Basic Real Analysis
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Lecture 37
Riemann Integration – Part 1

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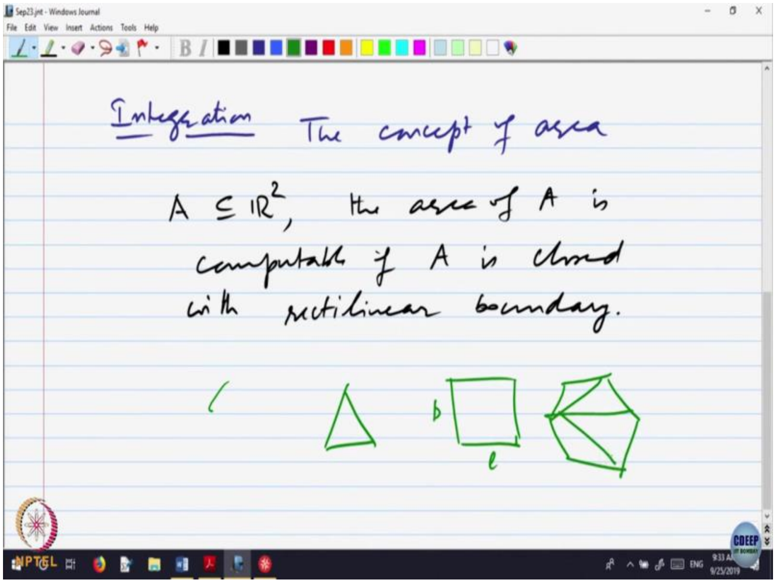

So let us begin today's lecture. We will be starting A new topic called integration. So, we will revise what is called Riemann integral and then say something about integration on functions of several variables. So, to start with the basic idea of integration is the concept of area. So geometrically, if you are given A set, say A contained in R 2, the area of the set is A is you can say compute or known, so A is computable.

If A is say closed with rectilinear boundary, so for example, if you have a if you have, say, A rectangle or A triangle or any other, so basically set with this boundary is rectilinear. So, you can calculate if it is a triangle you know the area is base into height, the formula which comes from a rectangle length into breadth and similarly you can divide, if you like, you can divide it into triangles and find out the area.

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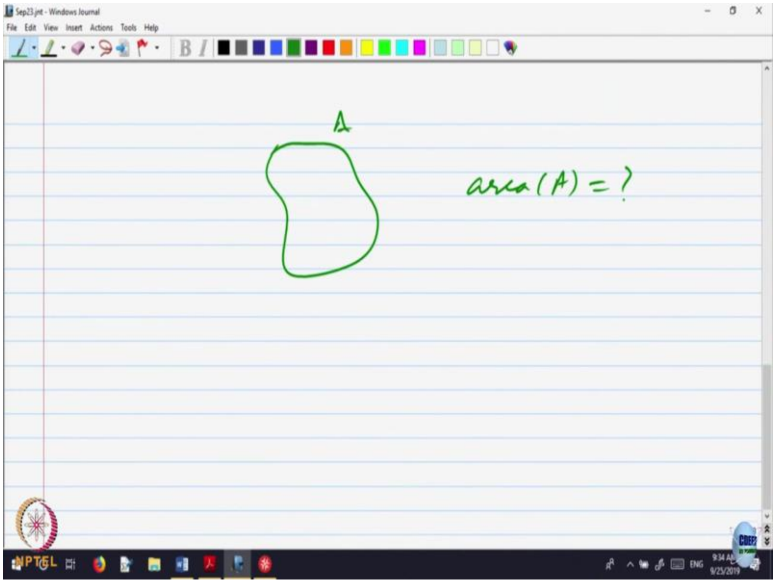

For a general set A in R 2, what is the concept of area of A? For what kind of sets one can define the notion of area? It is a another deep concept, how do we do that?

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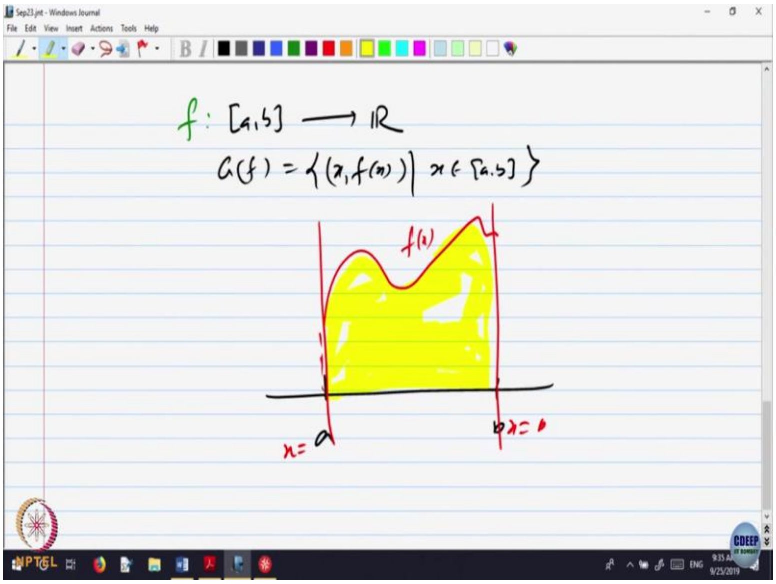

So, we will in our course, we will try to find out a particular kind of area, namely, will be given a function f on a interval a, b to R. So, we look at the graph of the function, so graph of the function is all points x comma f of x, x belonging to a, b and geometrically, so this is a and b and here is the graph of the function, so that is f of x. So, we would like to give a meaning to the notion of what can be called as the x is equal to a and x is equal to b, what could be way of interpreting this area, which is below the graph of the function bonded by the lines x is equal to and x is equal to b. So, in the picture f is drawn above the x axis just to understand it need not be.

So, now, the basic concept in all such problems is that is, when you want to find out this kind of area or in general when the boundary is not rectilinear, you try to approximate it by rectilinear things, because those things are known, so try to approximate. So, this area below the graph of the function one would like to proximate it by known objects, namely rectangles.

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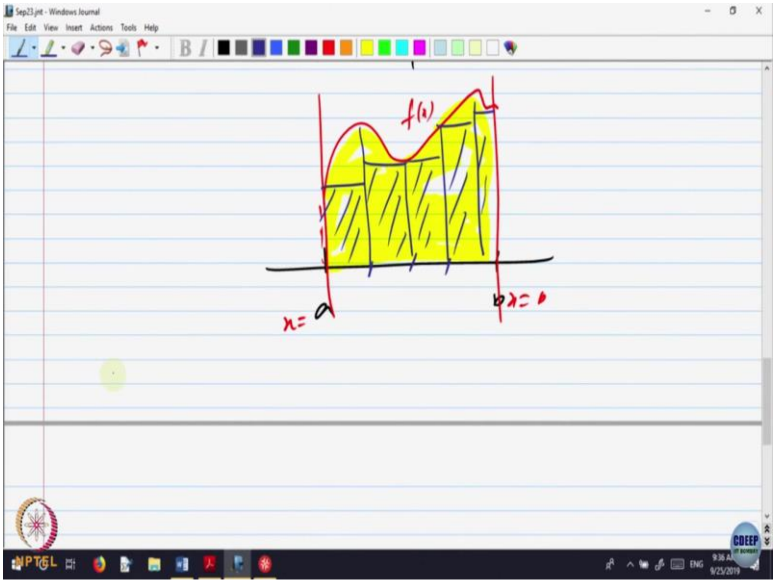

So, one would like to sort of fit in rectangles below it. So, you can sort of, so let us this is the kind of, so if we look at the areas of these rectangles. So, and add up these areas, then they give me sum estimate for what could be called as the area below the graph of the function. So, make these things more formal that means, we are going to select sum points in the interval a b, and then look at the rectangles with those as the width.

