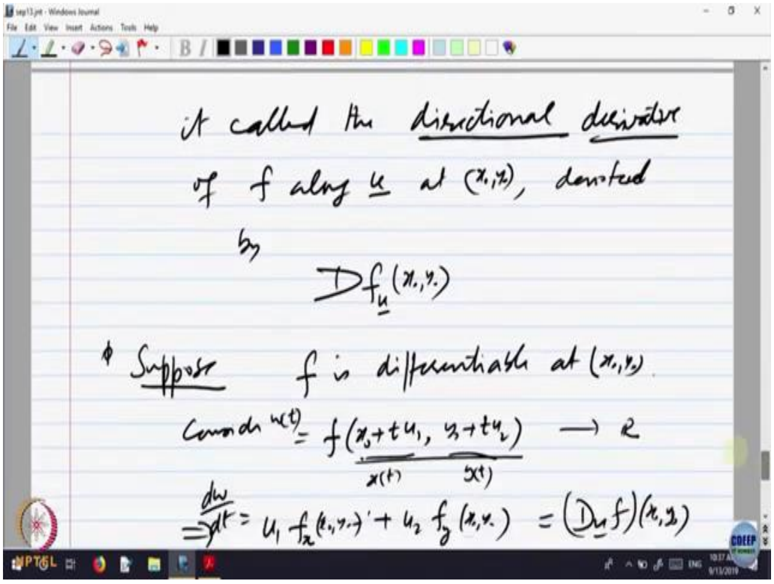
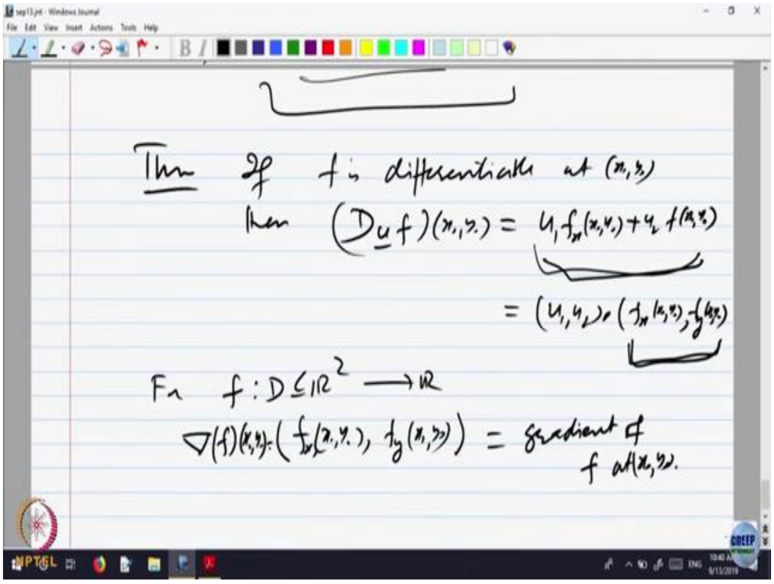
Basic Real Analysis
Professor. Inder. K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 36
Differentiability – Part VI

(Refer Slide Time: 0:16)





So, suppose F is differentiable. So, consider, now consider what directional derivative is what? It is nothing but the rate of change. So, it is X naught plus TU 1 comma Y naught plus TU 2. So, let us consider this function. So, it is a function of T goes to XT, YT a chain rule is coming into picture now, T goes to X. So, this is XT and this is YT F takes XT to some value in R.

So, if F is differentiable, the function XT that is differentiable, so composite function is differentiable chain rule applies. So, conditions of chain rule are met if F is differentiable implies what is the derivative of this? What is the derivative of this function? Let us call it as WT, T goes to one variable. So, what is the derivative of this one? Chain rule F of X F of X at X 0, Y 0, what is the derivative of this with respect to T, XT? So that is U 1 plus U 2, FY X 0 Y 0 that is a derivative of this function.

So, that is DW by DT, and what is this? We are looking at the derivative of this function, so, what does this equal to? AI If I look at as a composite function, if I and so, this is the rate of change of this composite function, what is the rate of change of the function we looked at? That was precisely, what was that there is a directional derivative. So, it says if I look at the apply the chain rule, this is nothing but the directional derivative of F X naught Y naught.

