Basic Real Analysis
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Lecture 40
Riemann Integration Part 4

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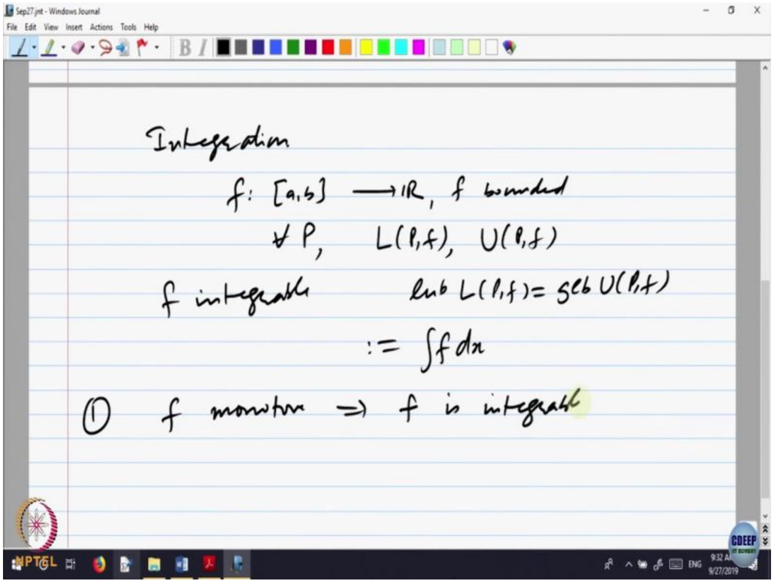

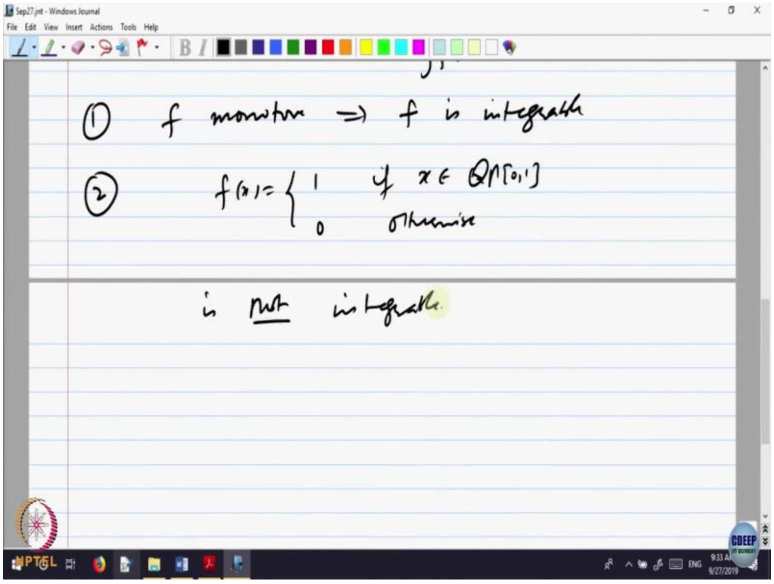

So, let us recall what we had done last time we have started looking at integration. So, given a function f on a interval a b to R, f bounded if you defined the integral of f via upper sums and lower sums, so we defined for every partition p we defined what is a lower sum? What is the upper sum and we said f integrable if and only or integrable definition if the supremum of least upper bounds of L lower sums is equal to latest lower bound of upper sums and that was denoted by integral f dx. So, we gave some examples f monotone implies f is integrable.

Then we gave some more examples, namely, the function f of x is equal to 1 if x belongs to rational say n 0, 1 and 0 otherwise is not because upper sum is always 1 lower sum is always 0 is not integrable.

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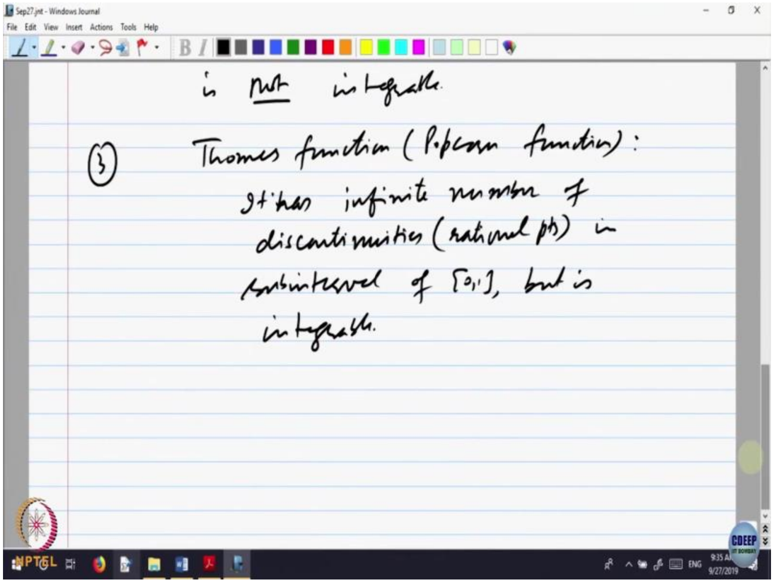