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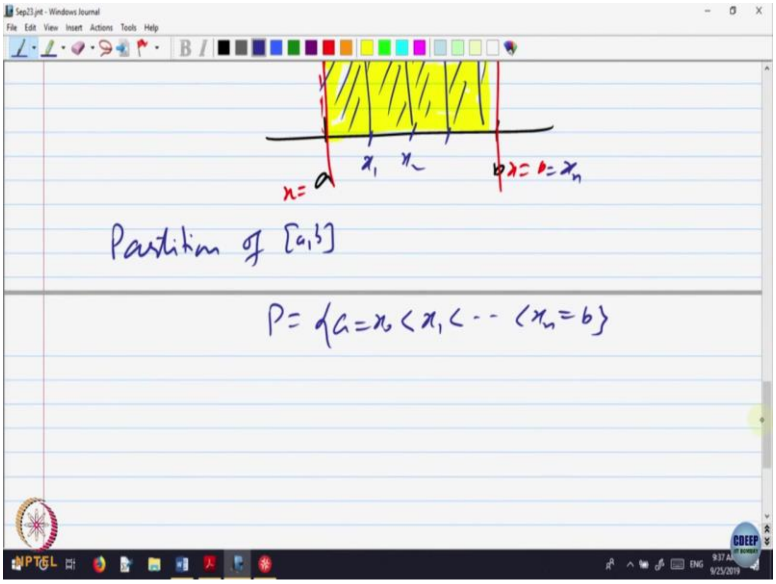

So, let us define what is called a partition of a, b. So, it is a collection of points P start with x is equal to, so a is x 0, take a point x 1, x n equal to b. So, this is x 1, this is x 2 and this is x n. So such as a so a partition of P will be represented by this kind of a set.

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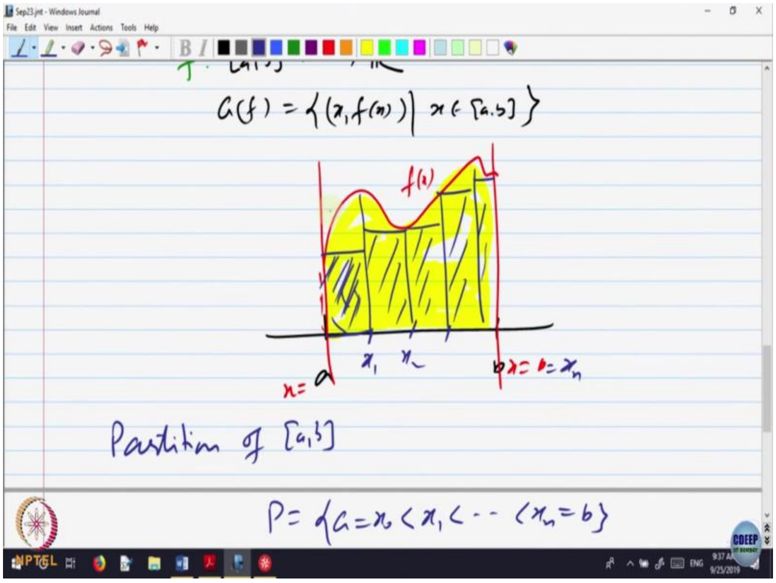

And now, what we want to do is we want to fit in rectangles. So, for example, this rectangle, if you look at, this is width is the sub interval and the height is for example, if I go slightly here then that rectangle will not fit in, so, the minimum value of the function.

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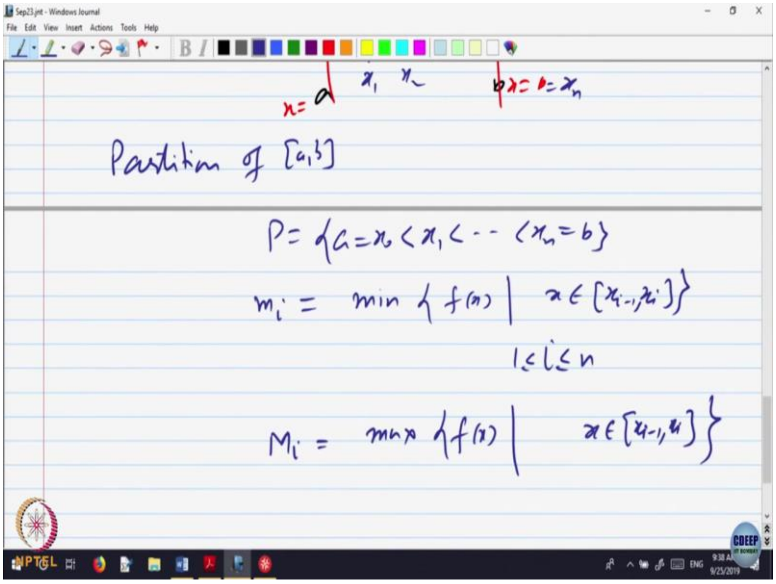

So, let us right m i equal to the minimum of f of x, x belonging to x i minus 1 to x I, so, i goes from 1 to n. And similarly, let us write capital M i to be the maximum of f x, x belongs to x i minus 1 to x i. Now, for this the minimum value of the function and the maximum value of the function in that interval to exist, we have to put some condition on the function, in general it may not exist. So, one has to put A condition on for this quantities to be meaningful.

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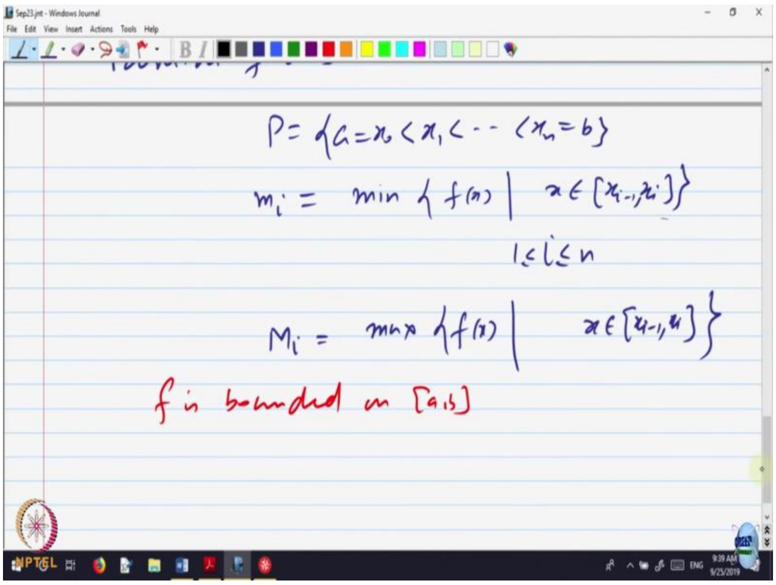

So, we put a condition that f is bounded on a, b. So, that is a condition that we put, so that f of x, x i x belonging to this interval, it will be bounded set, so it is a minimum and the maximum will exist, so no problem about that.

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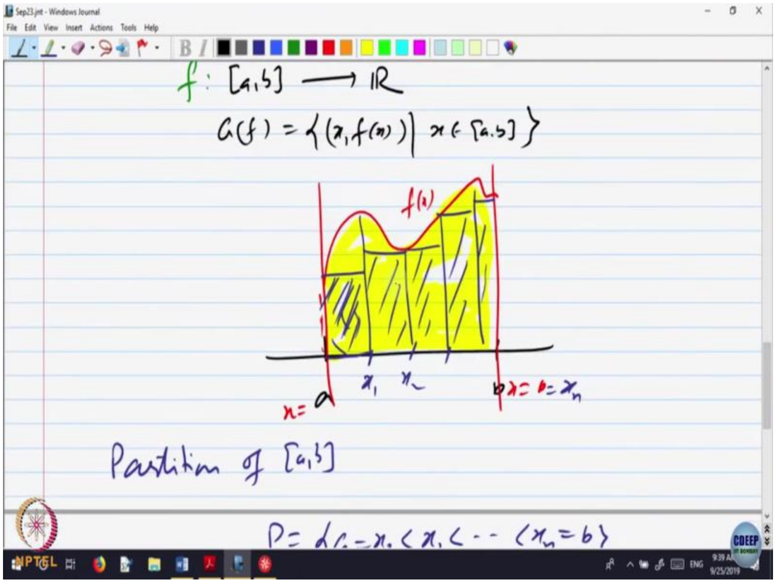

So, once you have got the minimum, so for example, this is the minimum value m i, so, area of the rectangle.

