that is just, I am just giving you a glimpse. If you feel interested, you should try to read some book. If you want to know a reference come and ask me I will tell you the reference where you can read these things. So, basically, we have tried to define integral.

(Refer Slide Time: 34:16)



So, let me just go back and look at we looked at integration via upper sums and lower sums. There is another way of defining this integral, so basically what we did was intuitively we try to capture the area below the graph of the function in lower and upper sums. But and we had to have the condition that if you want to define a upper sum, you want to define lower sum then your function must be a bounded function. Otherwise the infimum and supremum does not make sense at all.