So, if the function is differentiable, then the directional derivative exists and is given by this I do not have to go to the limit or anything because our limit is taken care by the chain rule now. Is that okay comfortable, everybody or not? So, I am looking at this function and what is the derivative of the as this minus divided by T, that is precisely the directional derivative, is not?

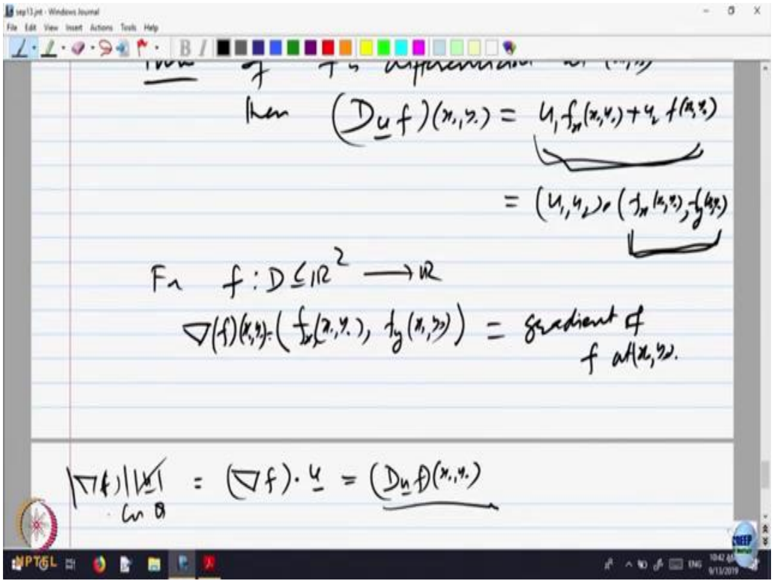
If I want to calculate the WT, if I want to calculate its derivative, what I have to do? This minus F at X 0 Y 0 divided by T, limit of that, that is by definition. So that by definition it is this, by chain rule it is this. So, both are equal, that all. So, it says if so, theorem says, if F is differentiable at X 0, Y 0, then the directional derivative of F in the direction of X 0, Y 0 is precisely equal to U 1 FX plus U 2 F of X 0 Y 0.

From here, let us a bit of vector, let me write this right hand side in terms of vectors. I want to write, so let us write. So, the I am saying this is equal to U 1 comma U 2, that are components of a given vector U dot product with FX X 0 Y 0 and second component FY X 0 Y 0.

I am creating a new vector given the function F if it has partial derivative, then this gives me a vector FX partial derivative comma FY partial derivative. So, this can be thought of as a dot product of these two vectors now. This can be thought of as a dot product of these two vectors, and this is a very important quantity associated with a function.

So, for F, which is on D in R 2 to R, this vector FX at the point X 0 Y 0 FY at the point X 0 Y 0 is a vector. It is denoted by inverted triangle of F at the point X 0 Y is 0 and it is called the gradient of F at X 0 Y 0, it is called the gradient of F at the point X 0 Y 0, for three variable give me a vector of with components three, so FZ will come.

(Refer Slide Time: 7:32)