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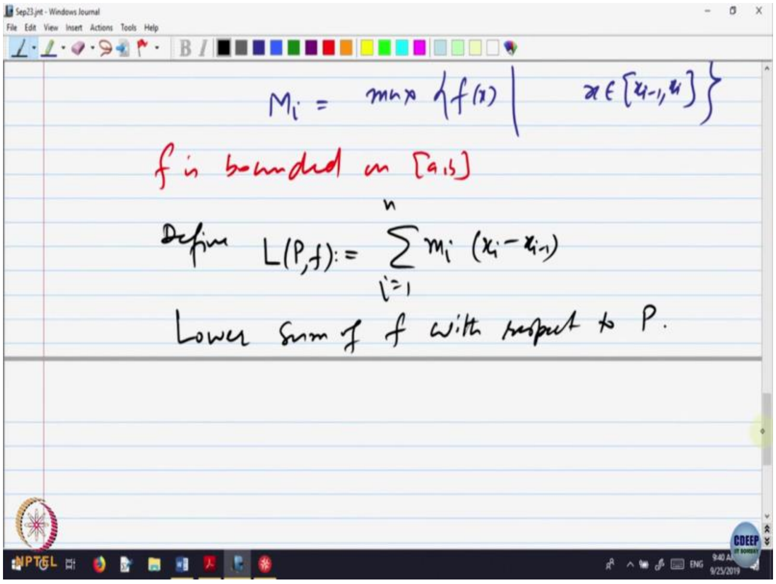

So, let us put those rectangles or define the minimum height inside times the length of the interval that is x i minus x i minus 1, and summation i equal to 1 to n. So, this is the sum of the areas of the rectangles which fit in below the graph of the function, so largest possible kind of rectangles below that. So, let us give them a name because the minimum value and it is with respect to a partition P for the function f. So, it is denoted by L P, f. So, this is normally called the lower sum of f with respect to the partition P.

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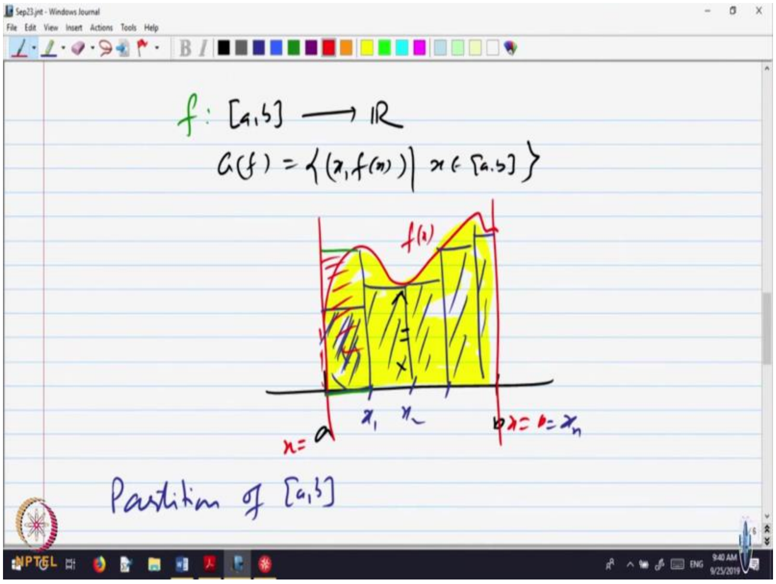

And one can also when we are estimating you can also go slightly above, so with the same base, I can look at this rectangle. So, that will also will estimate the area below the graph of the function, but slightly overestimate. So, let us do that also.

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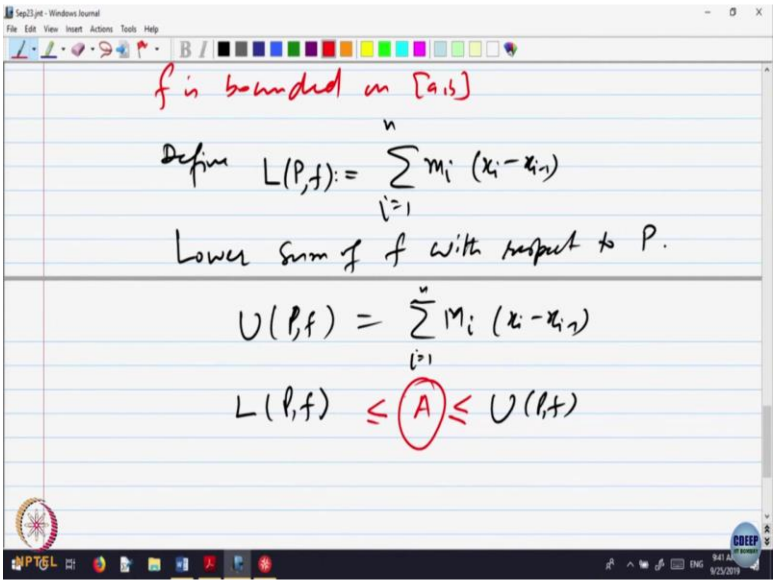

So, define and what is called the upper sum, so capital M i that is a maximum value into the length of the interval. So, intuitively this L P, f and U P f both are estimates for the area, concept of area. We do not know what is it actually, we are trying to sort of capture the geometric idea analytically but if we called that as say A, so this will be less than or equal to A less than or equal to U P, f.

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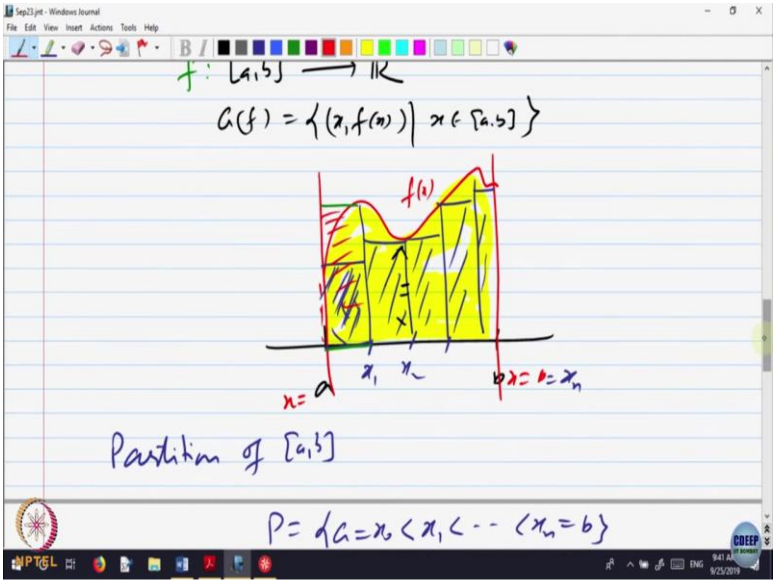

