Basic Real Analysis
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Lecture 34
Differentiability – Part IV

Right, so, let us begin, we had started looking at the notion of differentiability of functions of several variables.

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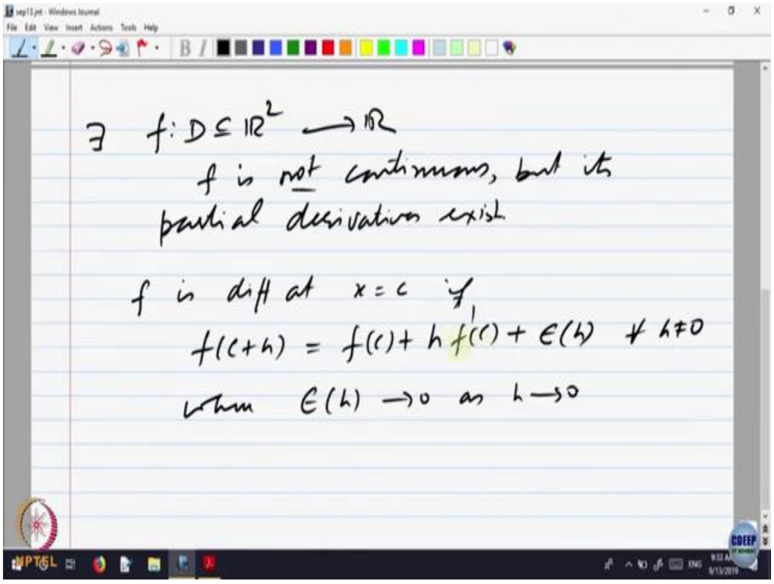

So, we gave examples for a function F defined in a domain D in R2. So, there exists functions F say that F is not continuous, but it is partial derivatives exists. So, that means the distance of partial derivative is not good enough, we need a stronger definition of what we should be calling as differentiability of functions of several variables.

So, let us go back to one variable for a function of one variable we said F is differentiable and at a point X is equal to C, if we can write F of X plus H or at C. So, let us write at that point C, C plus H equal to F of C plus H times F dash of C plus some error. If this is equal to this for every H not equal to 0 where the function epsilon the error epsilon H goes to 0 is H goes to 0. So, this is the analytical way of saying the differentiability of a function of one variable and this was the distance of derivative.

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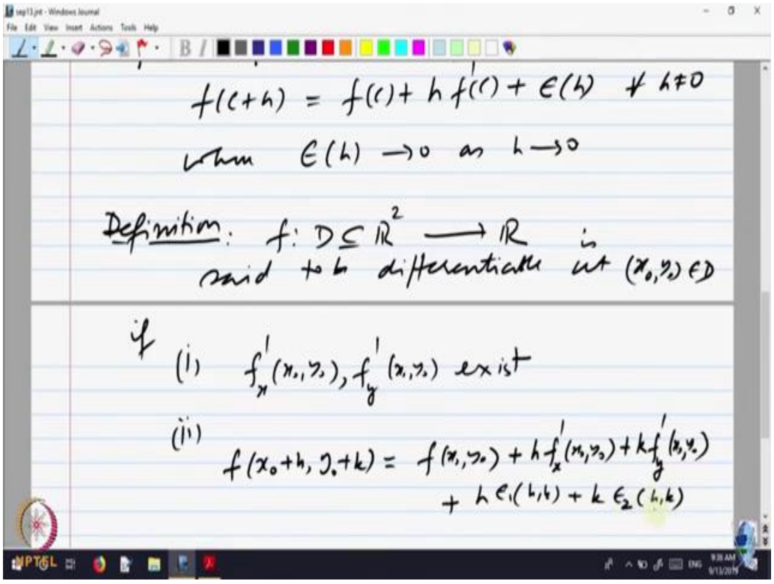

So, we will take this as a definition for two variables as follows, so let us write a definition for a function F defined in a domain, nice domain, so that everything is okay. But it can be an open ball around a point. So, F is a function of two variables is set to be differentiable at a point say X naught Y naught belonging to D if the following happens. One, both the partial derivatives exists F dash X at X naught Y naught, F dash Y at X naught Y naught exists both the partial derivatives exits.

And second, something like that error thing happens in both the variables. So, let us write what does it mean? It means that F of X naught plus H, Y naught plus K. So, that is the value of the function at a nearby point, should be equal to the value of the function at the point, plus in one variable we had H into F dash.

So, here there are two variables. So, each variable will contribute something. So, it is the increment in X variable is H. So, H times partial derivative of F with respect to X at X naught Y naught plus the second variable gives K times F partial derivative with respect to Y at X naught Y naught plus error functions. So, for both, so, there are two error functions H times epsilon 1 H, K plus K times epsilon two times H, K exists.

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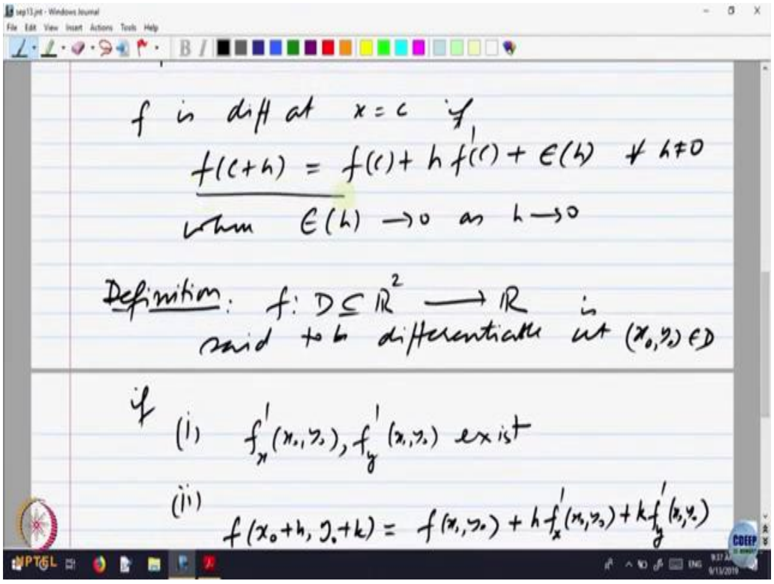

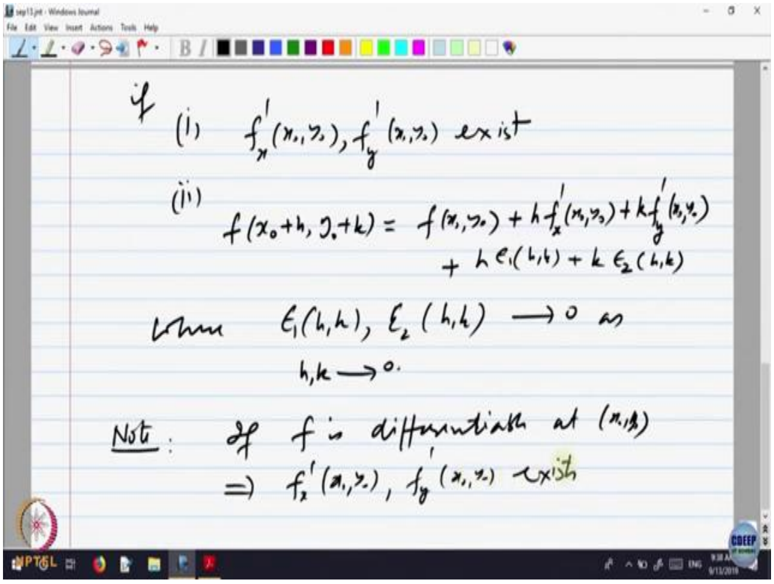

So, second condition is that value at a nearby point can be written as a value at that point plus H times F dash in the direction of X K times F dash in the direction of Y plus errors, like in one variable, but here that H times is incorporated inside this thing. So, where epsilon 1 H, K, their functions of two variables, they go to 0 as H and K both go to 0.

