Basic Real Analysis
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Lecture 38
Riemann Integration – Part II

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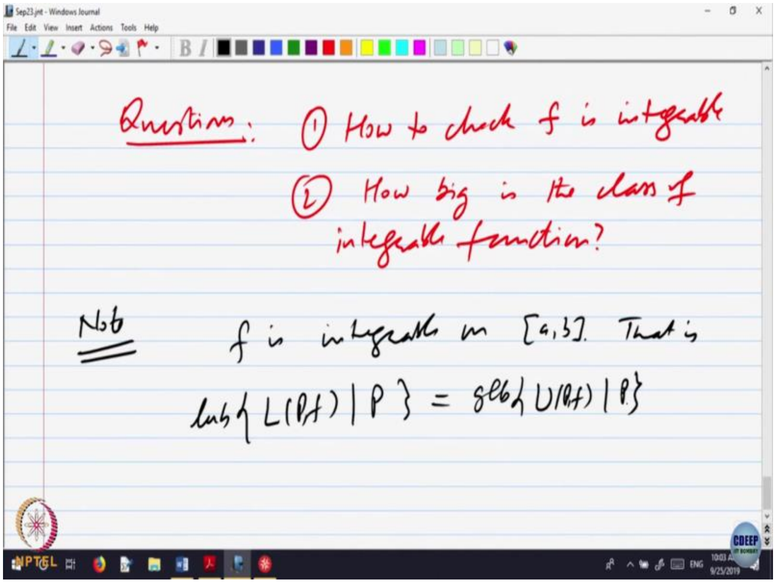

So let us try to answer these questions, so, let us note supposing f is integrable, that means that is the least upper bound of L P, f over P is equal to greatest lower bound of upper sums with respect to P, the partitions. So, given a partition P, what is the difference between upper and the lower sums? If f is integrable, then what is the difference? How big is the difference? If these two qualities are same, we should also say that the difference between the upper and the lower can be made as small as you want.

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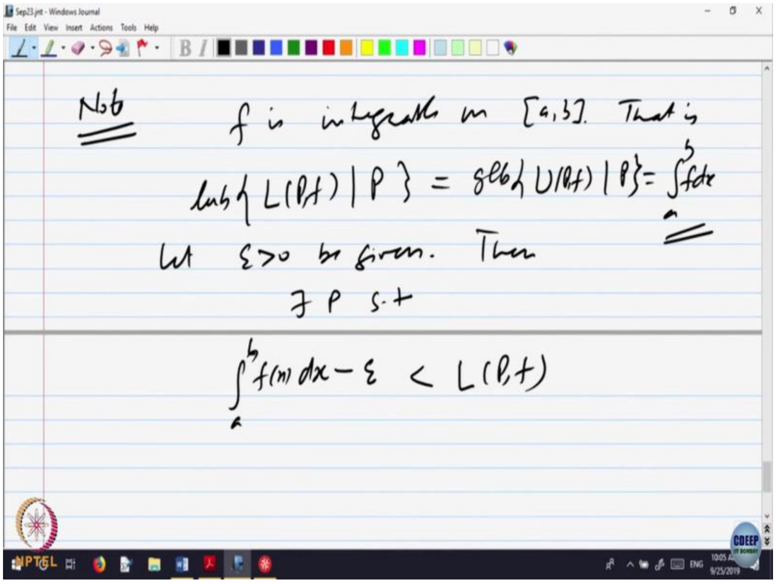

So, let us write this as, so let be given then, so this common value we agreed to denote it by integral f d x, that was a common value. Now, if this is the least upper bound, that means what? Look at a to b then a to b, f x, d x, these are least upper bound, so least upper bound it is a sort of the largest value, nothing smaller will work, it is a upper bound and nothing smaller will work.

Let us look at minus epsilon, this cannot be an upper bound for the set, that means what? That means what? then there should at least a partition P such that this should be this cannot be the upper bound that means there has to be an element on the right side of it. So, it has to be less than L P, f there is a partition, so that this happens, definition of least upper bound.

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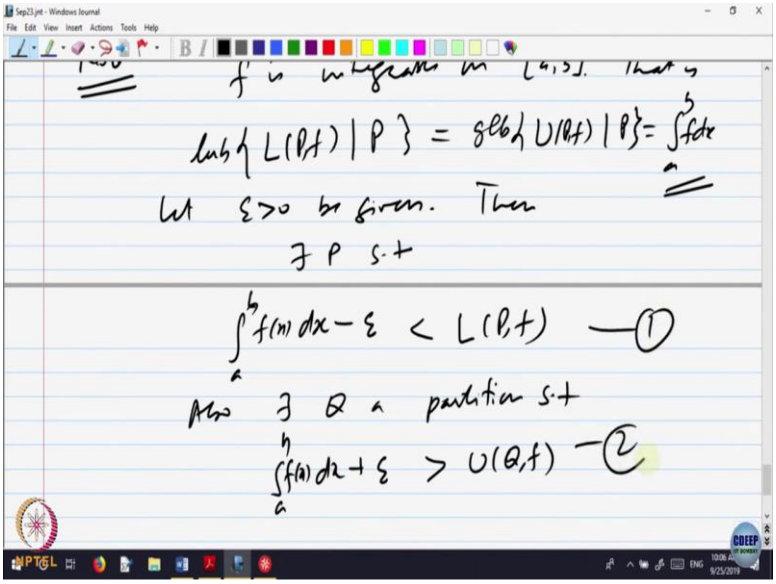

And similarly, also with respect to the other one there exists a partition Q a partition such that if I look at the integral that is a greatest lower bound plus cannot be, that way there has to be an element on the left side of it. So, has to be bigger than U Q f, is that okay? So, let us call this as 1, call this as 2, so this must happen.