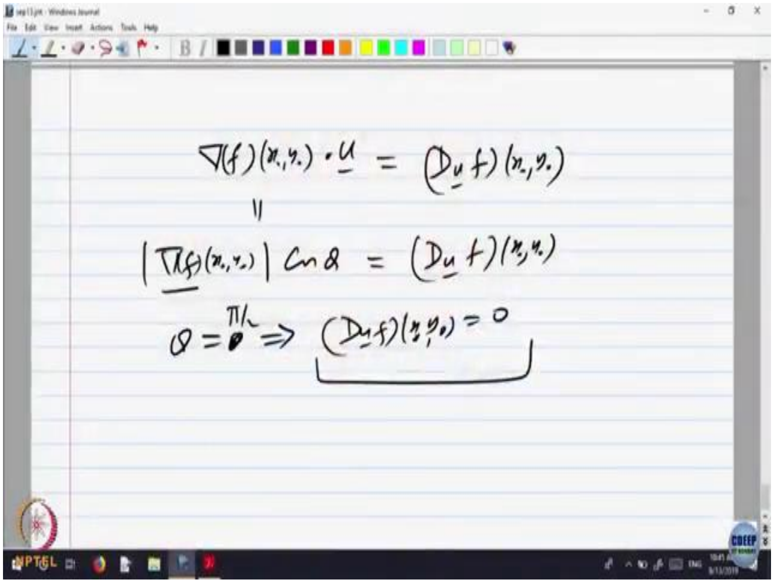


But the interesting thing is, so, if I write that way, gradient of F dot U at that point is equal to the directional derivative in that notation. Now, you see how things get interpreted. So, direction derivative in totally is a rate of change of the function in that direction and that is equal to the gradient times U, when okay.

Now here is a bit of vector algebra A dot B, you can write as norm of this, norm of U into, what is A dot B? Norm A, norm B into cos of the angle between them, so, cos theta, this is norm is 1, so we can forget that. Now, what is the theta? This is the vector, gradient is a vector, U is a vector. So, this is, theta is a angle between the two. So, and is cos of theta, when is cos theta minimum? Angle is pi by 2 and then it is 0.

So, it says the rate of change of the function is 0 at that point in the direction of the gradient when U his angle is 0, so direction of U is same as direction of gradient. So, it says the rate, so interpreting this in terms of rate of change, so rate of change of the function F is 0, when direction is that of the gradient, are you following what I am saying? No, so let me let me interpret it again.

(Refer Slide Time: 9:42)



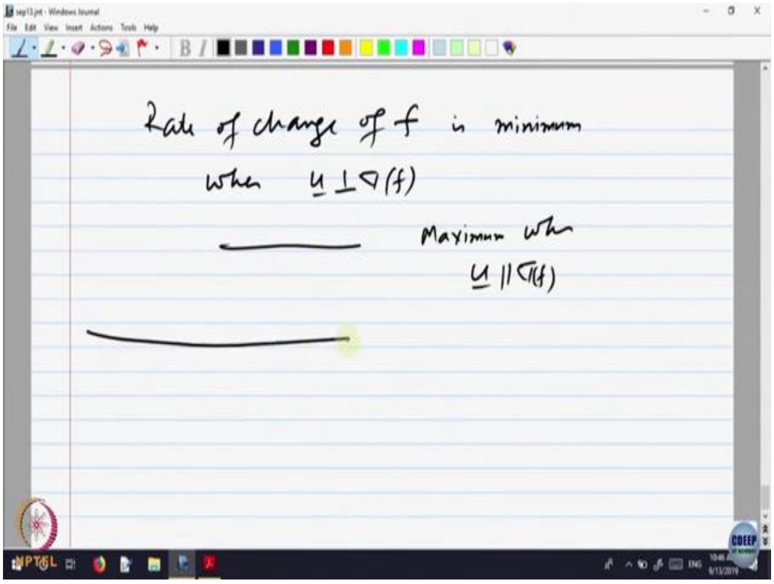
It says gradient of F at a point X 0 Y 0 is a vector dot product with U is equal to directional derivative F at the point X 0 Y 0. And this we said it is equal to gradient of F norm of gradient of F into cos theta, so that is 1 is equal to the DU F at X0. So, theta equal to 0 implies what or theta equal to pi 2 implies what? Cos is 0, implies DUF at X 0 Y 0 is equal to 0.

So, what is the meaning of saying this is 0? If I interpret it as the rate of change, so, rate of change of the function F in the direction of U is 0 when U is same as, direction is same as that of gradient because then only the angle is 0, angle between the gradient and U is 0 or sorry pi by 2 one perpendicular sorry, when it is perpendicular. So, rate of change of the function is 0 in the direction perpendicular to that of gradient.

And when it is maximum? When it is equal to one pi is, angle is 0. So, rate of change of the function is maximum in the direction of the gradient. So, this is a physical interpretation, you can imagine a plate, surface is a some kind of a plate and there is a vector given and the gradient vector is there at any point.