So, this is what the differentiability of a function of two way will means, we are trying to take care of each variable and similar to the definition in one variable. So, at a nearby point, so this is a value at a nearby point, the increment in the X direction is H. So, H times F dash in the direction of X increment and Y is K. So, K times F dash partial derivative with respect to Y, X naught Y naught plus the errors with respect to both the increments, which errors goes to 0 as H and K go to 0.

So, let us look slightly complicated but it is very much amenable in the sense that it is very much similar to the one variable definition. So, let us have some observations, so note, let us see whether this is good enough for definition or not. So, if F is differentiable at X naught Y naught then this okay. So, differentiability that if you require that the partial derivatives exist.

So, if the partial derivatives do not exist either of it at a point when the function is not going to be a differentiable. Implies F, X at X naught Y naught and the partial derivative with respect to Y, X naught Y naught exists.

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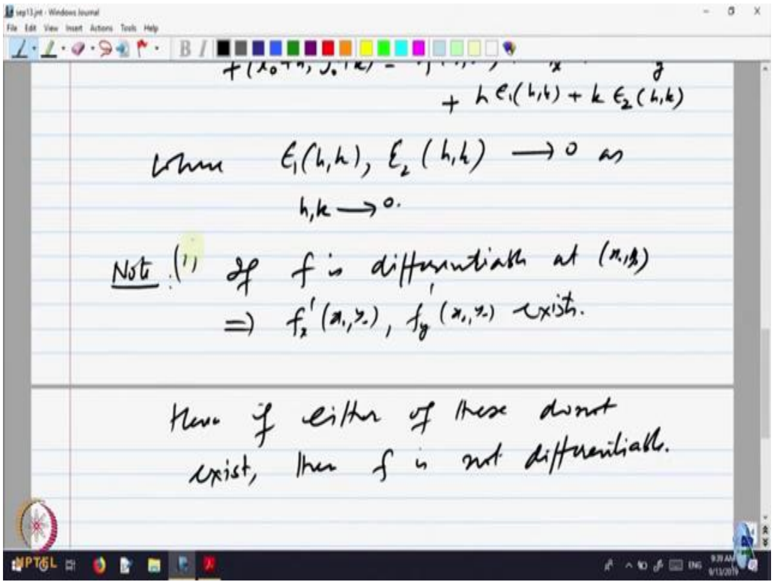

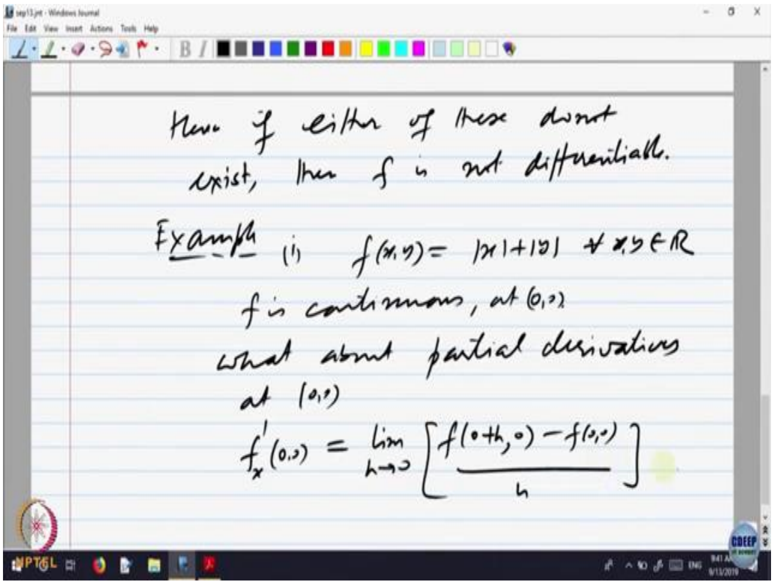

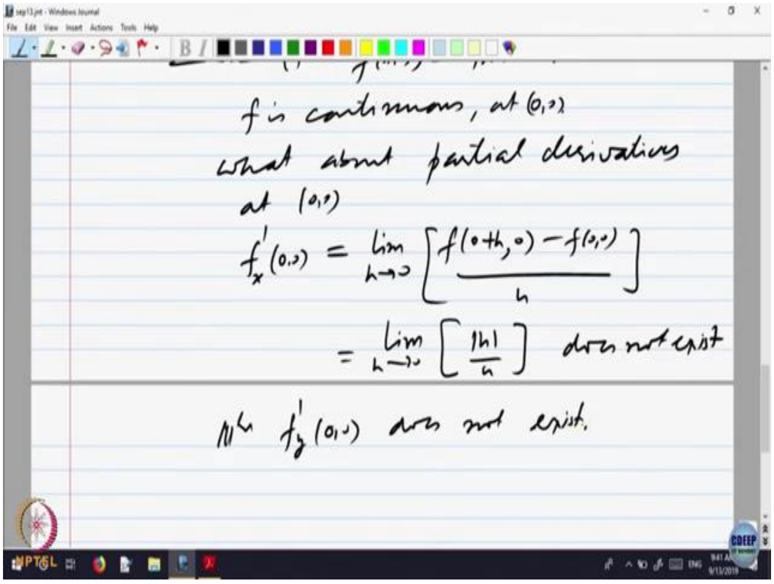

So, the consequence is a necessary requirement. So, hence, if either of these partial derivatives do not exist, then F is not a differentiable, so this is remark one. So, let us look at some example. So, example let us look at a simplest one, this is essentially extension of the one variable. So, let us look at the function F, X, Y equal to mod X plus mod Y for every X, Y.

So, the function of two variables, it is clear that F is continuous everywhere, let us say at 0, 0 also in particular. It is a continuous function everywhere continues, is it okay for everybody? It is continuous function mod X plus mod Y as a function of two variables it is continuous. Let us look at the partial derivatives, so what about partial derivatives? At the point 0, 0, so I want to calculate F dash of X at the point 0, 0. So, what will be that?

So, the partial derivative will be limit H going to 0, F of 0 plus H in the direction X 0 minus F at divided by H, There is a definition of the partial derivative at 0, 0, increment in the direction of X minus the value of the function at that point divided by. So, what is this? This is a limit H going to 0 what is F of 0 plus H.