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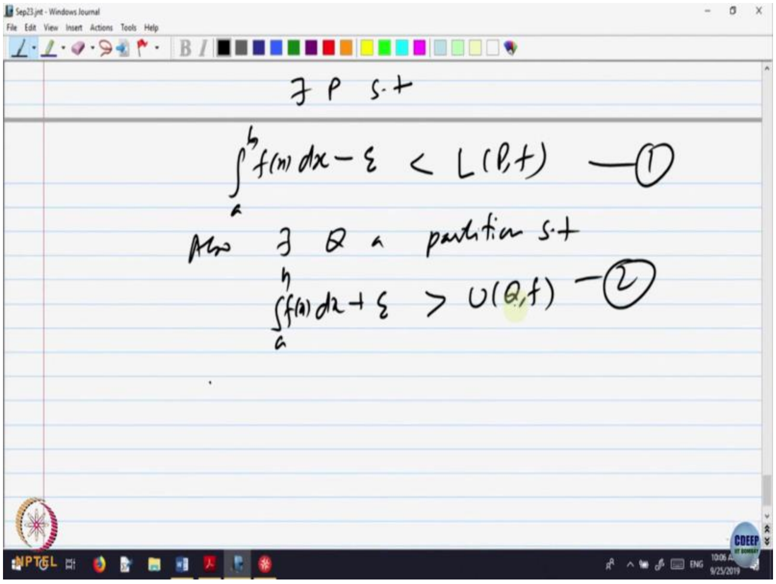

Now from here, I want to find out, this is a partition P this is a partition Q, these two may be different partitions, but I want to bring them to a common partition. So, how do I bring them to a common partition? Given two partitions P and Q, I want to have a partition, which is common to both common to both in the sense that the new partition should be a refinement of both of the original ones. So, the simplest thing is put all the points of P and Q together in one partition.

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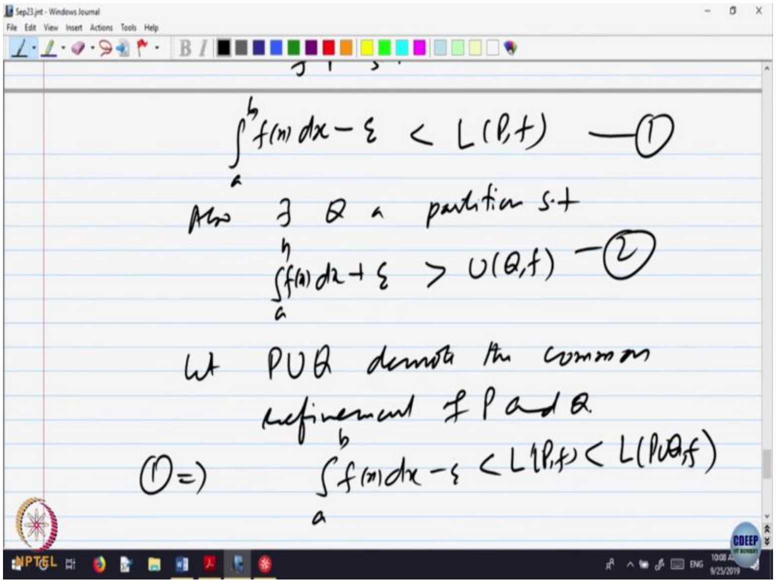

So let, P union Q denote the common refinement of P and Q. Then what happens when you refine, when you refine lower sums tend to when you refine lower sum tend to increase. So, from 1 integral a to b, f x d x which was less than this minus epsilon less than L P, f which is less than L P union Q f, because when you refine you get something more, were the lower sums.

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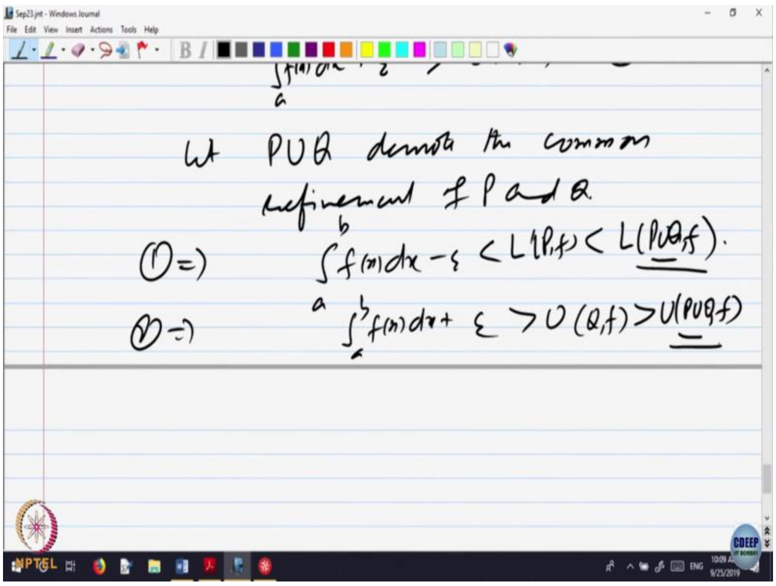

And similarly two implies integral a to b, f x, d x plus epsilon will be greater than the upper sum with respect to Q, which is bigger because upper sums tend to decrease with the refinement. Upper sum P union Q of f. Now, from these two now, partitions are same P union Q, P union Q and I know that the lower sum is always less than or equal to the upper sum for a given partition, the lower sum is always less than or equal to the upper sum.