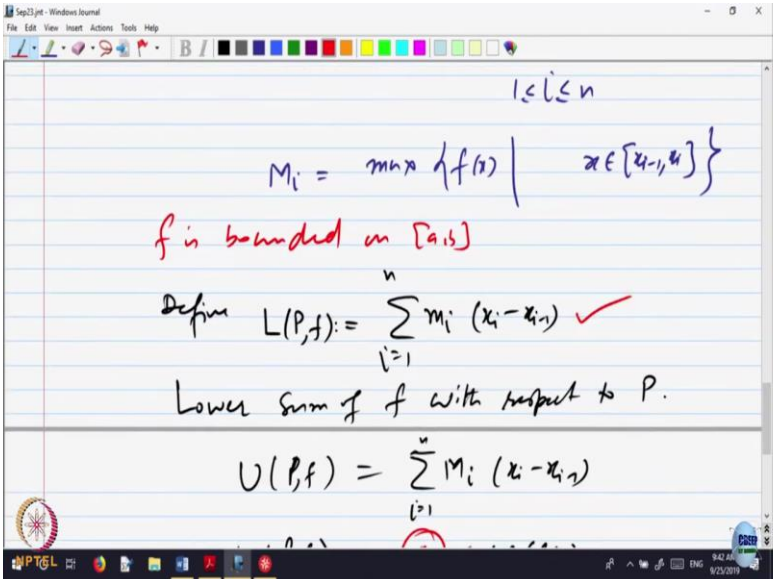

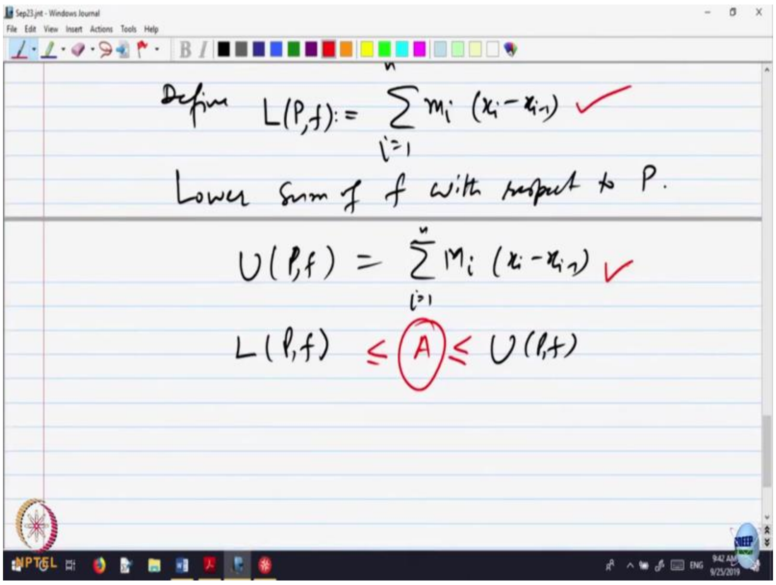

This is only purely geometric consideration that the, this this object called the area, we do not know what it is, but geometrically it is quite clear that the area this yellow portion will be below the sums of upper sum and the lower sum, but anyway, this is quite clear that lower sum will always be less than or equal to the upper sum, because the lower sum is the minimum value and U P, f is the maximum value. So small m i is less than or equal to capital M i.

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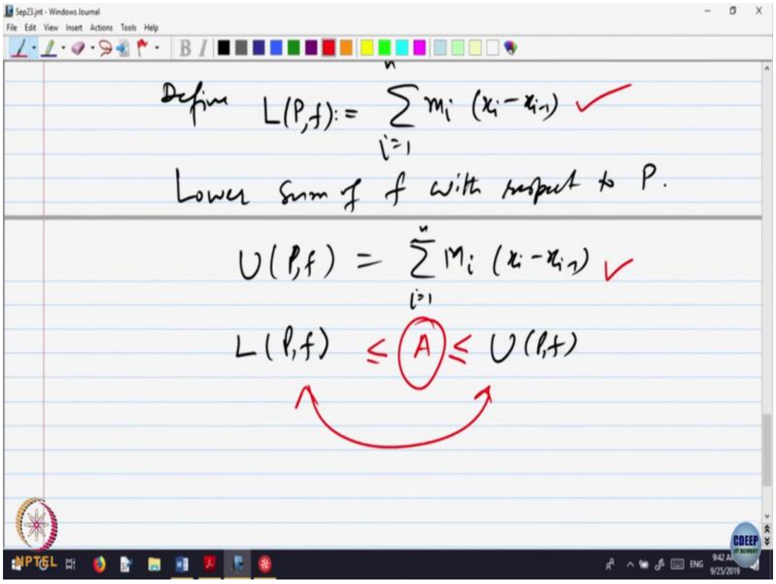

So this inequality is. And this is the thing we are try to capture in between.

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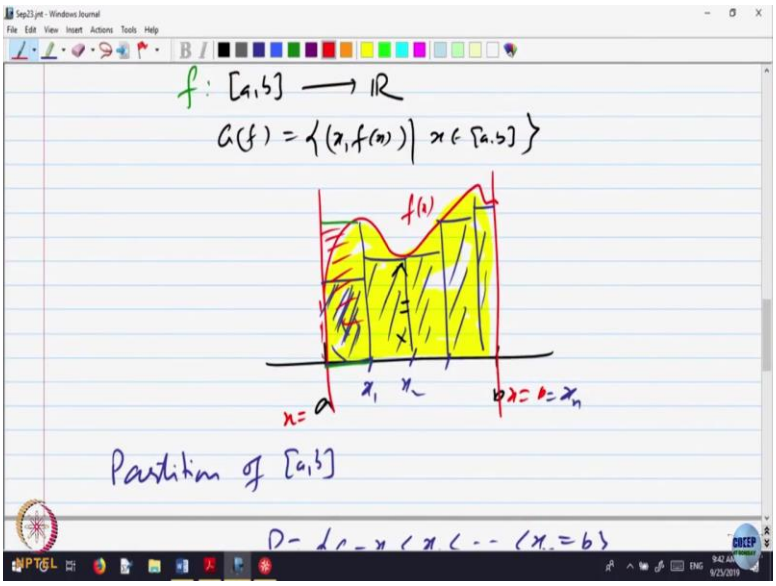

Now, the idea is that we try to sort of fit in these rectangles in such a way that the difference between the upper and the lower becomes smaller and smaller. So, that we actually capture that required area.

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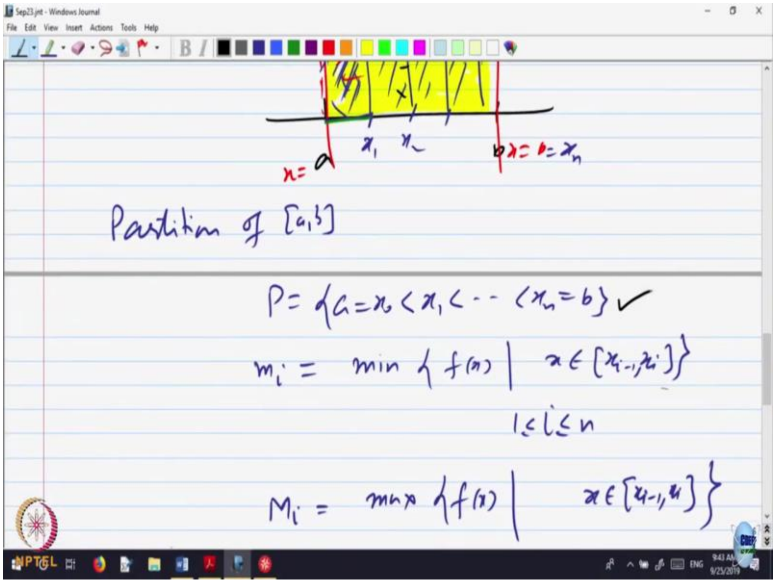