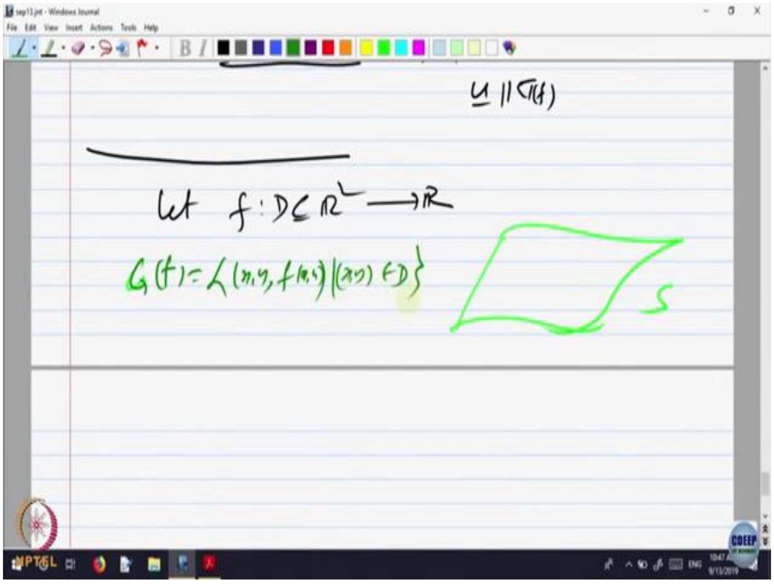
So, if at a point I want to know and you are measuring the temperature, function is a temperature at a point of the plate, and as you move on that plate in which direction the rate of change of temperature is maximum or minimum, you would like to know, you are walking on a hot plate and you do not want your feet to be burned. So we would like to seek the path direction in which temperature rate of change direction should be minimum. So, that is perpendicular to the at that point, find out the gradient, there is a minimum, maximum when it is parallel. So, these are physical applications, this kind of a thing, though it is purely mathematical.

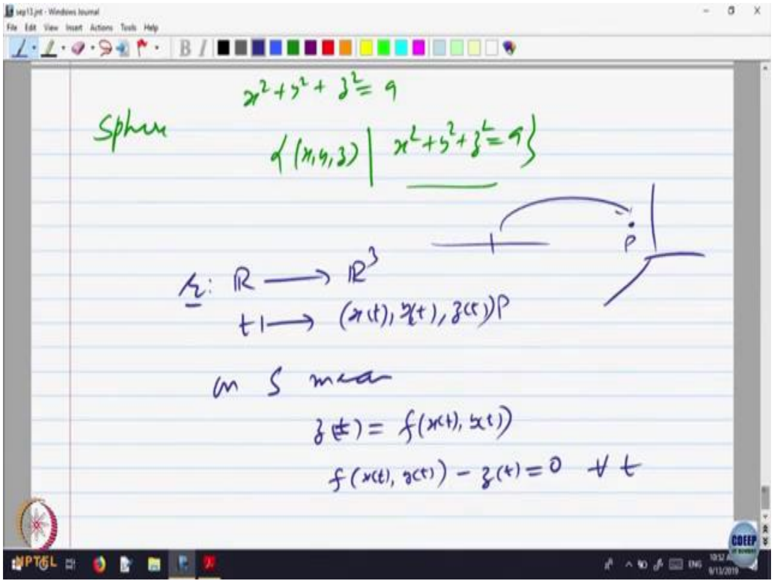
(Refer Slide Time: 12:50)



Something more I should say about this, here is another interpretation of this. So, let me just write rate of change of F is minimum when U is perpendicular to gradient of F, it is maximum when U is parallel to gradient of F. And see how these things lead to physical applications.

(Refer Slide Time: 13:31)





So, here is another one, so let us take F a function of two variables again. So, we saw its graph is a surface, to this is a surface S. So, graph of F is XY F of XY. Now in this, what is happening is, I was wondering whether I should introduce the notion of general surface or not. So, here is where Z is equal to F of XY, z coordinate is taken , it is going a bit away from what we.

So, this is kind of a surface, but the point is not every surface looks like Z is equal to F of XY, for example if you look at the sphere, what is the equation of sphere? Is all points X, Y and Z such that X square plus Y square plus Z square equal to some constant, that is the radius square?