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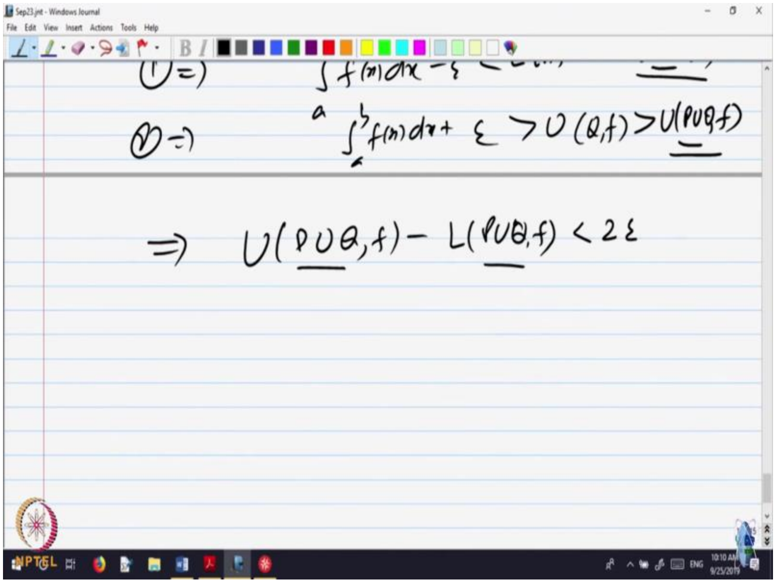

So, from these two, what do I get? Upper sums P union Q, f minus L P union Q f, from this what you will get? Can I say something is less than a same quantity minus epsilon the same quantity minus plus epsilon. So, this is it is less than 2 epsilon. Add and subtract epsilon if you want or if you want you can calculate, this plus epsilon, this minus epsilon, less than or equal to 2 epsilon, simple thing. So, that means what? What we have shown is, if f is integrable, so what we are saying is if f is integral, then there is a partition such that the difference between the upper and the lower must be less than epsilon for any given epsilon. And that is intuitively what we wanted actually.

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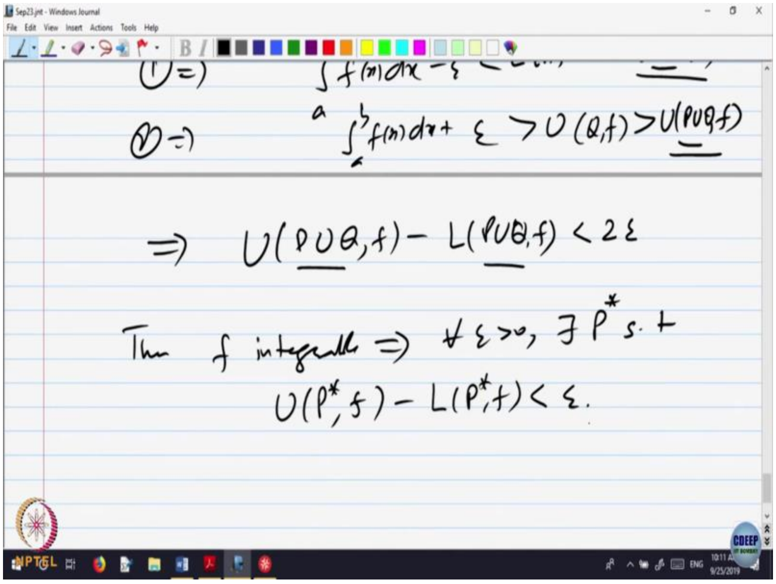

So it says so, thus f integrable implies for every epsilon bigger than 0 there is a partition P such that there is a partition, do not confuse with that same P, there is the partition what shall I call? P star, such that upper sums with respect to P star minus the lower sum with respect to P star is less than epsilon. So, this is one way of testing, whether a function is integrable or not, if integrable then I should be able to do that. Can I say the converse is also true?

Supposing f is a function such that for every epsilon there is a partition with this property. That means what? Lower is always less than or equal to upper and says the difference between upper and the lower is less than epsilon, so what happens to the when it goes on increasing, the lower ones, so, least upper bound, greatest lower bound, the difference I can make it small for any epsilon that means they must be equal.

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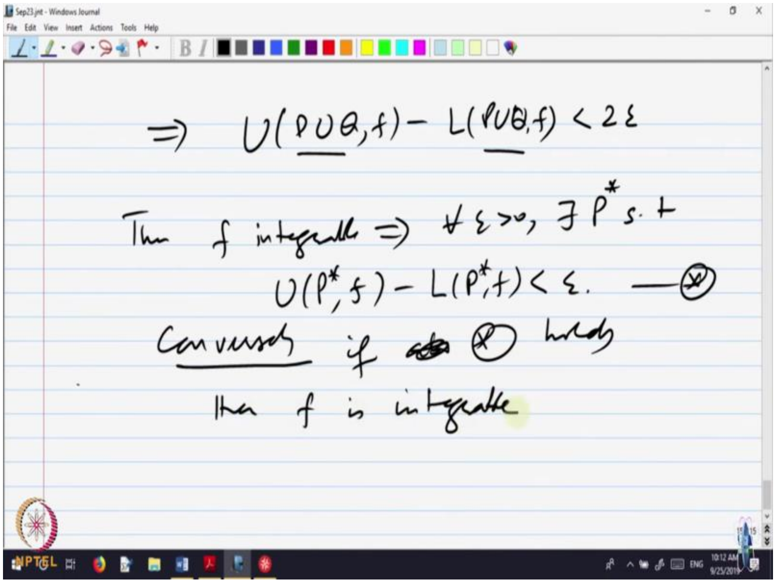

So, conversely, so converse also holds conversely if above holds, so if star holds then f is integrable, is that okay for everybody? If the difference between the upper can be made as small as you want, L P f is always less than U P f. So, the greatest least upper bound of L should be equal to the greatest lower bound of U that means a function is integrable. So, this is one way of checking whether a function is integrable or not.