So that is mod H, that is 0 divided H and that does not exist. Why it does not exist? Depending on it is positive or negative this limit is either 0 or minus 1 or 1. So left limit is not same as. So, it is mod X basically looking at it. So, done similarly, F dash of Y 0, 0 does not exist, so both the partial derivatives do not exist.

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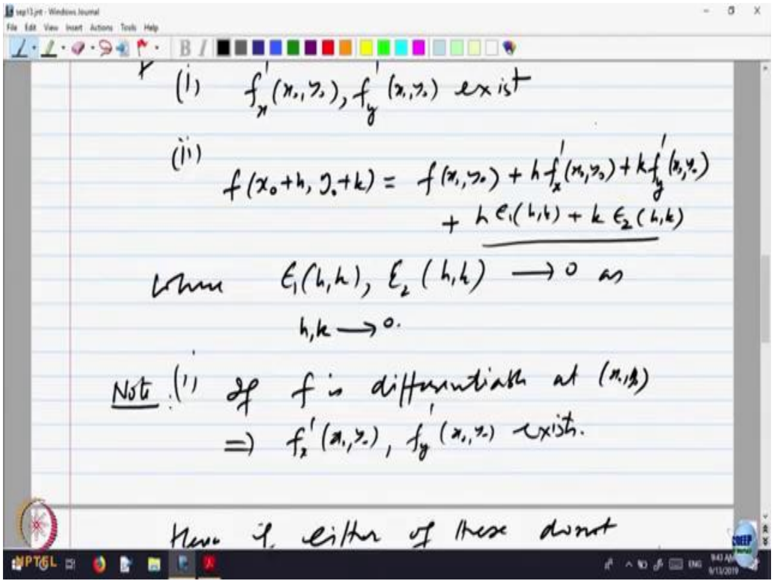

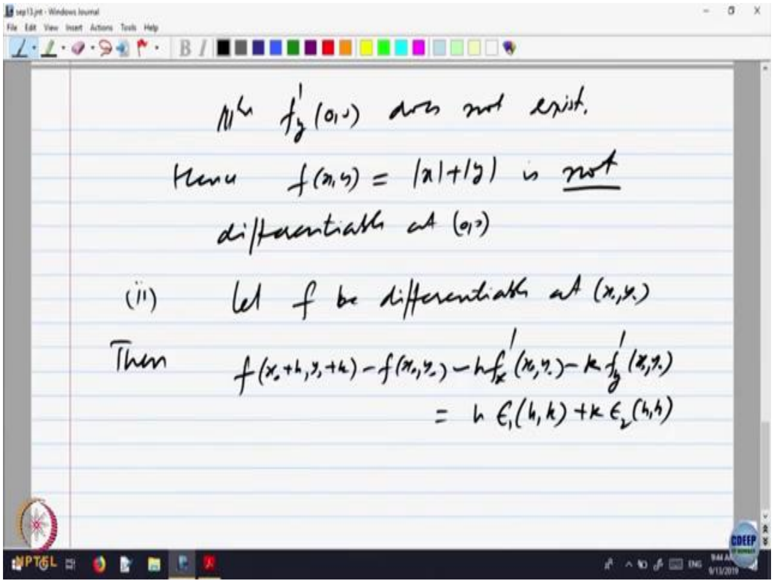

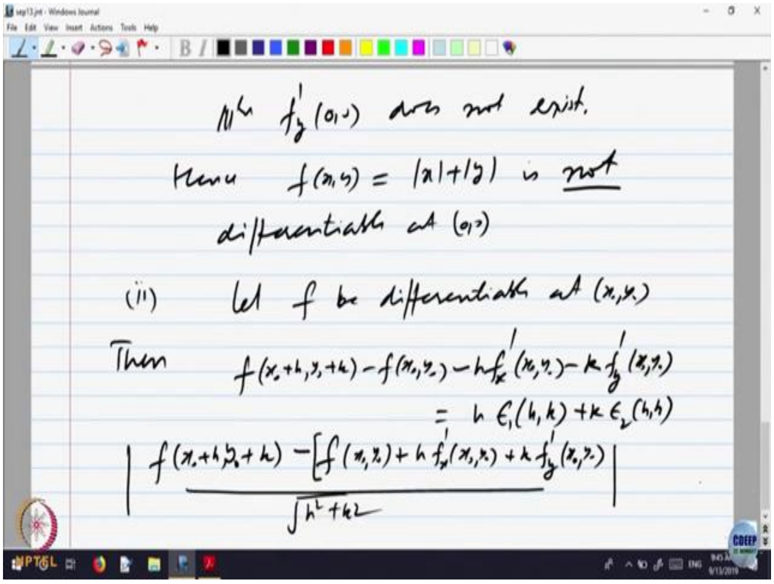

So, this function hence F, X, Y equal to mod X plus mod Y is not differentiable at 0, 0 because the partial derivatives itself do not exist. So, as per definition function to be differentiable, first of all partial derivatives must exist. Let us make another observation look at. So, let us suppose, let F be a differentiable at X 0, Y 0. Then we have with a nearby point X 0 plus H, Y 0 plus K.

So, let me bring all the terms on the one side except error terms, everything else on one side, like function of one variable. So minus F at X 0 Y 0 minus H times partial derivative in the direction of X minus K. F dash in the direction Y, X 0 Y 0. So that is looking back, so these terms I have brought at another side, the right hand side is H is equal to H times epsilon 1 H, K plus K times epsilon 2 the error function. So, epsilon 2 H, K.? I just taken some terms on the left-hand side and living error.

Now in one variable, when we look at the derivative is F at the nearby point minus F of X divided by the increment limit that is a limit of the secant slope. So, let us, so what is the increment here? In the H direction it is H, K direction it is, X direction is H, K direction K. So, total increment is square root of H square plus K square, if you take as a distance in R2.

So, let us divide by that, so, let us look at absolute value of F, X naught plus H, Y naught plus K minus F at X naught, Y naught, now let me put it this in the bracket so that it looks very much similar to one variable thing plus H times F dash X, X naught Y naught plus K times of F dash, with respect to Y at X naught Y naught divided by H square plus K square. So, I have taken divided by this and taken the absolute value.

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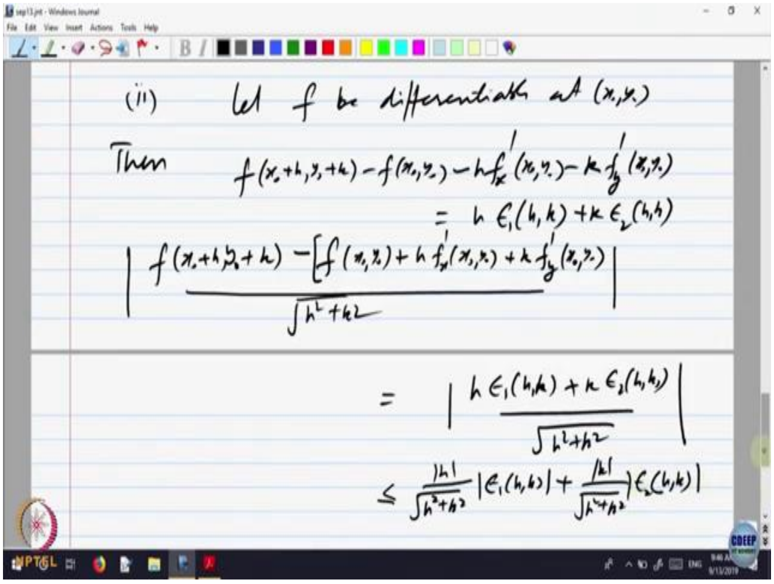