We also gave an example of Thomas function or called the popcorn function. So, this function has the special property namely, it has infinite number of discontinuities. In fact, this discontinuities are all rational points in every sub interval of 01 but is integrable. So, there is various examples of integrable functions, keeping in mind the number of discontinuities the function can have. So, as we were pointing out that this function is discontinuous everywhere monotone is discontinuous at all rational, accountable at the most countably many and Thomas function as infinite number of discontinuities in every sub interval and is integrable.

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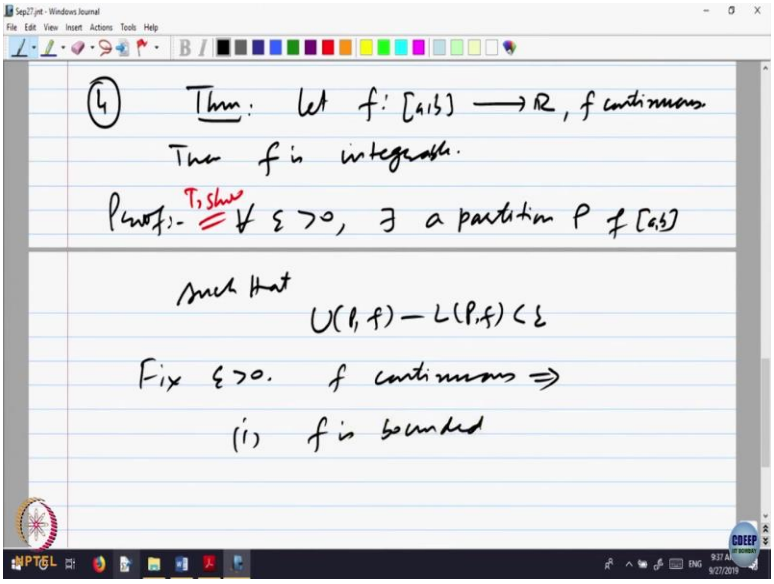