So, here there is a relation between X, Y and Z. So, a sphere, which normally we write as X square plus Y square plus Z square equal to say something say nine. What do you mean by that? There is a equation of a sphere, that means we are looking at the points X, Y and Z such that there is a relation. So, as an object in R 3 it is given by this equation.

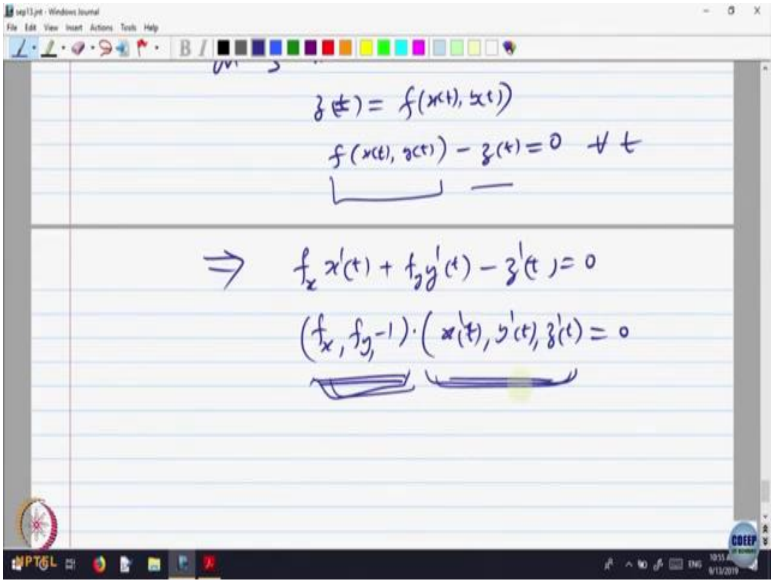
Now here Z is not, you cannot calculate Z in terms of X and Y because there is only a relation given. So, if you want to calculate it will be Z square equal to 9 square, 9 minus X square minus Y square. So z will be plus minus square root, if you take plus it will give you the upper part of the sphere, if you take negativity it will give you the. So this is what we call it as a implicitly equation of surfaces, Z is not explicitly known, it is only implicitly given.

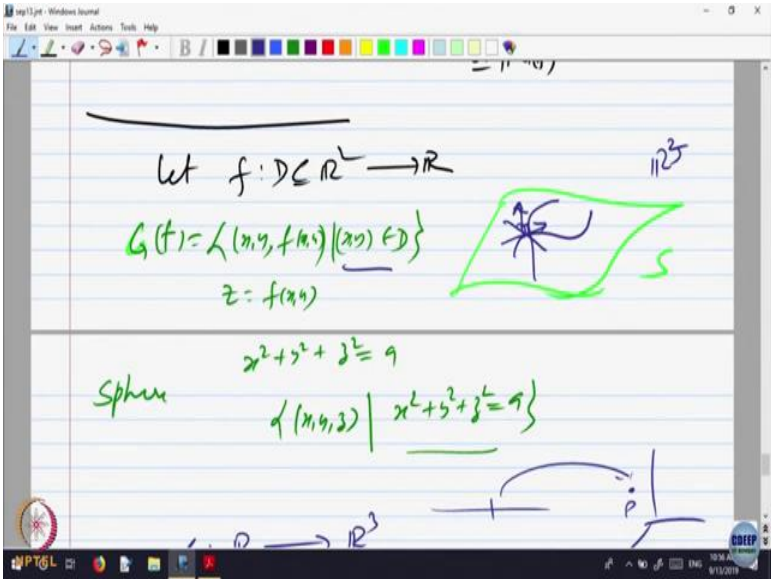
So, there are many surfaces which are of this type for example, if you take shape of an egg that is called ellipsoid that is also implicit. Some are explicit, some are implicit, what I want to do is, I want to look at a curve on this, I want to look at a curve on this surface. So, let us take, so this is an R 3, surface is in R 3. I want a curve on R, on the surface. So, what is a curve? What is it a curve? Is like a path of a particle.