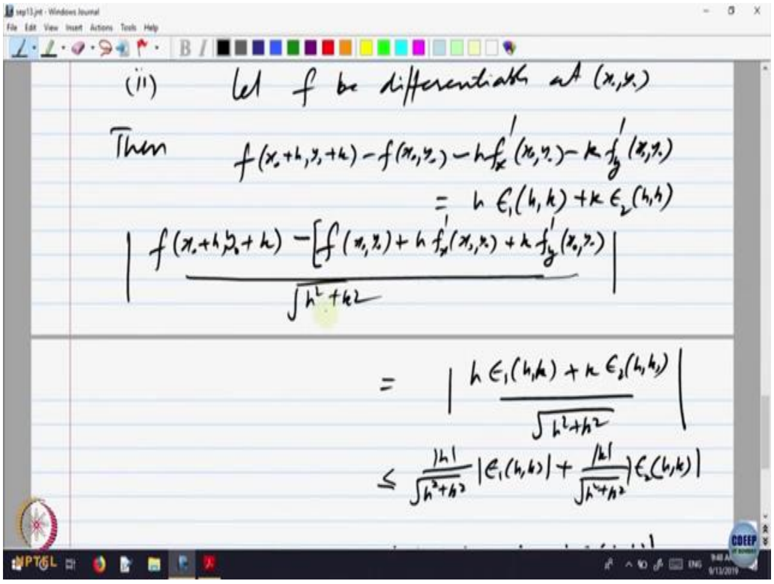

So that will be equal to, so in the right-hand side that will be equal to absolute value of H epsilon 1 H, K, plus epsilon 2 H, K divided by square root of H square plus K square, just divided and I am trying to bring it something similar to function of one variable.

Now this thing on the right-hand side I can separate that out is less than or equal to mod H divided by H square plus K square mod of epsilon 1 H, K plus mod K divided by H square plus K square, mod of epsilon 2 H, K you can track inequality less than or equal to. Now, let us observe does not matter actually, but mod H divided by this quantity is always less than or equal to 1 because denominator is always bigger than or equal to numerator that is less than equal to mod epsilon 1 H, K plus mod epsilon 2 of H, K.

Because mod H by this mod K by the square root of H square plus K square is less than or equal to 1 and this goes to 0 as a H and K, go to 0, 0. So, what we are saying is differentiability implies that this quantity on the left-hand side goes to 0 as, H and K go to 0, 0. So, this is very much similar to the one variable definition, and so it does not require and this does not require the knowledge of epsilon 1, epsilon 2 and so on.

So, it says F differentiable implies that this quantity, the partial derivative should exist, once they exist, you form this quotient and that should go to 0. So that will tell that if this is a differentiable that differentiability implies this and the interesting thing is one can prove the converse and say that if this condition is satisfied, then function is differentiable.

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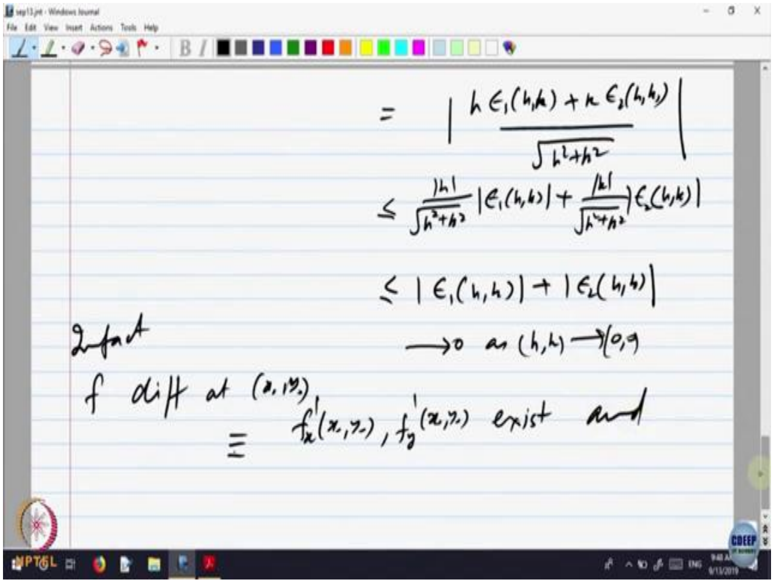

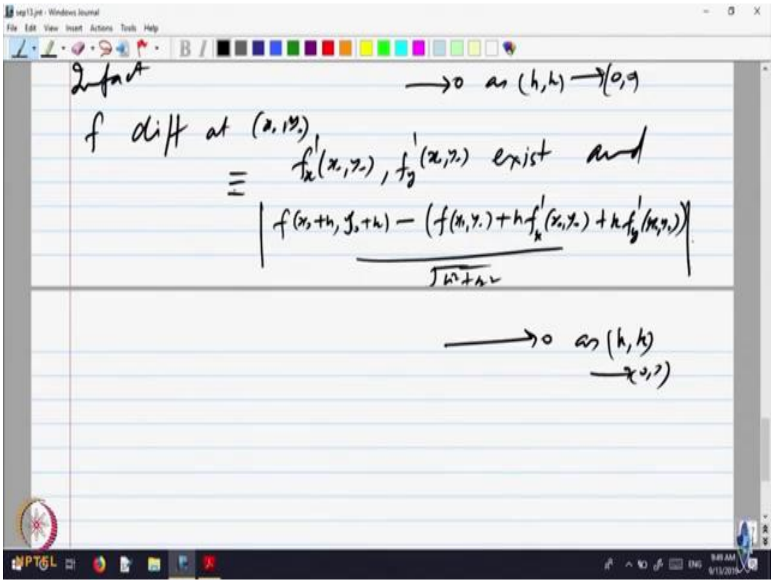

So, let us, will not prove that, but we will just list it. So, F differentiable, so, in fact, at X naught, Y naught is equivalent to saying the partial derivatives exists and this quantity, so minus. F at X naught, Y naught plus H times the partial derivative with respect to X, I am just writing again and again, so that it gets clear and plus K times partial derivative respect to Y divided by square root of X square plus K square absolute value goes to 0 as H and K go to 0. 0

This is not, we have just now shown one way it is quite easy to show, other way also it is not difficult one has to just manipulate a few things. So, we will leave that will assume that. So whenever, we want to check whether a function is differentiable or not, will check these two conditions are satisfied or not. So, this is like existence of partial derivatives and something goes to 0.