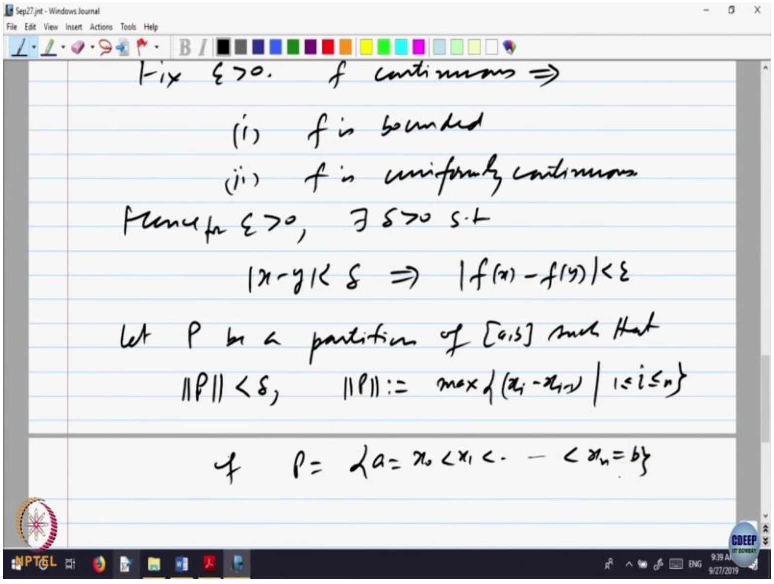

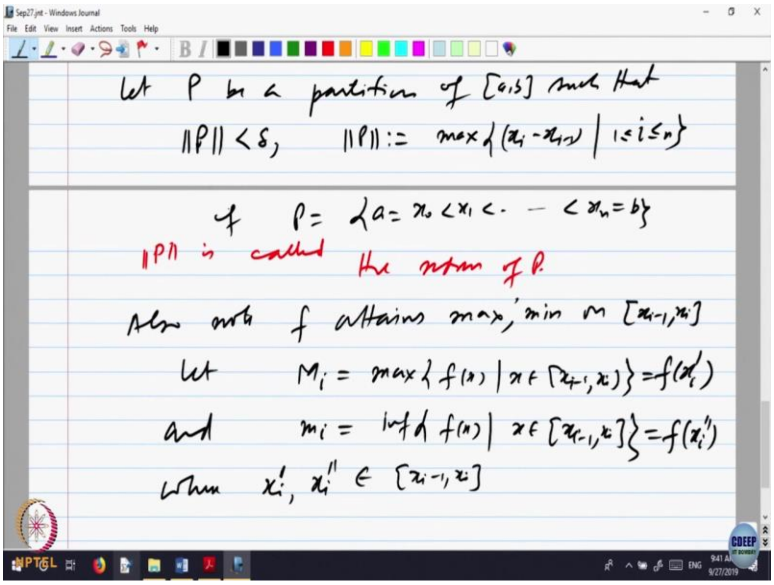

So, let us prove a important class of functions which is integrable namely. So, theorem let f be a, b to R f continuous, so let f be a continuous function on an interval, close bounded interval a b, then f is integrable, so let us prove that. So, basically what we want to show it is integrable we are not interested in computing the integral. So, we will use the criteria that given any epsilon, so we will use a criteria that for every epsilon bigger than 0 there exist a partition P of a b such that upper sum minus the lower sum is less than epsilon.

So, this is we will show so, this is what we will show and we had observed that this is a necessary and sufficient condition for a function to be Riemann integrable. So, let us start with an epsilon of fix. So fix, what does continuity imply? f continuous implies 2 things one f is bounded, in fact it attains maxima minima, we know that and second, we proved that f is uniformly continuous. Every continuous function on a closed bounded interval is uniformly continuous.

So what does uniform continuity say? So hence we are already given an epsilon so given epsilon, so for epsilon that is 0 there is a delta such that whenever x minus y is less than delta that implies fx minus fy is less than an epsilon, whenever 2 points are close, their values are close. That is what uniform continuity says. So, mathematically that says that for any epsilon bigger than 0 there is a delta so that whenever x and y are close by a distance delta the values are close by epsilon, so that is quite useful in the sense that now we can select a partition, so that the points are always inside distance delta.

So, choose a partition, so let p be a partition of a b such that, so it divides into sub intervals such that… I did not introduce this notion is less than delta. So, what is the norm of the partition? Norm of P a partition is defined as a maximum, maximum of the length of sub intervals, so xi minus xi minus 1, I between 1 and n if P is the partition say a equal to x 0 less than x1 less than x and equal to b.

Here definition of length, definition of the norm of a partition. So, every partition divides the interval AB into sub intervals. So, look at the maximum length of the sub intervals and that maximum length is called the norm. So, this is what is called so, this is called the norm of the partition.

So, uniform continuity says given epsilon there is a delta say that whenever two points are close fx minus fy is close So, if we choose a partition whose maximum whose norm is less than delta, then that will mean that for any two points in a sub interval, the length will be less than the distance will be less than delta and hence the values will be closed by epsilon. So, now also observe also note f attains maximum and minimum on each sub interval because it is continuous function.

So, let capital Mi which is the maximum of f of x we attained at some point to let us call it as x belonging to xi minus 1, 2 x i let us say this value is attained at some point say xi dash and where xi dash is a point in between and small mi is the infimum of fx x belonging to xi minus 1 to xi and let that be attained at some point say xi double dash where xi dash and xi double dash belong to.

So, just saying that f is a continuous function. So, look at this restriction on the closed bounded interval xi minus 1 to xi, it must have a maximum value somewhere in that interval it should have a minimum value in that interval and must be attained. So, those points we are calling as this.