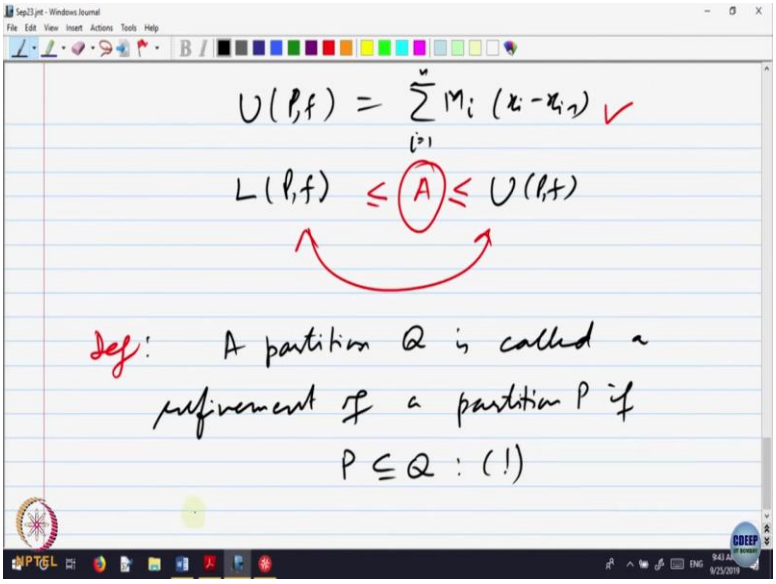

So, to do that let us write what is called the notion of refinement. So, a partition Q is called a refinement of a partition P if, is not a very sort of, if let me write P is a subset of Q. So, what does that mean? See we have written P as a set, the partition, in set theory we do not write a set like this actually, but we just denoted P as a partition to indicate that properties of the points. So, P and Q are two partitions. So, what does that mean? That P is a refinement Q is a refinement of P means, so let us explain this quantity.

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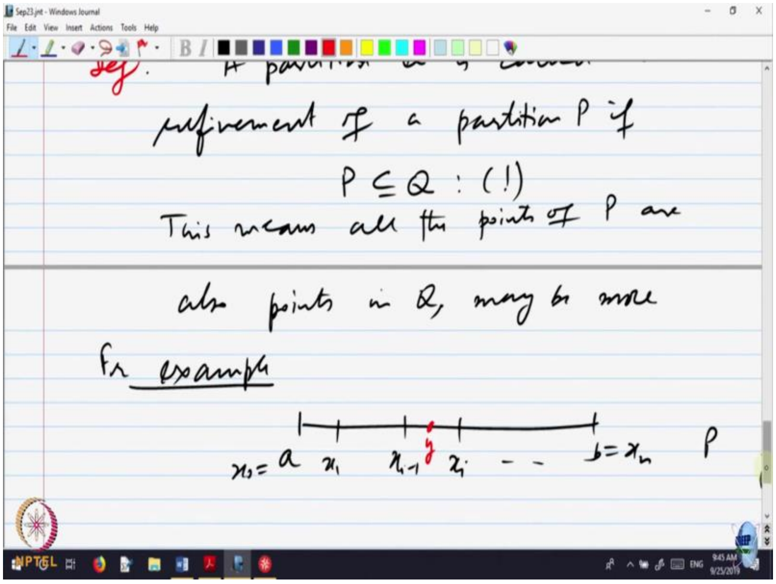

So, this means all the points of P are also points in Q, maybe some more. So Q all the points in the partition P, there are also points in the partition Q, but Q may have some more points, additional points. So, for example, one obvious way of, so this is a, this is b, and this is say x 0, x 1, x i minus 1, x i and so on and that is x n.

So, that is a partition P. So, if you want to generate I want to make a partition, which is a refinement of P, then I may add sum points say, for example, y here. So, P is x 1, x 2, x n and Q is x 1, x 2, x i minus 1 an additional point, in between. So, you are throwing more points in the partition P to get a refinement. So, that is a idea.

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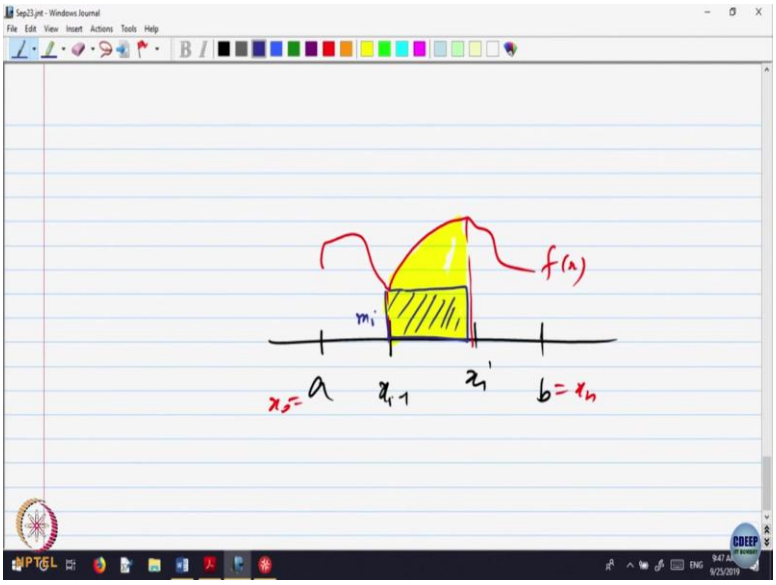

Now, why are we are doing that? So, the reason for that defining refinement is the following. So, let us look at, let us look at a upper sums and lower sums. So, this is a and this is b and let us look at x i minus 1 and x i, and let us look at the graph, so let me put it this way, so that is the, so in between as draw it nicely to indicate.

So, this is my function f of x.. So, what is if I considering the lower sum for this partition, P, so this is x 0 and that is x n. So, in the lower sum, one of the rectangles that will be coming in this partition, for this is the area below the graph of the function that we are trying to approximate. So, the lower small m i times, so this will be, so this will be m i times the length that will be smaller.

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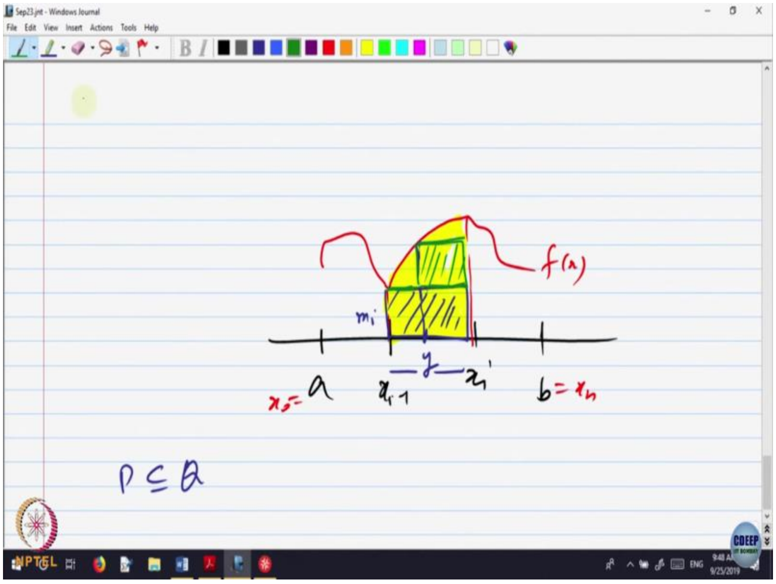

Now, supposing I have added one more point where we find the partition P and the new point is in this interval, say somewhere, so let us say the new point is here somewhere, so, let us call that point as y. Now when you want to consider the lower sum with respect to this new partition, so Q is a refinement, so, P is partition given and Q is a refinement, let us look at the corresponding lower sums with respect to P and with respect to Q.

So, the contribution in the lower sum with respect to P will be this rectangle. Now, when I want to consider with respect to Q, I will have to look at the minimum value in this interval and in this interval. So, what is the minimum value in this interval? The first one that is same as the earlier one, what is the minimum value in the smaller interval? That is here. So, the contribution for the lower sum with respect to the new partition will give me additional approximation additional rectangle which will add up in the lower sums. So, this picture tells me that the lower sums tend to increase if I refine the partition.