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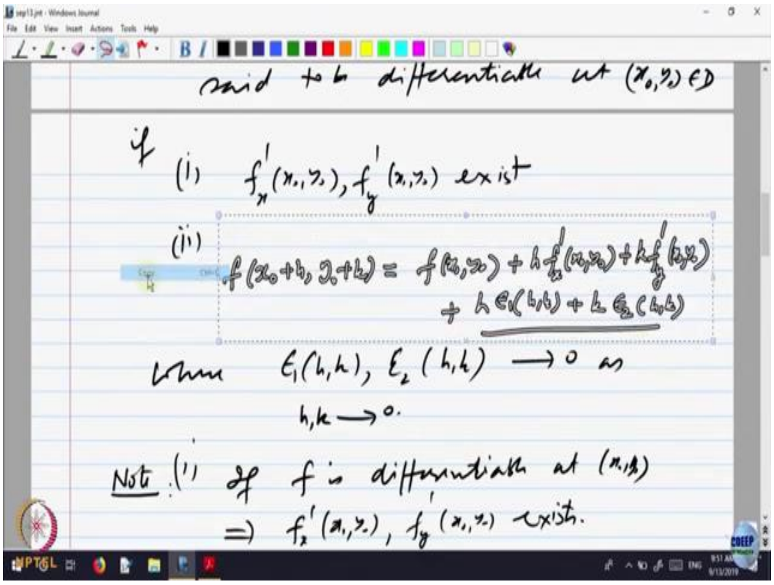

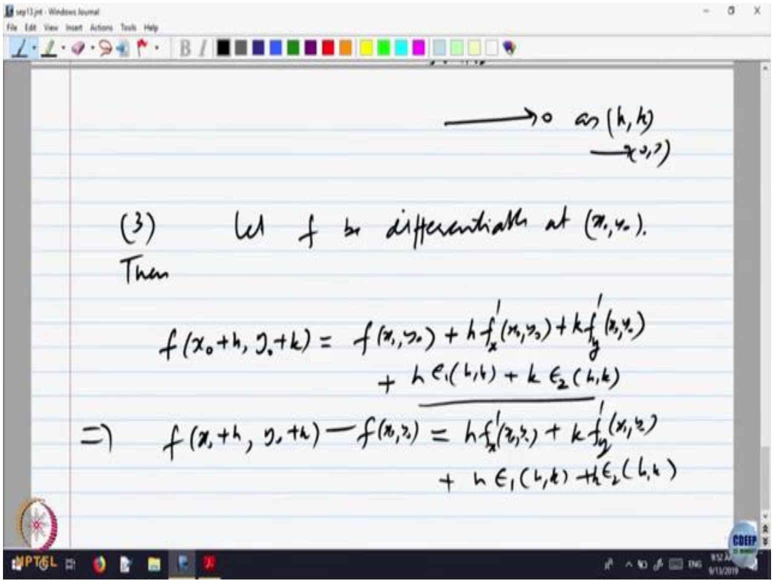

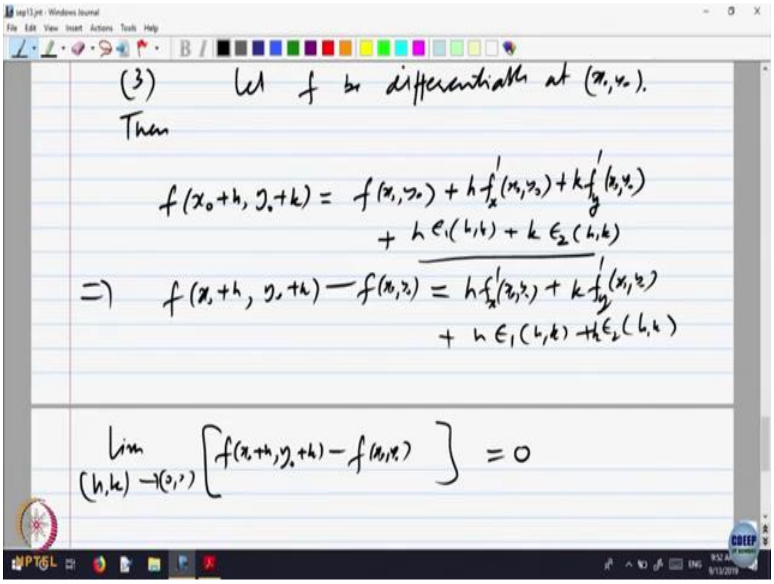

Let us look at something more consequence of this definition three. So, let F be differentiable at X naught, Y naught. Then once again let us go back to the definition and look at this equation, or let us go back to the original thing that was this one condition two. So, let us rewrite this, what I want to do is I just want to bring one time on the left-hand side, everything else on the right-hand side. So, let me do that and see if I can copy that equation without much effort let us write it.

So, this is what we got. So, let us shift, so implies F of X naught plus H, Y naught plus K minus F at X naught, Y naught is equal to H times something F, X I have just X naught, Y naught plus K times F, Y X naught Y naught plus H times epsilon 1 plus K times epsilon 2. Now let us take the limit of both sides as H and K go to 0, 0, what happens? So, limit of the left-hand side.

So, limit H, K going to 0, 0 of the left-hand side, what is the left-hand side? F at the nearby point minus F at X naught Y naught is H time something, H goes to 0. So, the first term goes to 0, second term is K times f dash partial derivative that goes to 0, H times, K times everything goes to 0. So, this limit is equal to 0 because all the terms on the right-hand side go to 0 as H and K goes to 0, 0.

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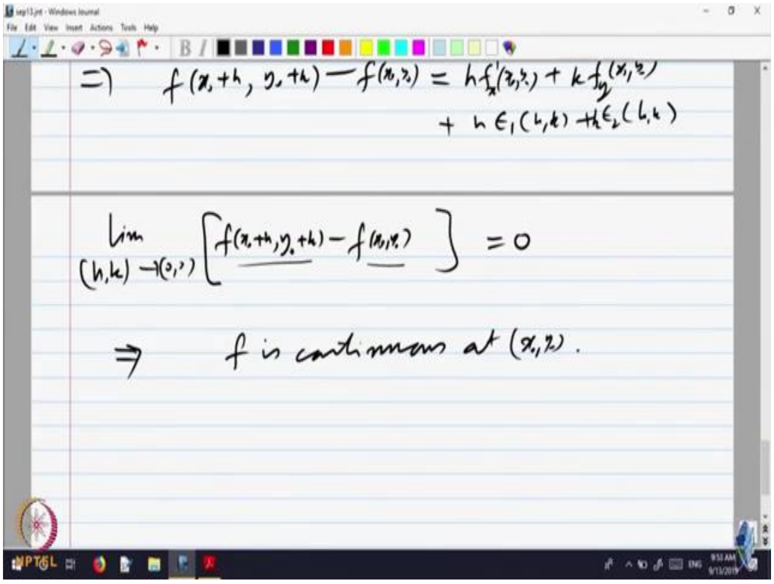

So, what does this mean? You are saying F at a nearby point minus F at that point the distance goes to 0. That means a function is continuous at the point X naught, Y naught. So implies F is continuous 0 at X 0, Y 0. So, this notion of differentiability does imply the function is continuous. So, this seems to be a good enough definition for differentiability. There is another way of checking something is differentiable or not and that is only a sufficient condition.