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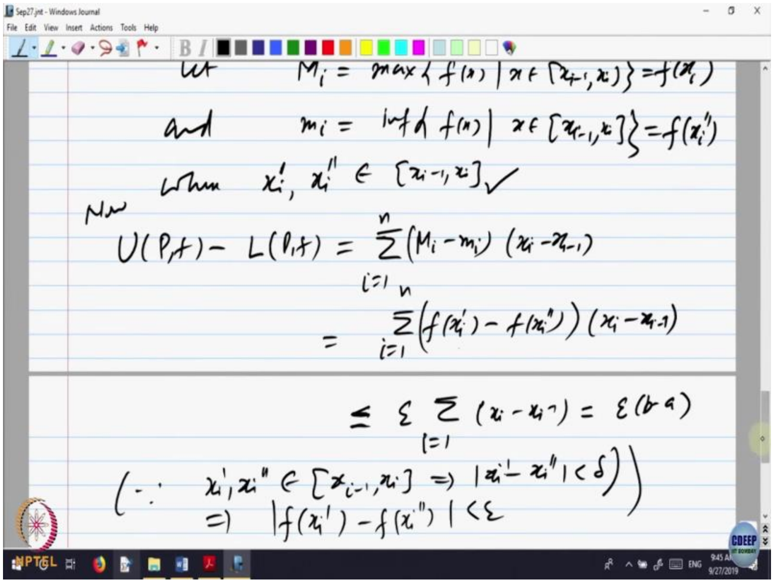

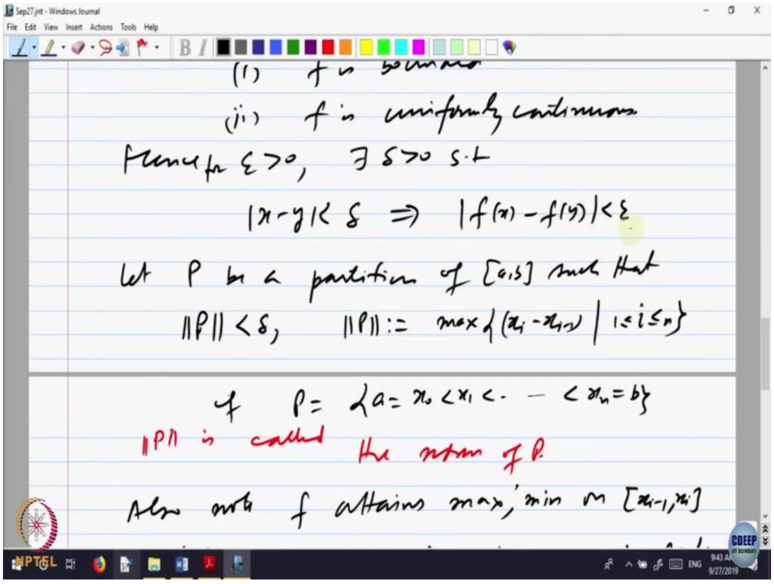

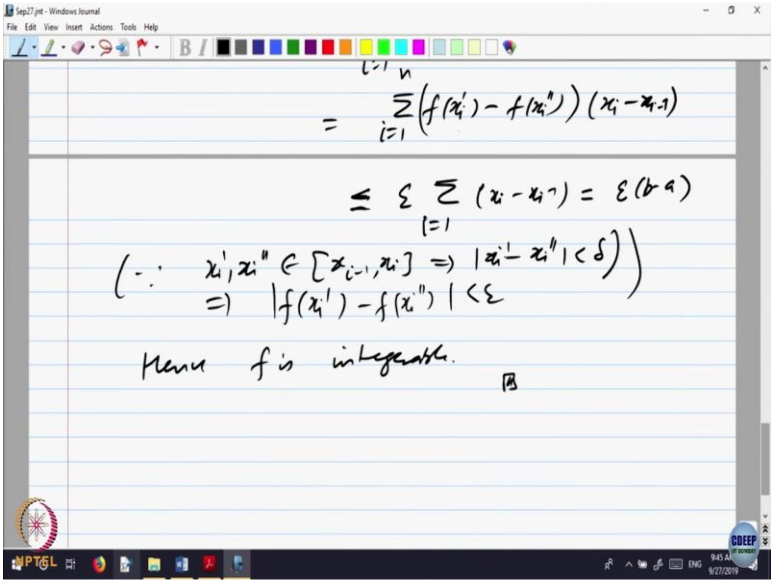

So, why we are doing all that is because now let us look at the upper sum with respect to f minus the lower some with respect to f. So, what is that, so, that is equal to capital Mi minus small mi 1 to n maximum minus the minimum into the length of the interval xi minus xi minus 1.

Now, this value is taken at that point, so this is i equal to 1 to n so, f of xi dash minus f of xi double dash in to the length of the interval. So, that was and now these two points xi dash and xi double dash are inside this interval and so that means the distance between them is less than delta because norm of the partition p is less than delta and whenever that happens, we know that by uniform continuity these values are small.