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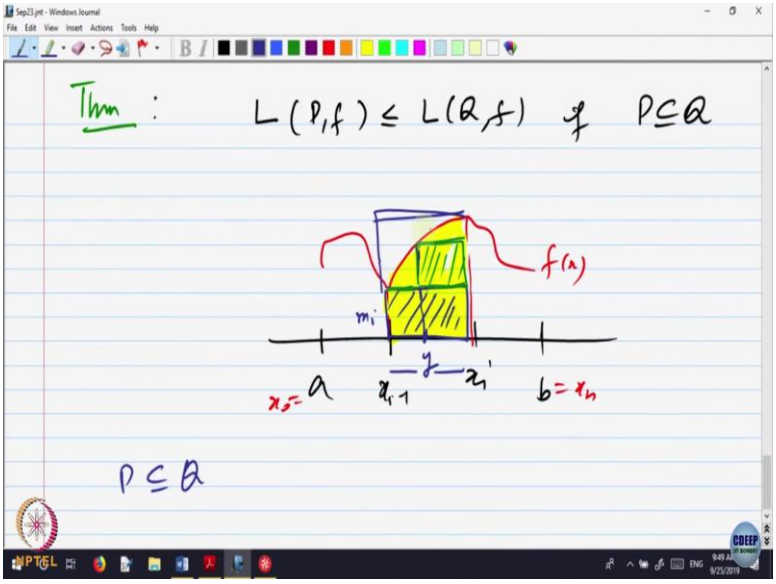

So, here is a theorem which we will like to state that the lower sum with respect to P and Q, is less than or equal to lower sum with, sorry with of the function f, Q f if P Q is a refinement of P, and you can also look at the upper sums in this picture what will be the relation between them.

For the earlier partition, the upper sum will be, for the big interval the value is here, so that will go, so, that will be the rectangle the bigger one from here to here, but when you consider the lower, sorry upper sums I am looking at. So, the maximum value will change and it will go like this and maximum is here, so this will be the value,. Let me remove this, remove that.

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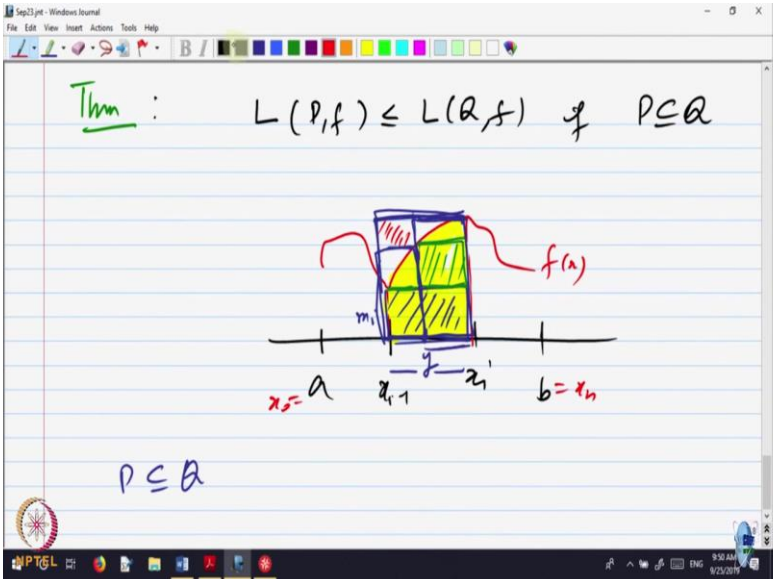

In this, bigger interval, the maximum value is here. So, this will be a rectangle we will be looking at. When I cut it up, up to here, so in this portion on the right hand side, so the maximum value is this, so I will be looking at this rectangle for the refinement but the for the other part of the partition from x i minus 1 to y the maximum value is here, so I will be looking at this rectangle. So, what will happen from the upper sums, this portion in between this will be removed, so the upper sum will tend to decrease as you refine the partition.

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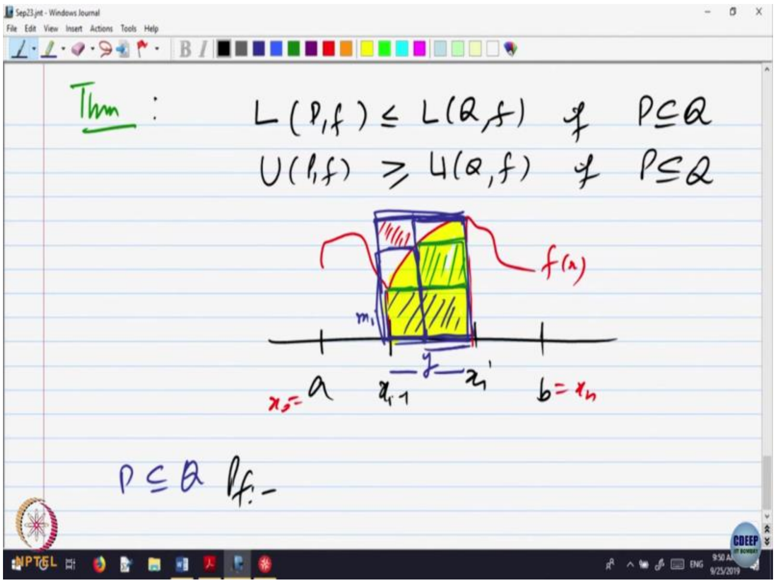

So, let us write that also, that the upper sum P f is bigger than or equal to lower sums, sorry upper sums with respect to Q, if P is a refinement of Q, is it clear what is happening? Now mathematically why does that happen?

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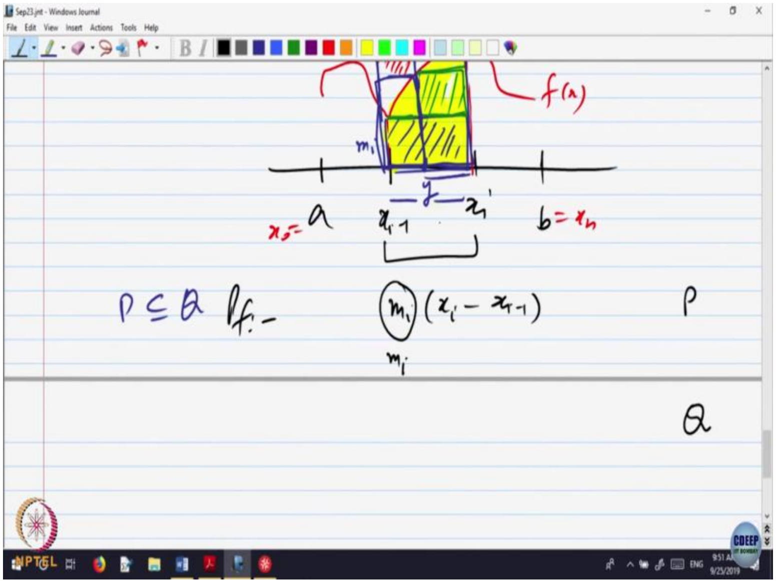

So, one want to write the proof of this, so, what you be looking at? You will be looking at the term m i times x i minus, so this will be the only term, one in the earlier when you are looking with respect to P, I am looking at that portion only one point is extra added. And when you look at the new one, so let me look at, so and what is this m i?

That is the minimum value of a function in this interval, when I want to look at with respect to Q, I will be looking at the minimum value between x i minus 1 to y. So, the interval will be smaller, it is a part of the earlier. So, what is the relation between the minimum value of the function in a bigger interval with the minimum value of the function in a smaller part of that interval? So, that is the relation we have to look at, that is what will change.