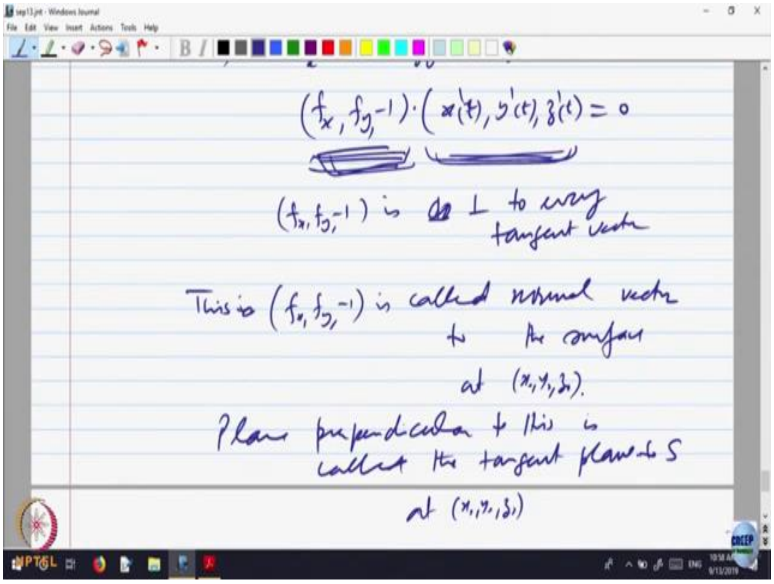
Imagine a fly moving around in the room. At some, you start observing at T equal to 0, it is at somewhere T equal to 1 it is somewhere, T equal to 2 somewhere moves around, to locate its position, we have to know what is the coordinates XT, YT and ZT. So, a curve is a function from R 2, R 3, T goes to XT, YT, ZT. So, let us call it as a function R, any curve is a part of a particle, if you can imagine, a sometime you are observing what is a position of something you think it as way.

So, at T it gives a point which is in here, so, this is the point. I want this curve to be on the surface, then what should happen? If it is to be on the surface, then what should happen? It should satisfy the equation or the surface. So, on S means Z is equal to Z of T is equal to F of XT, YT because Z was equal to F of XY. So, Z of T must be equal to that, and that I can write as F of XT, YT, ZT sorry, for XT, YT minus ZT equal to 0. I brought everything on the one side, for every T to be on the surface, function of T.

(Refer Slide Time: 20:07)







Now mathematics enters into picture, his function is 0 everywhere, so what happens to the derivative of this? Also should be 0, what is the derivative of left-hand side? Implies what is the derivative of left-hand side? FX into X dash T, sorry into X dash T of T plus FY Z dash of T minus Z dash of T equal to 0. I want to rewrite this now as a vector equation.

So, it is FX, FY minus 1 dot X dash T, Y dash T, Z dash T equal to 0.

Student: (())(21:27)

Professor: This equation? Yes, I am just differentiating this with respect to T, this is chain rule and that is ZT only, so Z dash , oh this one, oh sorry, yeah, I meant Y dash I wrote as Z dash, sorry, sorry That is a mistake, so that is Y dash, solved. Everybody happy? Yeah.

So, this quantity multiplied by this quantity is 0, what is this quantity? Geometrically XT, YT, ZT was the point on the surface, this is a derivative. So, this is the tangent vector to the curve at that point and says this vector is perpendicular to the tangent vector. So, what should be this vector? It should be normal to the surface; this vector is perpendicular to whatever curve I take. So, in the picture at this point , there is a vector which is not changing, because the vector is FX of Y, it depends only on F.

There is a vector and of course, that component is minus 1, so whichever their direction but at this point if I take the tangent, then this vector is perpendicular, if I take some other curve, take the tangent still it is perpendicular. So, this is a vector which is perpendicular to all tangents to all the curves. So, what this vector should be? We should call it as the normal to the surface, we should call this vector as normal to the surface, all the tangents will lie in a plane that will be the tangent plane to the surface.

So, we are going back to saying derivative helps us to define tangent. So, what we are doing? We are saying that derivative in the following sense the partial derivatives for a function of two variable, partial derivative, we assume there can differentiable.. So, partial derivatives exist and are continuous and so on.