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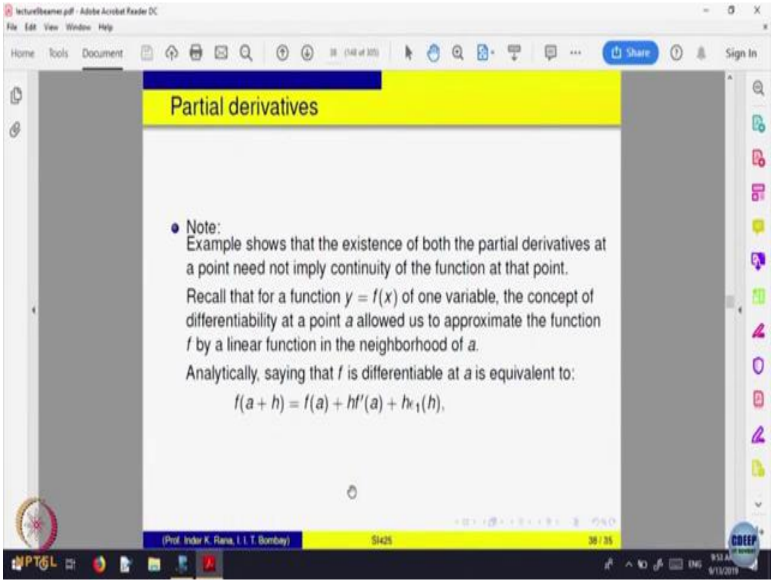

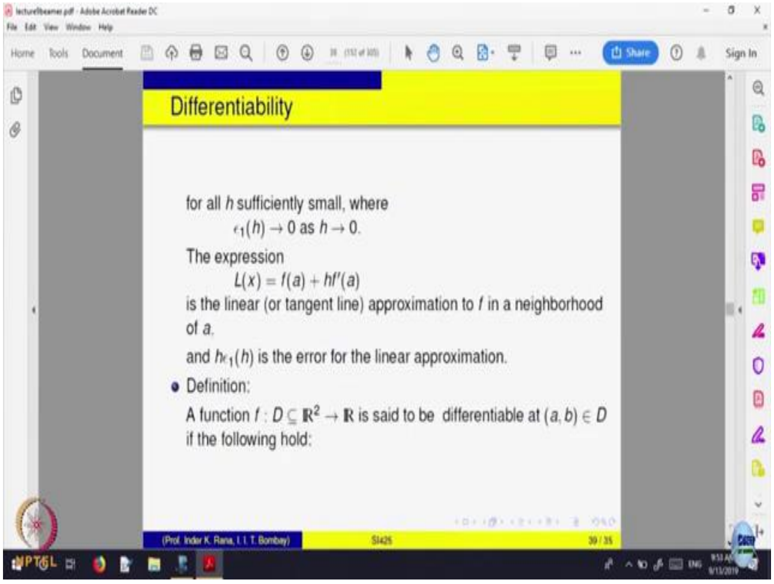

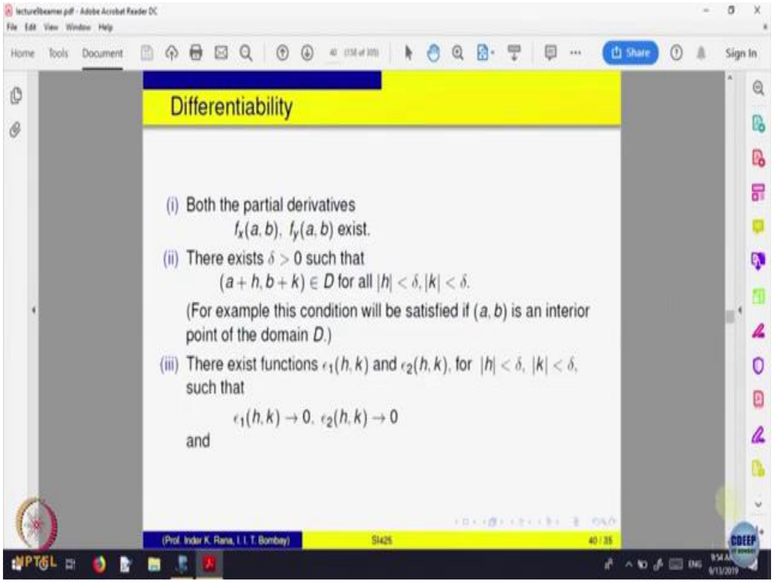

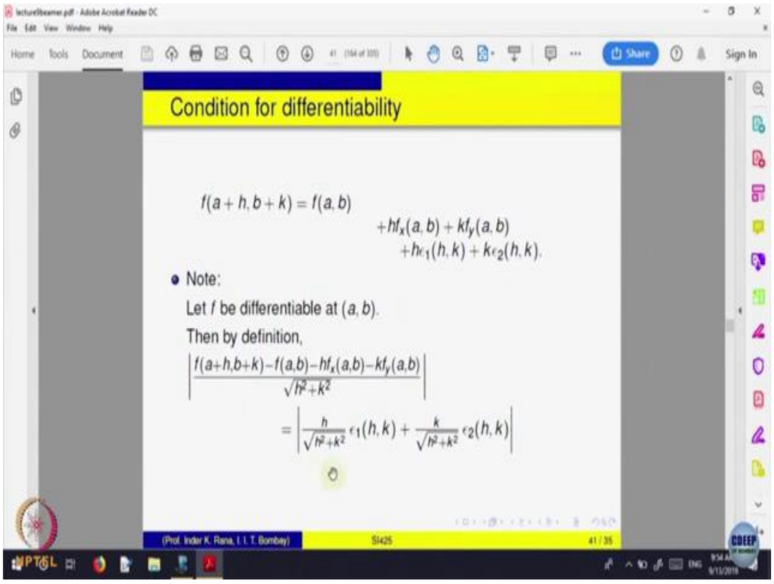

So, probably let me look at the slides and show you. So, there is one variable definition and so this is the two variable definition that we just now said, here the point taken is A, B. So, at a nearby point, there should exist error functions, epsilon 1 and epsilon 2, such that the value at a nearby point is equal to the value at that point plus the increment in the X direction, H partial derivative, K partial derivative, H times epsilon 1 plus K times epsilon 2. And we said that this is equivalent to by taking everything on the one-side, dividing and taking the limit.