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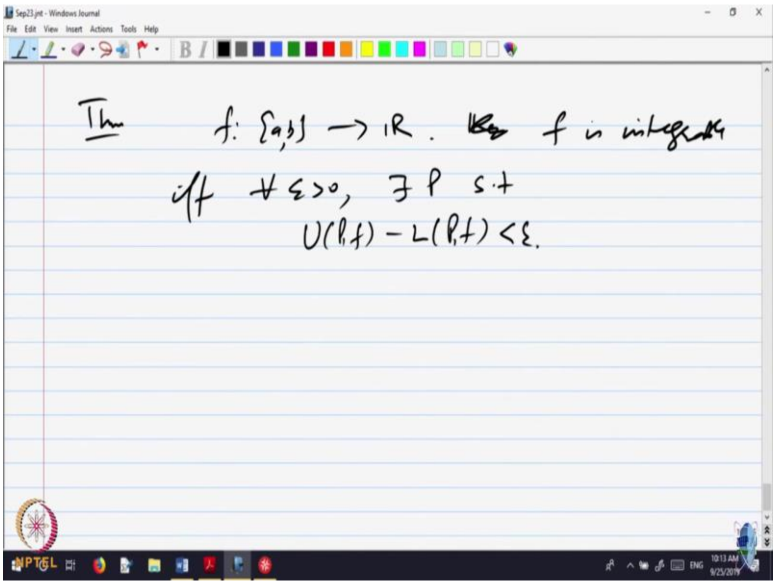

So, the theorem says we did not write it is as a theorem, so let us write, the theorem says f is a function from a, b to R, then f, f is integrable if and only if, for every epsilon bigger than 0 there is a partition P, such that U P, f minus L P f is less than epsilon. See keep in mind we are not trying to find what is that common value, this only says this is a way of checking whether something is integrable or not.

Go back to something a sequence is convergent if and only if Cauchy so, limit exists if and only if it is Cauchy, it is something like that checking the limit will exists for a Cauchy sequence for a sequence you can check just check it is Cauchy, if you are not interested in finding the limit. Same here to check whether a function is integrable or not we have to just make the difference to be small, as small as you can that is good enough.

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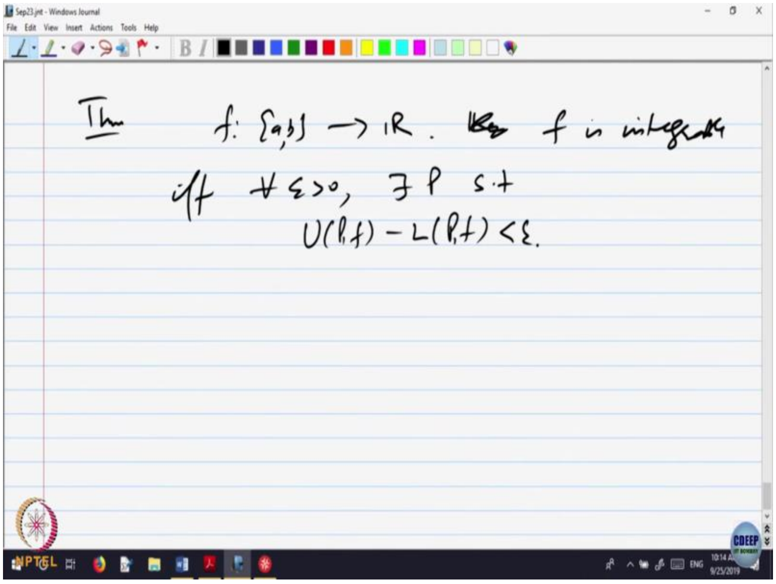

So, let us let us look at another one. Now, supposing another way of looking at the same thing would be, supposing I can find some real number alpha, which lies between all upper sums and all lower sums. Whenever, I give any partition and construct the lower sum, construct the upper sum I know the lower sum is always less than equal to the upper sum, but somehow I am able to guess a number alpha such that the lower sum is less than or equal to alpha is less than or equal to the upper sum for every partition.

Then what that alpha should be? That should be the integral, that should be the common value, because I can make them small and small.

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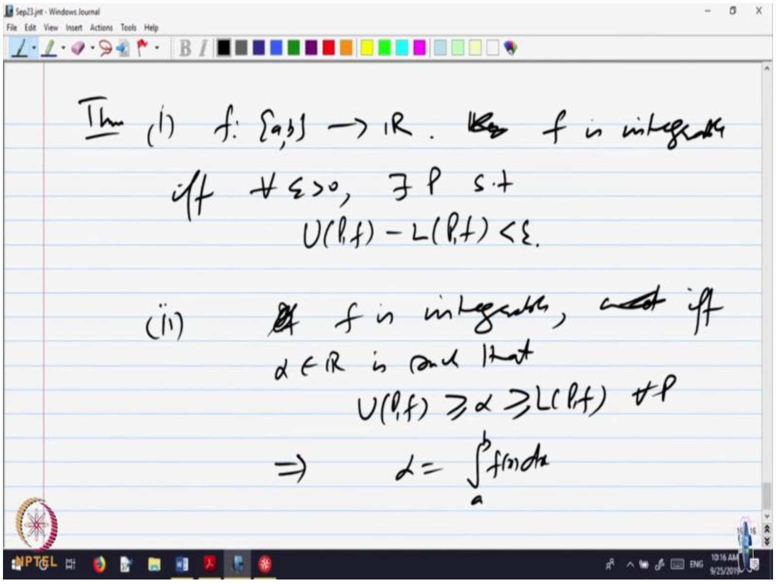

So that is a second part. So let us call it one, so I would like you to write down if f is integrable and alpha belonging to R is such that upper sum is bigger than or equal to alpha is less than lower sum for every P then that implies this alpha is equal to integral a to b f x and d x. Actually, you can say if and only if converse is also obvious.