So, that is all the reason we did that. So, let us imply so, this is less than or equal to delta time sigma i equal to 1 to less than delta so, this is less than epsilon times xi minus xi minus 1 because xi dash, xi double dash both belong to the interval xi minus 1 to xi implying that the distance between them is less than delta and that should be okay to say that implying that f of xi dash minus f of xi double dash is less than epsilon by uniform concluding. So, what we are done is the interval a, b we have found a partition saying that the distance between any two points in the sub interval is less than delta and that delta is corresponding to the uniform continuity of the function.

So, the values for any two points will be less than epsilon. So, all that is used to bring it here. So, what is this quantity? So, consecutive terms will cancel out so this is epsilon times b minus a. So, given epsilon bigger than 0, we have found a partition p, say that the upper sum minus the lower sum is less than epsilon times v minus a.

So, constant time something does not matter, we could have started with epsilon divided by b minus a, so hence f is integrable. So, that proves to the theorem that every continuous function is integrable and historically this was first approved by Cauchy you will find the Cauchy coming in your various courses. He was a French mathematician who contributed a lot of things in a lot of branches in mathematics.

Real analysis, complex analysis, algebra, statistics you will have Cauchy distribution coming, statistics and probability and here is the first one who proved this theorem gave a proof that every continuous one, in fact, he was the one who actually defined rigorously, what is the notion of a function being integrable and he, that was a time when mathematicians did not bother much about continuity or discontinuity, they thought every function is continuous kind of thing. So, he assumed not only continuity, he even assumed the fact that we are using that it is uniformly continuous.

So, that was only I will not say mistake, but there is oversight, that what we call now as uniform continuity he thought that is continuity at that time and he gave a rigorous proof of this. So, the class of integrable functions is quite large it includes all continuous functions on the interval a, b. Now, the problem is, so we had started looking at the two questions that what is the class of integrable function, so we have given a lot of examples and…

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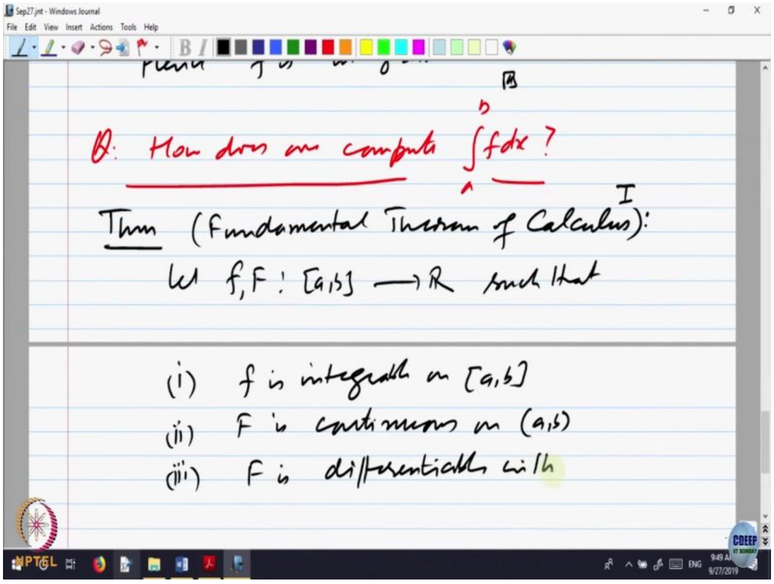