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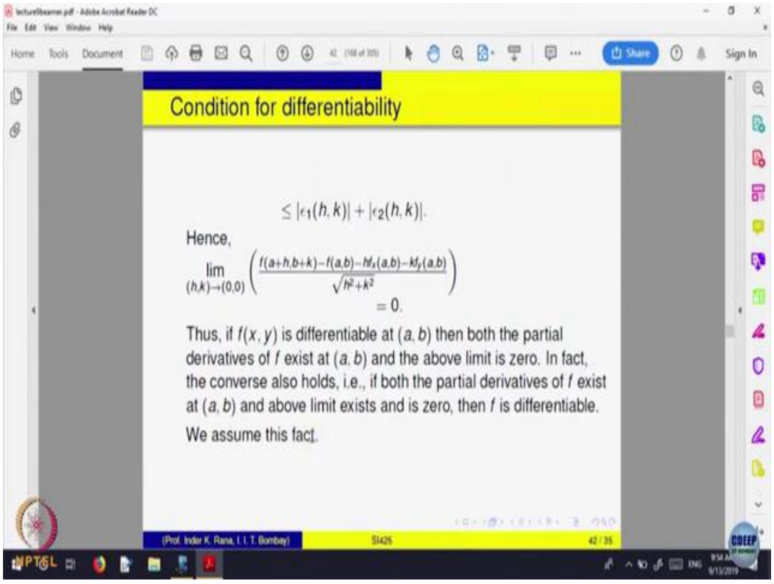

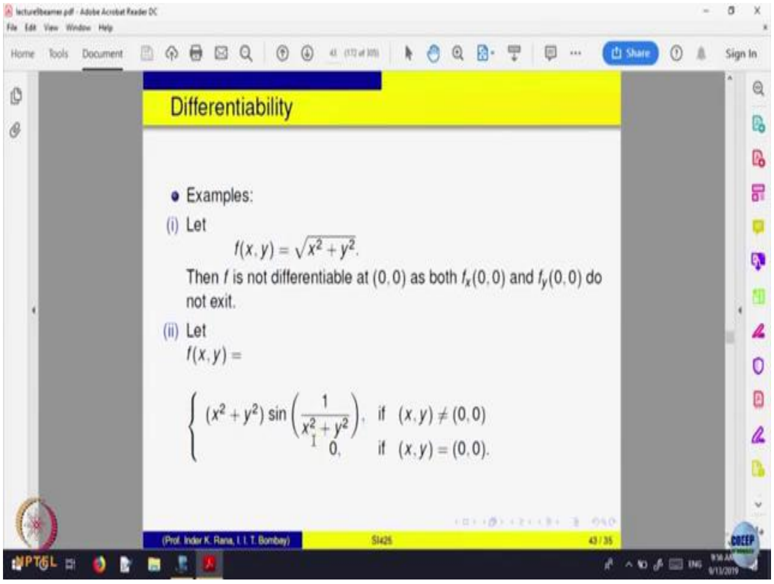

So, it goes to 0. So, differentiability implies that this goes to 0 and conversely is also true. So, we will, we assuming that fact. For example, here is another example, look at this function F, X, Y equal to X square plus Y square, square root. This function is continuous at 0, 0. As X goes to 0, Y goes to 0, this goes to 0, 0, obviously, it is a continuous function.

If you want to write epsilon delta definition you can write down, epsilon equal to delta that will be okay, this is not differentiable at 0, 0 once again because a partial derivative do not exist when you want to calculate partial derivative at 0, 0, you will put Y equal to 0, square root of X square. So, what will be that? That will be mod X, so, that is not differentiable at 0, similarly. So, this is not differentiable at 0, 0 because the partial derivatives do not exist, we already had one example, let us look at this example. This is F, X, Y equal to X square plus Y square, sin of one over X square plus Y square as X, Y naught equal to 0, 0.

Because we are dividing at 0 the value is 0, is this function continuous at 0, 0? Because value at 0 is 0. If I take the absolute value sin is bonded by one, so square root of, this is just X square plus Y square which goes to. So is bounded by X square plus Y square which goes to 0, 0. So, limited 0, 0 is 0 which is the value of the function, so, this function is continuous.

Let us find whether this is differentiable or not, so what about the partial derivatives? How do I find partial derivative with respect to X? So, Y equal to 0, this is X square sin 1 over X square. So is that differentiable with respect to X?

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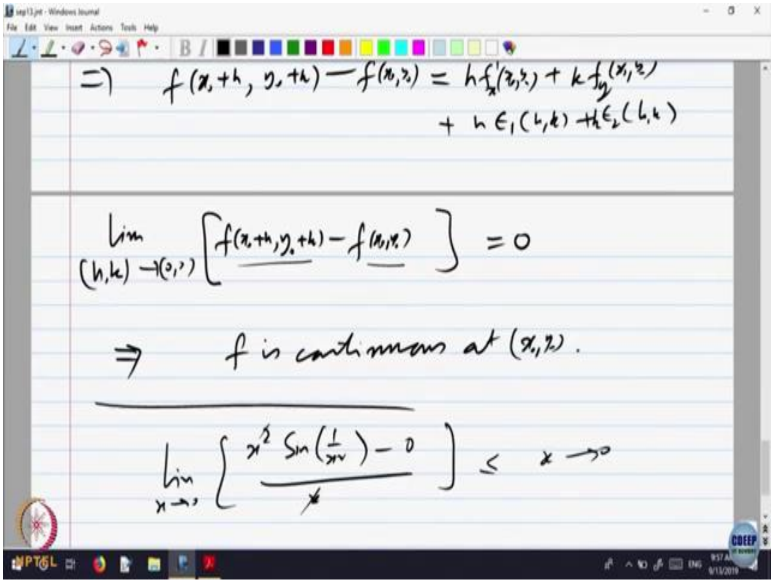

So, X square sin one over X square, Y is 0. So, minus the value of 0 is 0 divided by X limit of this X goes to 0 that will be the partial derivative at the point 0 with respect to X. And that this X cancels, still X is there, so it is less than or equal to limit of X. So, X dominates power is two here, sin is bounded by 1, so which goes to 0. So, even as a function of one variable X square sine one over X square is differentiable with respect to X at the point 0.

So, that says this is also differentiable. So, partial derivative exists at 0 and is equal to 0 with respect to X or this function, similarly the partial derivative with respect to Y also exists and is equal to 0 at 0, 0. We want to check whether this as a function of two variables is differentiable at 0, 0 or not.

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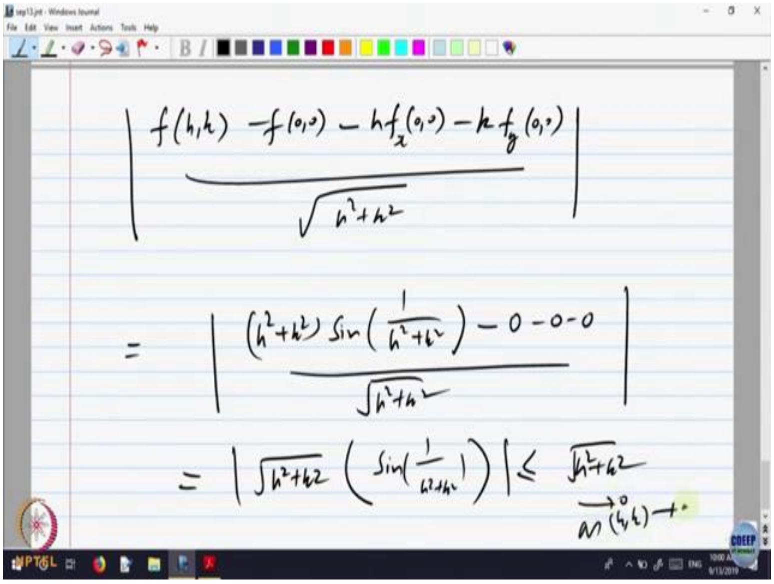

So, let us try to apply that criteria that we develop just now. So, I want to look at F at a nearby point, near 0, 0. So, H, K minus F at 0, 0 minus H times partial derivative, I do not have to put dash actually when I writing partial, they are both symbols, I should not be putting, 0, 0 minus K times partial derivative with respect to Y at 0, 0 divided by H square plus absolute value of this.