Namely that f is integrable if and only if this happens, because if this nothing such a thing alpha happens and the difference between upper and the lower will be small. So, this is the way you check whether a function is integral or not let us look at some examples before we go to computing.

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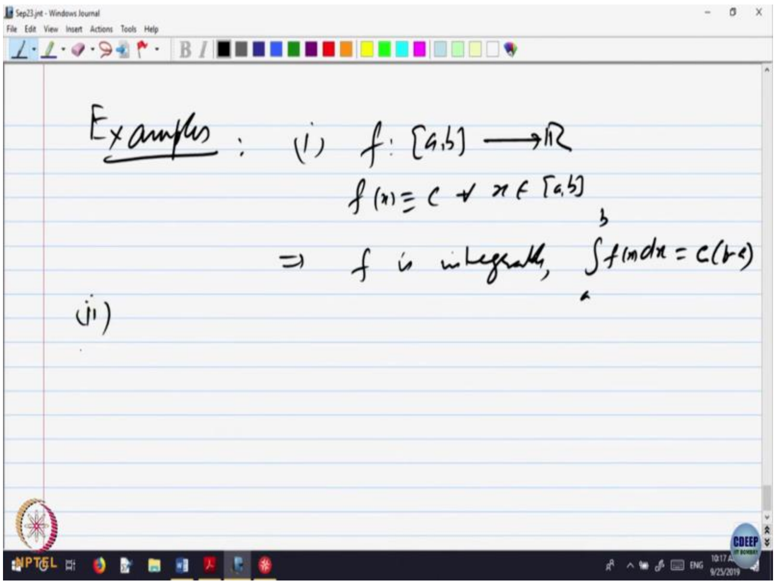

So, let us look at some examples, let us look at a function f from a, b to R, f of x is a constant is a constant function, then what is the lower sum? c times b minus a, upper sums is also b minus c times b minus a, whatever be the partition, the upper sum is same as the lower sum, that is c times b minus a. So, that implies f, so, is integrable, an integral of f x, d x is equal to c times b minus a.

So, this set of intuitively tells me that for rectilinear figure the graph of the constant function is straight line, so the graph below that, the area below that is a rectangle actually. So, area of the rectangle with height as c length as base as b minus a should be this. So, the integral is actually making sense that it is a area below the graph of the function in that sense.

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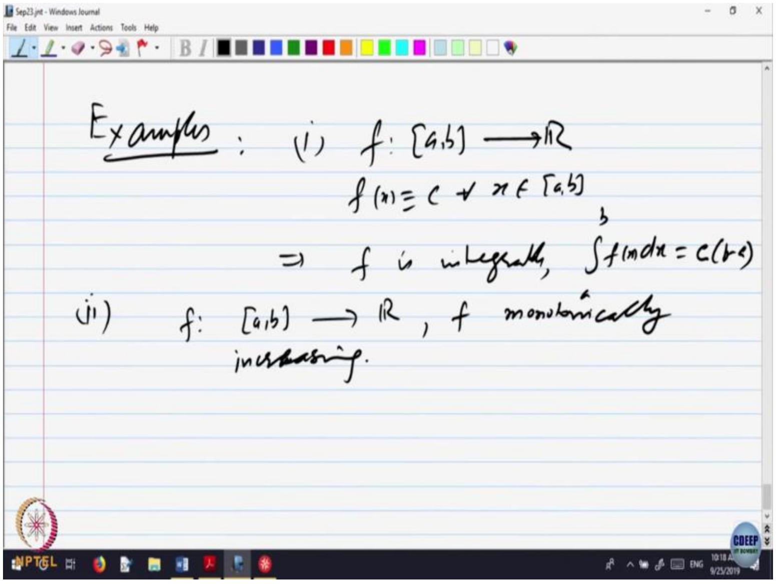

So, let us look at f another thing, so, let us look at f monotonically increasing is a monotonically increasing function. Now, I want to find out what is the upper sum, and what is the lower sum? And can I make it small? So, how do I do that, is monotonically increasing. So, what is the minimum value of the function in any interval if it is monotonically increasing? The minimum value will be the value at the left endpoint of that interval, maximum it will be the value at the right endpoint.

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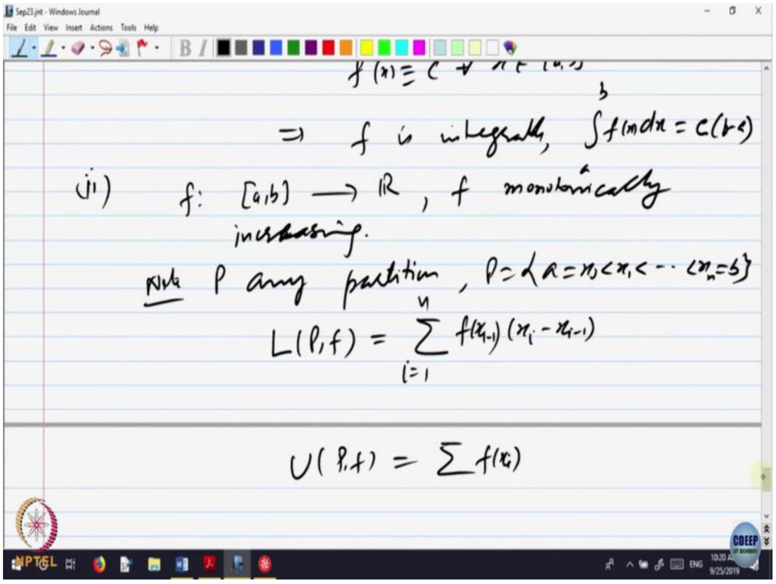