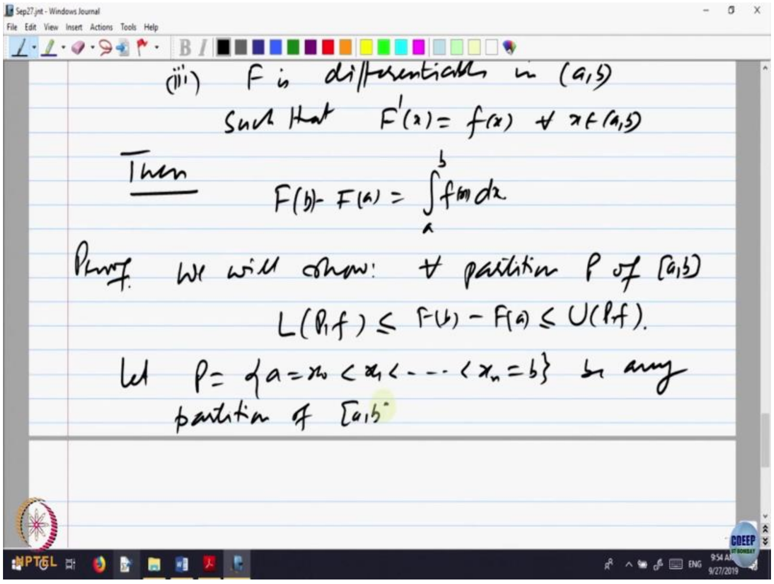

Now, second problem is how does one compute integral. So, let us look answer that question, how does compute fdx a to b? Can we have a way of computing for a given function, what is that integral? So, there comes a theorem which is very important, and that is why this integral becomes important that is called the fundamental theorem of calculus. So, let us state and prove what is called Fundamental Theorem of Calculus. So theorem, let me call this as part one.

There are two parts of this, so it says let f and capital F be two functions on define in the interval a b to R with the following properties, one f is integrable on a, b. Capital F is continuous on at least the open interval a, b and f is differentiable with, continuous in the closed interval I am sorry, we should say f is continuous in the closed interval and is differentiable at least in the open interval and differentiable capital F is continuous and differentiable in interval a b such that is derivative is the function f of x.

So, small f and capital F are related with each other in the following way that the derivative of capital F is equal to small f and the interval open interval a b and of course, we are putting conditions like small f is integrable, capital F is continuous on the interval a, b so it will be integrable then, if all these conditions are satisfied, then f b minus f of a is equal to integral a to b fx dx. So, the function capital F and small f are related with each other by this equation.

That if small f is the derivative, so look at this, so this integral, if small f is the derivative of capital F, then integral of small f is just f b minus f of a, that means what, there is a big advantage that for a given function, if you are able to find that it is a derivative of something, then integral of that function is just f b minus f of a, you do not have to go to partitions, you do not have to go to limits or anything. So, this is and this was again approved by Cauchy rigorously and historically, it was a very significant theorem. At least it lets differentiable functions derivative with the integral.

So, in that sense, people started looking at derivative as an integral as the processes of reverse of each other kind of thing, there is a reason because of this theorem, but the importance of this theorem lies in the fact that it helps you to compute integrals? If we are able to recognize that small f is the derivative of a function capital F and then the integral is… So, let us prove this theorem, give a proof. So, we want to compute f of b minus f of a and we know what is f of b minus f of a and we want to show it as an integral.

So, essentially that means, so that means essentially that this is a number we should be trying to show, lies between the upper and the lower sum whatever be the partition, we can show that for every partition P of the interval a b, f b minus f of a is in between then that must be the integral.

So, that is what we are trying to show. So, we will show for every partition, P of a b, lower sum is less than or equal to F b minus F of a is less than or equal to the upper sum or that is enough to show that because this is only number between upper and lower, they are shrinking. So this number must be equal to the integral of the function. So, let us start with a partition. So let P be a partition and so let us say a is equal to x 0, x n equal to b, b any partition of a b.

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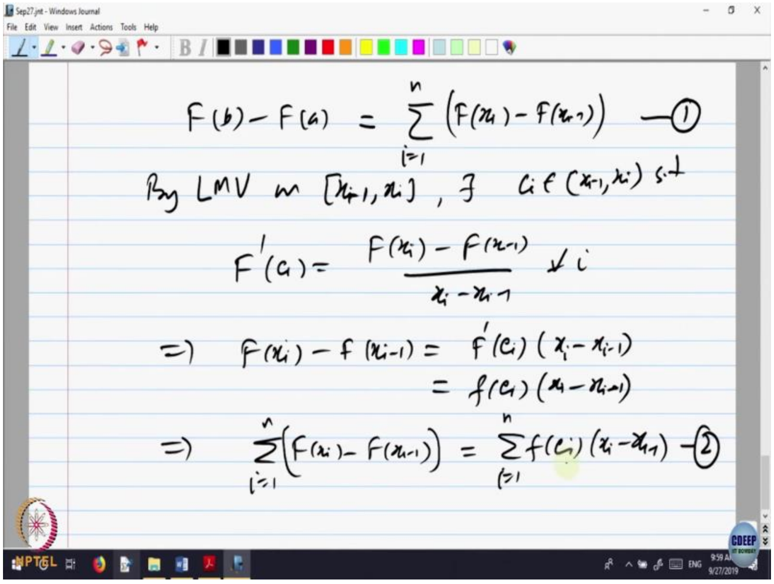