Then this vector is perpendicular to this that means this vector FX, FY minus 1 is perpendicular to, is perpendicular to every tangent vector. So, we call this as, so this is called, this meaning FX, FY minus 1 is called tangent vector, to the surface at this point X 0, Y 0, Z 0, some confusion?

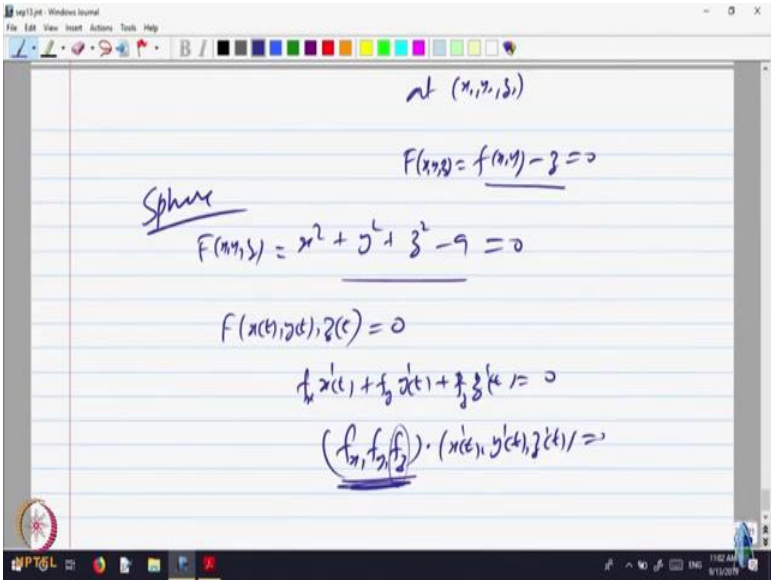
Student: (())(25:00)

Professor: Is called sorry, not normal sorry, is called the normal, vector to the surface at. Plane perpendicular to this is called the tangent plane, is called the tangent plane to the surface as at X 0, Y 0, Z 0. So, I am just trying to bring out the similarity between one variable and several variables. In one variable, we had continuity, we had differentiability, continuity meant that there is no break in the graph of the function and differentiability was something stronger which implied continuity and said that at every point you can draw a tangent.

So, same way we are saying in three variables differentiability you can define and differentiability implies continuity and differentiability means that at every point on the surface, you can have a, on the graph of the function that is a surface, you can have a tangent plane and that we are coming via normal. And saying there is a normal, normal is a gradient vector. Gradient vector is a very crucial one, it tells you how the things are changing on the surface, name gradient, itself should indicate English word gradient, should indicate.

So, rate of change is maximum. If it is, it is maximum and it is parallel to the gradient, and minimum and it is perpendicular to the gradient, gradient is also the direction of the normal geometrically to the surface at that point, you may be wondering why this minus 1 is coming because our surface is explicit, Z is equal to FXY? If you take a sphere, if you take the sphere X square plus Y square plus Z square equal to 0.

(Refer Slide Time: 27:33)



So, let me just, let me just indicate if you take the sphere is X square plus minus 9 equal to 0 say, that is my FXYZ that is a surface is given by a function of three variables, which is implicit. Z is equal to FXY I could write as FXY minus Z is equal to 0 I could write this as FXY, there Z was explicitly known in terms of, here Z is not explicitly known, but still even if I take a curve on this surface, it will still have that thing, same property X square, Y square, X square. So, what is F of XT, YT, ZT that will be curve on the surface that equal 0 on the surface, so that is equal to 0.

So, what is derivative of this? FX, X dash, FY, Y dash FZ, Z dash T equal to 0, that is same as saying FX, FY and FZ dot X dash T, Y dash T, Z dash T equal to 0. Still you can, what is this now? That is a gradient for this, what is the gradient? FZ is minus 1, that is why this minus 1 is coming here, there is nothing special happening, for explicit, when Z is explicit that value will be 1 or minus 1 depending on whether you write Z minus or this minus. In general, this will be the gradient vector, which is a normal to the surface..

So, this is very useful in doing what is called differential geometry later on that normals and so on. So, we will not be doing into that, my idea of bringing the directional derivative was to indicate that there is something called rate of change in any direction possible. And how it interprets geometrically. So gradient vector is one which is crucial. So, let us stop here.