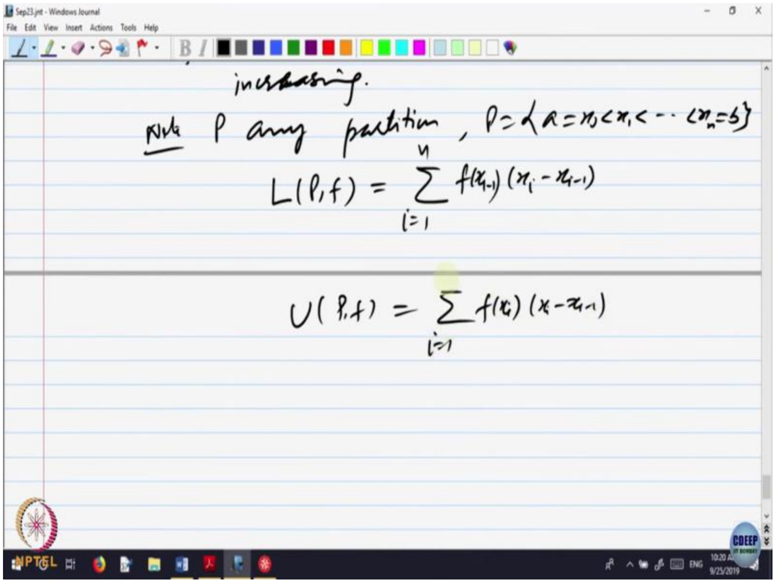

So, let us write for any partition P not for P any partition, what is a lower sum? So, let us say P is the partition which is a equal to x 0 less than x 1, so that is equal to sigma, so minimum value, so, at the interval x i minus x i minus 1 that is a length, so f of x i or lower sum to left endpoint. So, I should be writing as your endpoint, this is for the interval is x i minus 1 and what is upper sums? So, that is sigma f of x i, x i minus x i minus 1, so that will be i equal to 1 to n.

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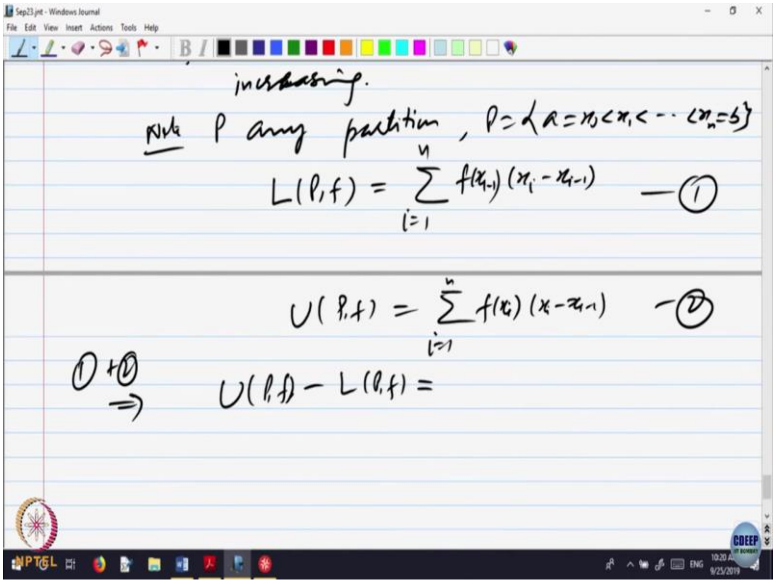

So, what is the difference between the upper and the lower? This minus this, so what is that equal to? So, that is, so from 1 and 2, I cannot compute exactly. So, let us do one thing, let us specialize the partition, so that I know what is x i.

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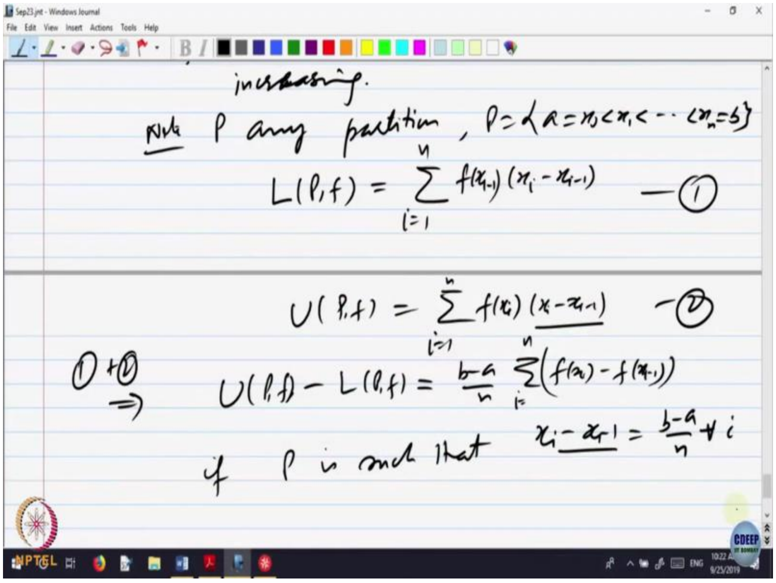

So, let imply if the partition P, see this is a partition, let us say that what we are doing is, we are choosing the partition a, b such that the length of each sub interval is b minus a divided by n, that means what? I am dividing the interval a to b into n equal parts. So, that means P if P is such that x i minus x i minus 1 is equal to b minus a divided by n, for every i. Then what will be this? Then what will be this difference equal to? What will be the difference equal to? Each one is b minus a divided by n, so that comes out, the length of the sub interval comes out, so it is b minus a divided by n into sigma f of x i. So, that is, what is sigma of f i? The consecutively they will cancel or whatever it is, let us keep it as it is does not matter, f of x i minus f of x i minus 1, i equal to 1 to n.