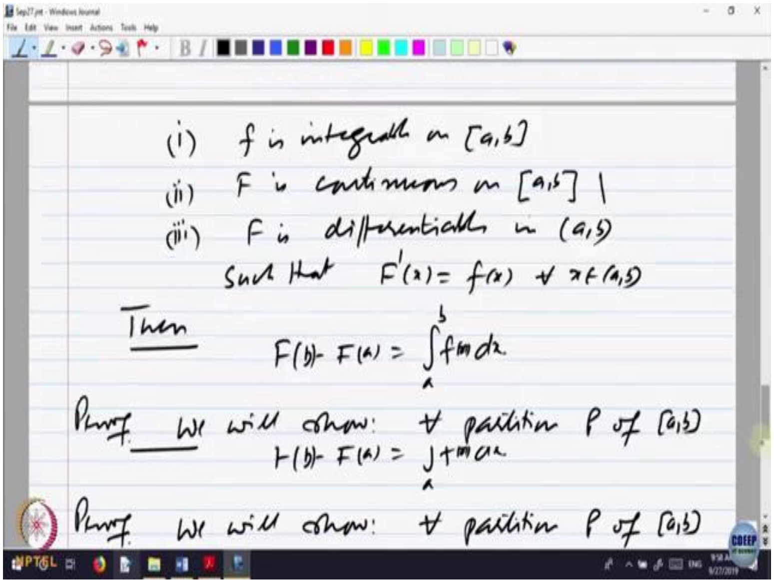

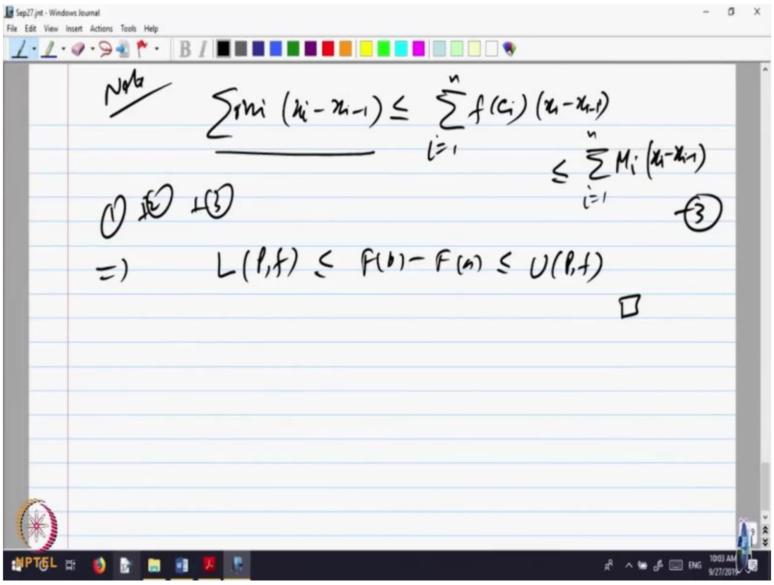

So now let us (())(23:38) what we are trying to do is I want to bring in inside the upper and the lower sums so I had to go to the upper and the lower sums with respect to the partition P somehow I have to steer my arguments towards that idea. I should bring in this, so now look at F b minus F of a this is what we want to compute and now well this is how the partition comes in the same as f of x i minus f of x i minus 1 i equal to 1 to n. I add and subtract the consecutive values of f and the consecutive points so, now partition points are inside now. So, let us call this as 1.

Now, what is now the next step should be that this partition this f of x i minus f of x i minus 1 should be related with small f then you will get some summation kind of thing in sums of small f now, so and small f is a derivative of capital F. So, what is the, which is the theorem which relates a function with the values of the derivatives.

So, that is all 1 thinks about. So f b minus f of a in terms of the derivative and that is Lagrange's mean value theorem. So, by Lagrange's Mean Value Theorem on xi minus 1, to xi, there is a point ci belong into xi minus 1 to xi such that F dash at ci is equal to F at xi minus 1, F at xi minus F at xi minus 1 divided by xi minus xi minus 1 for every i.

So, once that is a case that means what? So, that implies that F of xi minus, F of xi minus 1 is equal to F dash of ci at and the length of the interval. So, that is Lagrange's Mean Value Theorem essentially see how Lagrange's Mean Value Theorem is playing a part at all these places and this derivative of small f capital F a small f.