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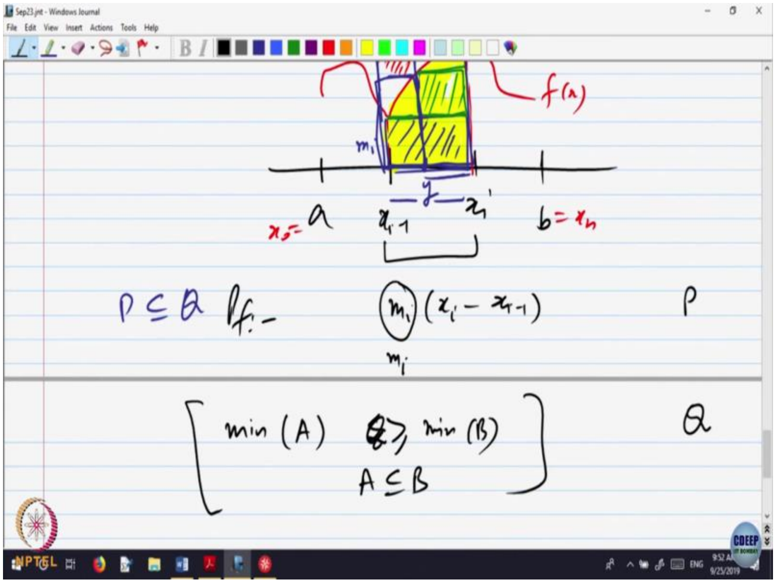

So, the basic concept in this proof will be looking at the minimum, so this is a general problem, minimum value in a set A and minimum in B if A is a subset of B, what is the relation between these two? We are given a subset A and a subset B, such that A is a subset of B, then if A is a subset of B what is the minimum, its relation with respect to the minimum on B? B is the bigger set, so what is the relation?

Minimum of A will be less than or equal to no, no, so, it is bigger than or equal to this. Minimum tends to increase when you shrink the set, because on the bigger thing, minimum could have been something which may I have be eliminated in B. So, minimum tends to increase. Yes.

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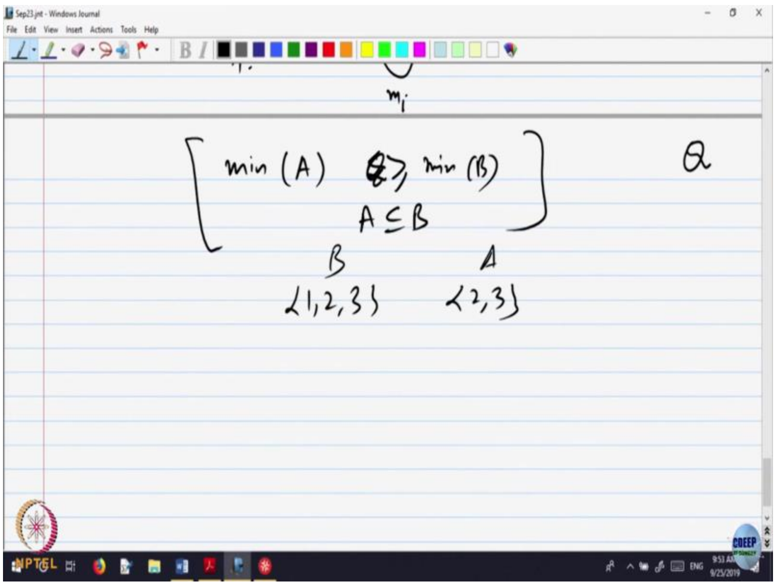

Let us let us look at just look at an example, look at the sets 1, 2 and 3 and look at the set 2 and 3, simple, you can always verify such things. What is the minimum of this set A and this is a set B? A is subset of B. So minimum of 2, 3 is 2, where the minimum of 1, 2, 3 is 1.

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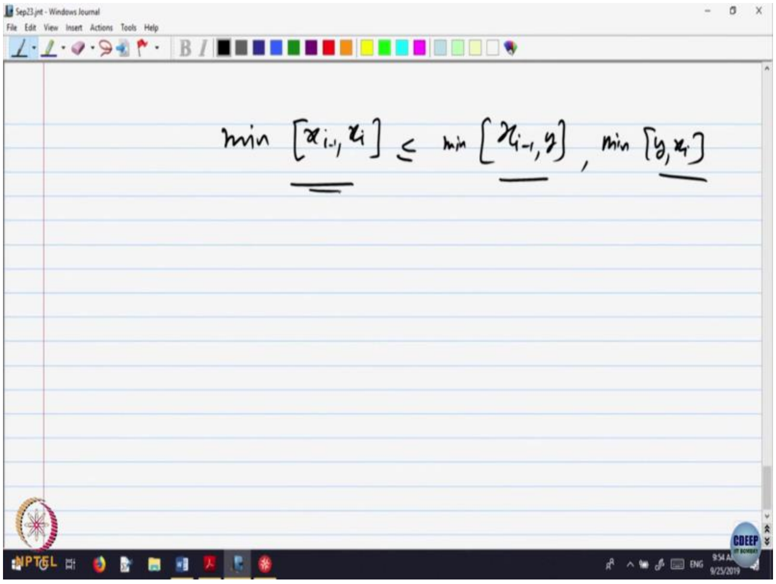

So, that is the main property that we will be using and writing a proof of this, that the minimum tends to, so, what we want to say is that the minimum x i, so minimum in this interval x i minus 1 to x i. So, what happens to the minimum value? We are looking at minimum x i minus 1 to y and or the minimum x i x y to x i. So, we either of this will be bigger than or equal to this, both this minimums will be bigger, because this is a part of this and this is a part of this.

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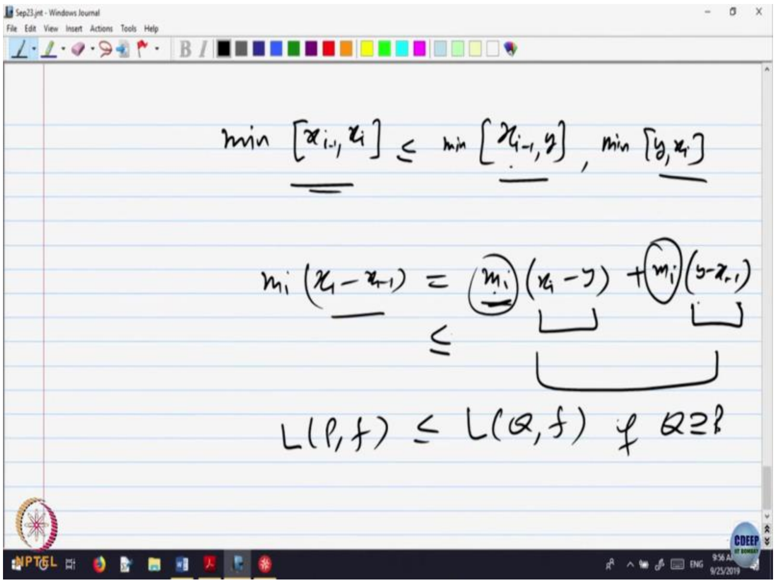

So, when you look at m i that sum m i times x i minus x i minus 1, you want to go to the smaller intervals, we write m i times x i minus y plus m i times y minus x i minus 1, this we can separate into two parts, add and subtract y, and this quantity is less than, so this is bigger than this and this also is bigger if I replace the minimums in those intervals, are you following?

See for a minimum for this respect to Q, I have to look at the base, this is a base, but the minimum in the smaller thing is bigger than or equal to, so, this m i will be smaller than minimum in the smaller interval, so this will give you less than or equal to. So, that will be the terms coming in the lower sum. So, that will give you the lower sum with respect to P is less than lower sum with respect to Q, if Q is a refinement of P. And when you look at the maximum, is the other way round.