If that goes to 0, then the function is differentiable at the point 0, 0 as a function of two variables. So, what is this equal to? What is a function at a point H, K? So, this is H square plus K square sin of one over plus K square. F at 0, 0 is 0, partial derivative at 0 is 0, partial related to Y is 0, so square root of X square plus K square. So, this quantity is equal to, this cancels out, only square root is left. So, H square plus K square absolute value sin of one over square root of H square plus K square.

Now, this is H square plus K square is a square root of H square.

Student: (())(28:21)

Professor: So, oh right, the square root does matter I think that is immaterial okay, yeah that does not affect, I think, so that is. So, and that is less than or equal to sin is bounded, so it is less than equal to square root of H square plus K square and that goes to 0 as H and K goes to 0, 0.

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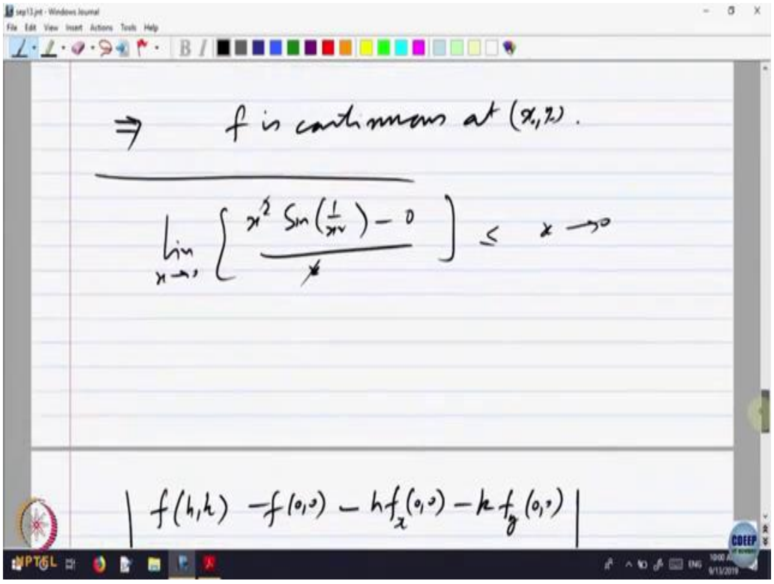

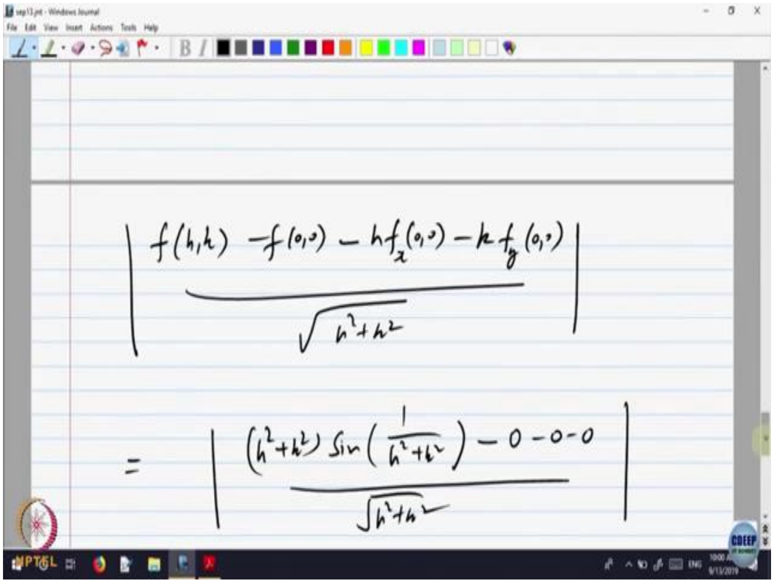

So, this function, so F is differentiable at 0, 0. So, there is a function as a function of two variables. So, we have checked it by checking that the partial derivatives exists both the partial derivatives exists and this quantity goes to 0 as H, K go to 0, 0. So, this is a differentiable function.

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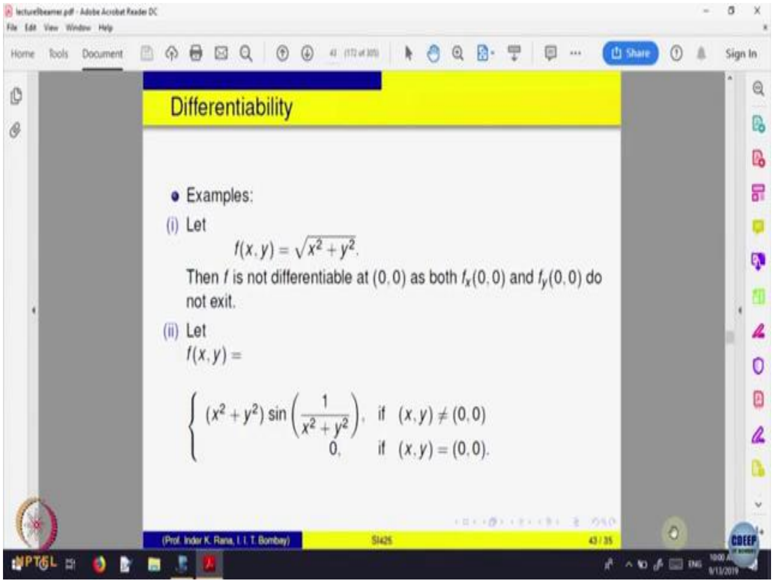

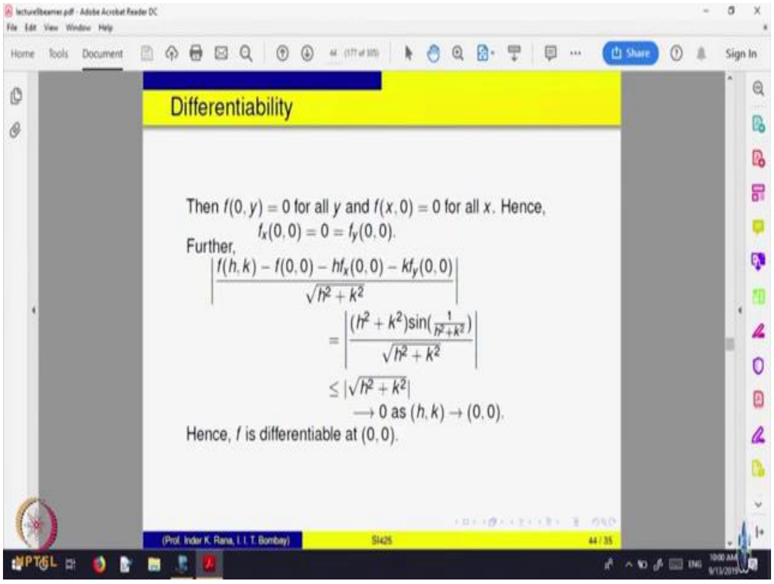

So that is this function, you just now checked is differentiable, so you can check that it is differentiable.