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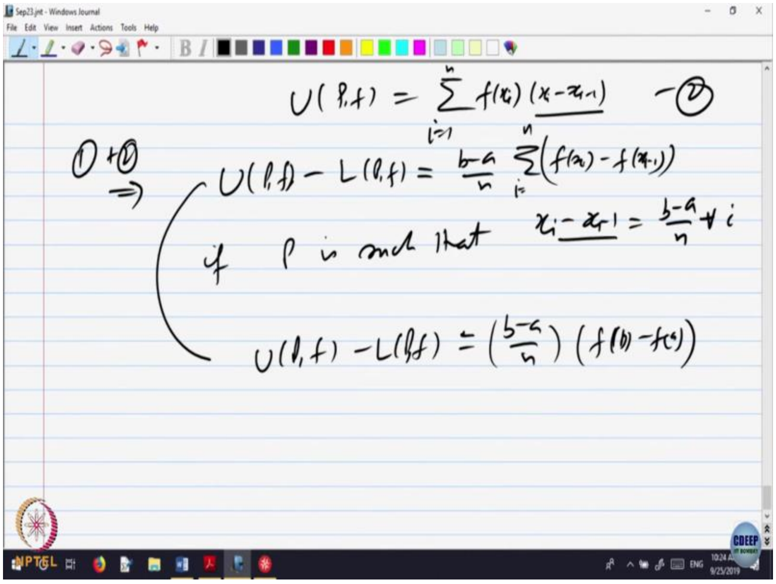

Now, let me take it slightly further from here. So, that is same as saying U P, f minus L P, f is equal to b minus a divided by n. Now, this I want to remove, the sum I want to remove, I want to make it a constant, f of x i, so, what is the largest value of f? If you like is a value at the end point b.

So, I can say this quantity is less than or equal to f of b plus f of b twice So, I am saying that this quantity is, if you like, 2 times maximum value, what should I write? Does it cancel out? Yes, what cancels, I do not have to do anything actually. So what is there sum? f at the end point that is b and the last endpoint, f of b minus f of a, consecutive terms will cancel out, so I do not have to do anything.

Now, why I am doing all this? It says, if I choose this particular partition P of dividing the interval into n equal parts, the difference between the upper and lower, 1 over n is coming here. So I can make that this quantity as small as I want.

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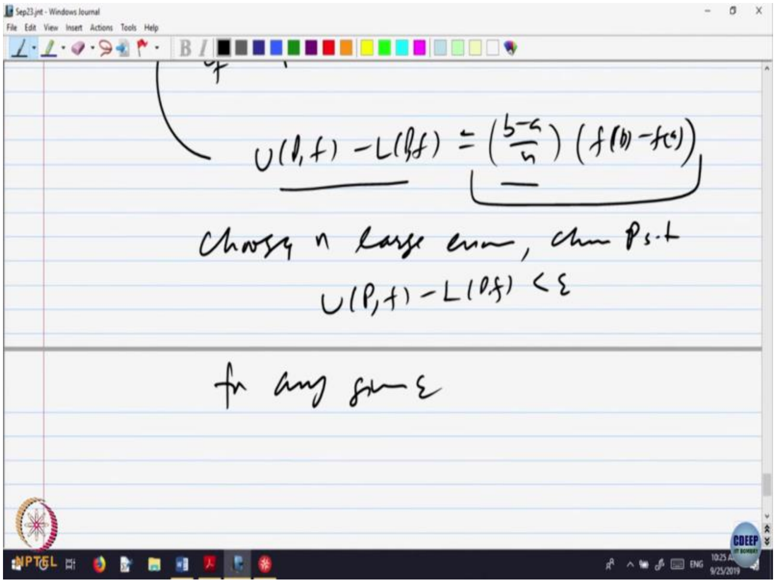

So that says, so choosing n large enough or we can just say that choosing n large enough U P, f minus L P, f will be less than epsilon, for any given, choose P such that this is possible. When P n is n equal parts, the upper minus the lower is than some constant divided by n. So, if I make the number of points, number of subdivisions large enough then upper minus the lower will become small. So, that says every monotonically increasing function is integrable, we do not know the integral, but it says it is integrable.

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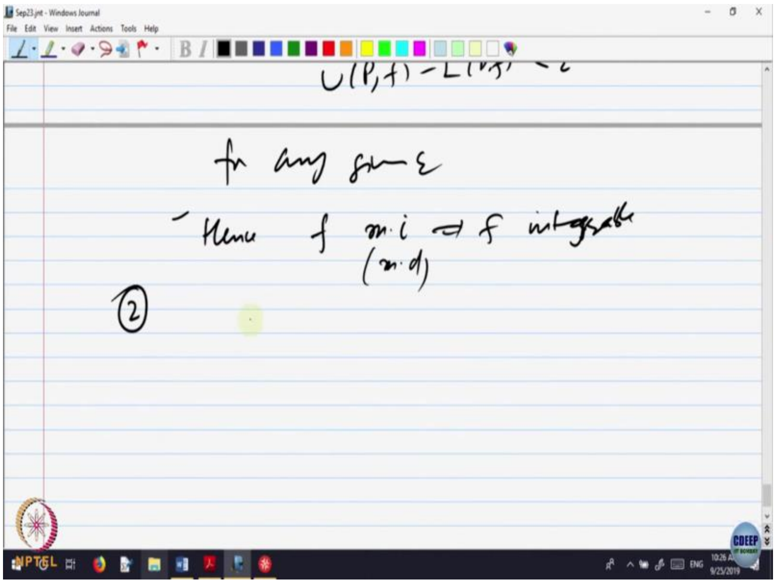

So, implies hence f monotonically increasing implies f integrable. So, let us look at second example for this, same will be monotonically decreasing also, instead of f b minus f of a it will come f of a minus f of b, so that is not a problem.