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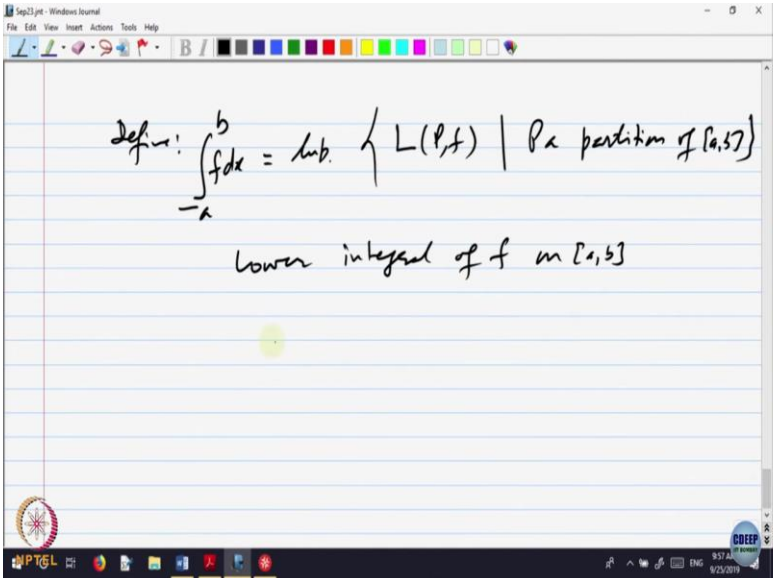

So, the lower sum tends to increase and upper sum tends to decrease, so let us define look at the lower sums with respect to P a partition, so as you refine these lower sums will tend to increase. So, if I look at this set of lower sums, this is a set which is non-empty of course, you can take the partition to be the end points only, it is a non-empty set which is bounded above. Why it is bounded above? (())(25:25) function is bounded f, so m times B minus A.

So, all the lower sums will be less than that, so it is non-empty set which is bounded above. So, what should happen? Completeness property for a non-empty set which is bounded above, or should have the least upper bound of this set should exists. So, that we denoted by f d x, a to b and it is a lower sums increasing, so one writes a bar below it, is a lower sums. So, it is called the lower integral of f on a, b. Now, we are already taken this the supremum with respect to the partitions, so least upper bounds.

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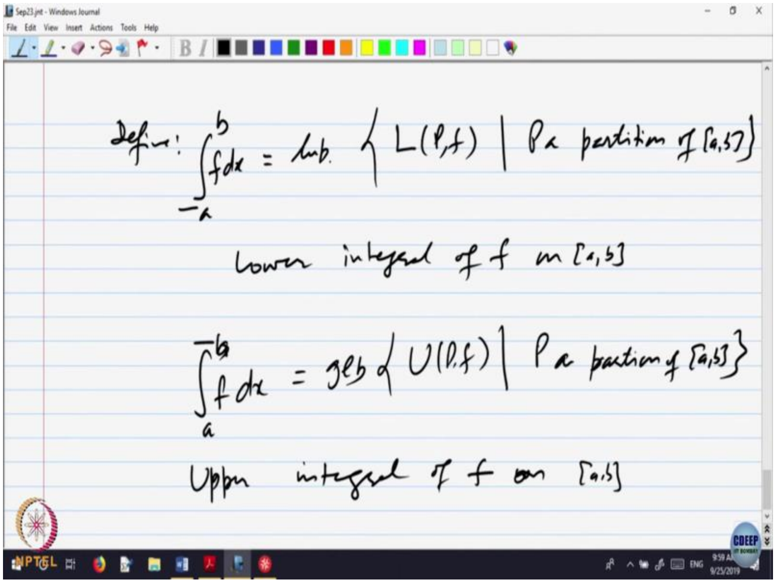

And similarly you can look at upper sums P a partition and they are decrease, so they decrease and it is bounded above, bounded below, so greatest lower bound of this will exist and that is called the upper integral of f, so let us call it as upper integral on a, b. So, what we are saying is, you will have approximations from below, you will ever approximations from above, the lower approximations tend to increase, upper tends to decrease.

So, they will come to a value, but those two values may not be equal, these two values may not be equal. So, unless they are equal we cannot say that we have captured the area below the graph of the function.

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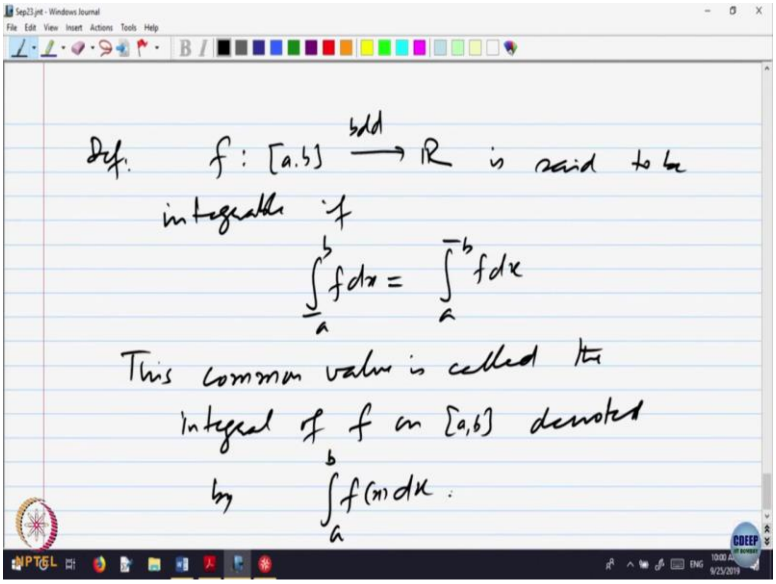

So, one makes a definition, so the definition, f on a, b to R is said to be integrable. So, it is a bounded function that we have assumed, if the lower integral a to b f dx is equal to the upper integral a to b of f d x, one writes f x but you need not this does not matter actually, f of d x, and this common value is called the integral of f on a, b.

So, whenever these two are same will call it as a common value to be the integral of f denoted by or normally writes f x dx to indicate a to b. So, this is called the integral of the function. So, geometrically this indicates what is this defines, so, you can take this as the definition of, geometrically to be the area below the graph of the function.

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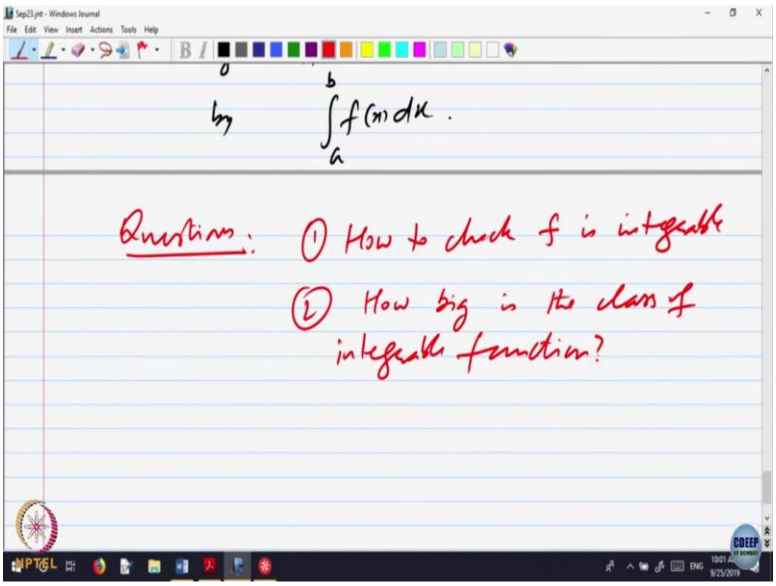

So now, the question comes, so the question we would like to deal with that the following. How to check f is integrable? One and second, how big is the class integrable functions? So, given a function, how do I check it is integrable? The definition says, we should look at, the partitions look at lower sums, look at the upper sums and see whether they, come closer and closer as you refine the partitions and so on. Is that the only way or can we have some other way of testing whether some function is integrable or not.

And secondly, how big is this class of integrable functions? That means, we want to look at examples of functions or the classes or functions which automatically become integrable. So these two questions are important from whenever one defines a new concept, give examples of that concept and check when given an object when that object will have that property. So these are general things, so let us look at some of them.