So, it is f of ci xi minus xi minus 1. Now, keep in mind I said apply Lagrange's Mean Value Theorem on this interval one should check the conditions of Lagrange's Mean Value Theorem are applicable capital F is continuous everywhere, so it is continuous on the interval xi minus 1 to xi, it is differentiable in the open interval.

So, it will be differentiable in the open intervals xi minus 1 to xi each one of them. So, Lagrange's Mean Value Theorem is applicable for each closed bounded interval xi minus 1 to xi, so that is why those conditions were put both of these ones and now, so let us we want to go to 1 we need summation.

So, implies summation of xi minus f of xi minus 1 equal to 1 to n is equal to sigma f of ci i equal to 1 to n xi minus xi minus 1 and that by 1 is equal to F b minus F of a. So, let us call this as 2. So, what we have done is, we have computed F b minus F of a in terms of the function small f it says this summation is equal to f at some point ci, ci is a point in that interval into the length of the interval.

So now, what is F of ci that is the value of the function at the point ci. So, it will always bigger than or equal to the minimum value and less than or equal to the maximum value. So, using that implies that if I look at the minimum value small mi xi minus xi minus 1 will be less than or equal to summation f of ci.

So, this is not imply note, this is observation minus xi minus 1 equal to 1 to n and that is less than equal to capital M, I put summation, so let me put summation everywhere either I put summation in between, so less than or equal to summation capital Mi of xi minus xi minus 1. Because there is a value at some point in between in the interval big, so f of, small f of ca is bigger than mi is less than capital Mi, so now just combine what is that in between quantity 1 and 2 say that is F b minus F of a.

So, 1 plus 2 plus you can call it as 3 if you like imply that, so this is a lower sum is less than or equal to F b minus F of a is less than or equal to the upper sum, because this thing is a lower sum, left hand side in 3 this is lower sum, this is the upper sum. So, lower sum is less than this summation and that summation is by 2 equal to sigma of capital Fi and that by 1 is F b minus F of a.

So, putting these 3 equations together we get this and that completes the proof. So, this is one of the important theorems in integral or in all of calculus or analysis as well as differentiation and integrations are concerned. It relates differentiation and integration in the way that if you know that the derivative of a function then you can compute the integral of that function.

So, as a consequence of this, all you get every derivative formula gives you a integral formula. derivative of sin is cos, so we integrate cos we must get back sin as integrals of. derivative of any function, any derivative formula will give you a corresponding and that is one of the reasons that as soon as differentiation is done in undergraduate courses and integration is started immediately 500 formulas appear and you are sort of supposed to remember them and start computing integrals.

The culprit is fundamental theorem of calculus and historically this was very important, because not only it said that for every continuous function you can have the integral and if you know that this function is the derivative of something, then you can compute the integral and that led to a lot of research in what is called theory of Fourier series. So, it gets related historically with the theory of Fourier series.

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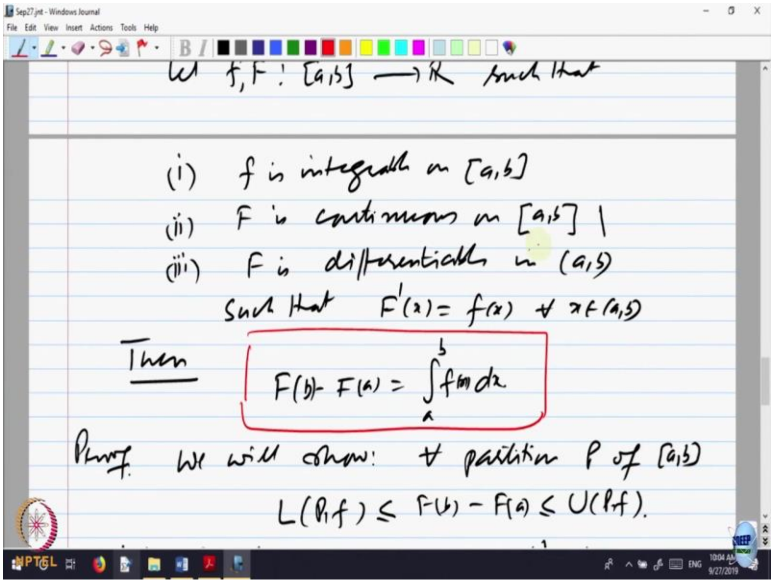

So, let me not go into that so, basically the important thing is as far as computation is concerned, so this is what is important, you can compute integral of f, if we know that keep in mind F dash is equal to small f so automatically says small f is continuous because capital F is continuous, small f is differentiable.