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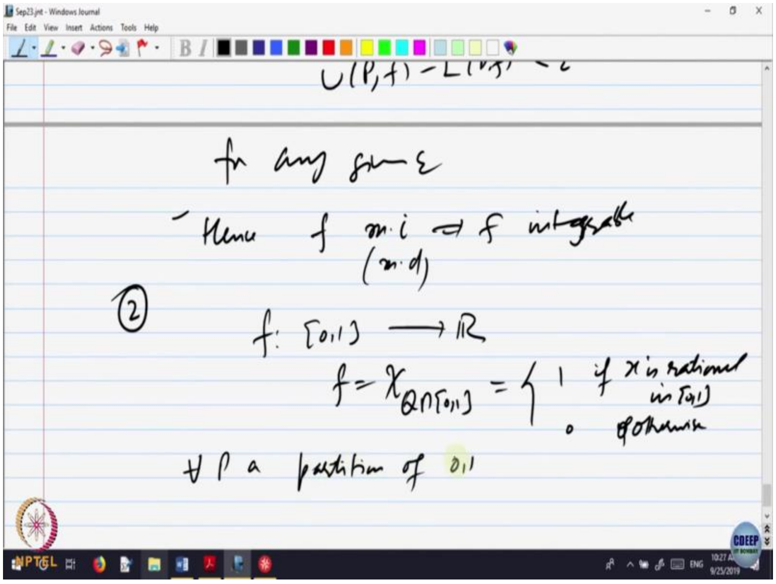

Let us look at another simple example, say f is from 0, 1, to R, f is equal to the indicator function of rationals in 0, 1. So, remember we define what is called the indicator function of a set. So, that means this is equal to 1 if x is rational in 0, 1 and it is 0 otherwise, that was the indicator function. So, for any partition for every P or for every P a partition of 0, 1.

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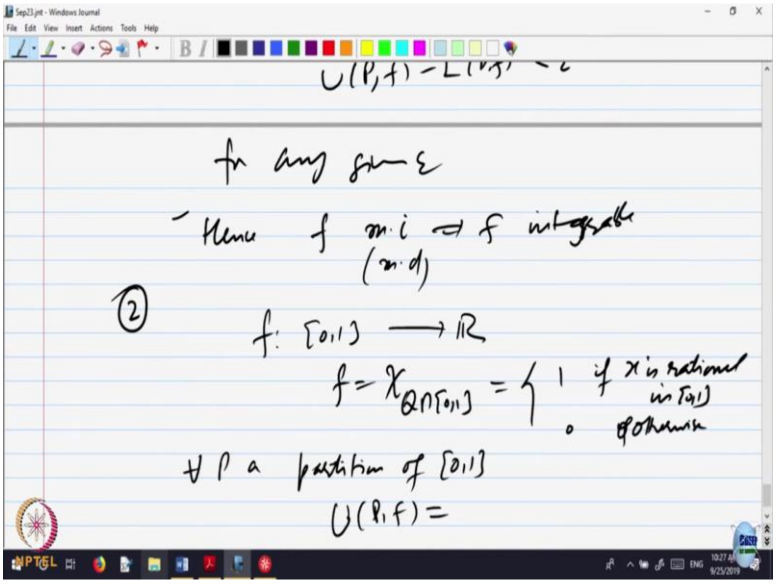

What is U P, f? I want to calculate the upper sum, that means given any partition, it will have sub intervals, I want to look at the maximum value of the function in any sub interval. So, what is the maximum value of the function? Function take only two values 0 and 1, and any sub interval will have a irrational in it. So, for any sub interval the maximum value will be 1, how so ever small the interval may be. So, one times, so upper sum will be capital M i that is

1 multiplied by x i minus x i minus 1 summation, so consecutive terms will cancel out b minus a and that is equal to 1, length of interval is 1, is that okay for everybody? So, upper sums is 1 for every partition.

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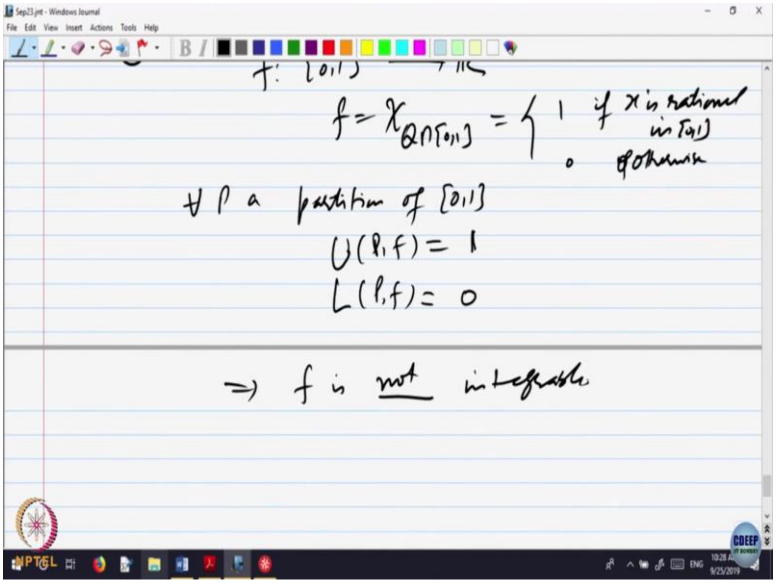

What is the lower sum? So, let us calculate the lower sum. So, the lower sum with respect to any partition, the smallest value the function takes is 0 and that takes it every rational every interval has irrational inside it. So, the minimum value of the function in any sub interval is 0, whatever be the length of interval does not matter, so this is 0 for every partition. You are using the fact that rationals are dense and irrationals are also dense. So that implies this f is not integrable, this function is not integrable.