Basic Real Analysis
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Lecture 33
Differentiability – Part III

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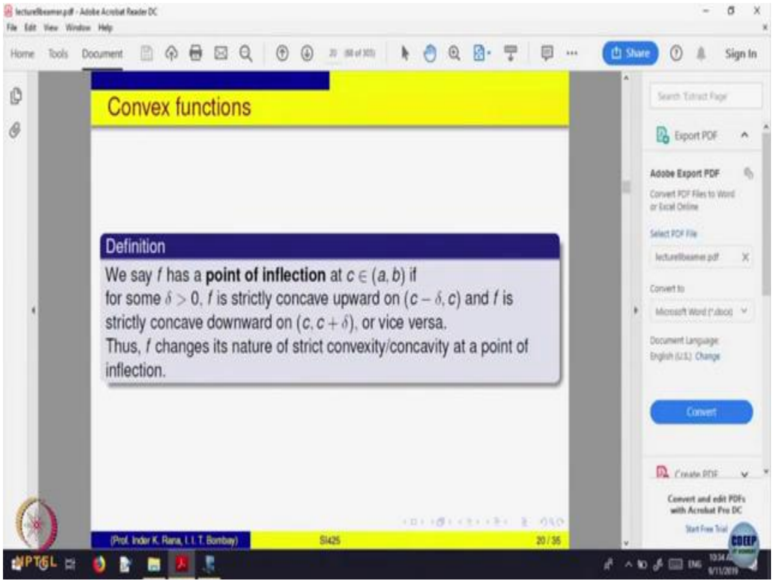

All these properties, there are points where the nature changes, they are called points of inflection. For example, if you look at Y equal to the function X cube. How does the graph of the function X cube look like? It goes from the down it is negative, X cube is negative on the negative side at 0 and then it starts increasing, but the nature here it is, the nature here it is different.

It is concave down, it is concave up there and at 0 there is a tangent which is horizontal because derivative of F of X cube is X square at X equal to 0 slope is horizontal. So, at 0 there is a tangent, so it goes smoothly and cup. But the nature of the function on the left and on the right are totally opposite to each other, concave down to concave up.

So, you say the point X is equal to 0 is a point of inflection. So, you can define in general a point of inflection to be a point where the function changes its nature from concave up to concave down or vice versa. All these are properties of the function; it helps you to draw a picture of the function how does the function look like?

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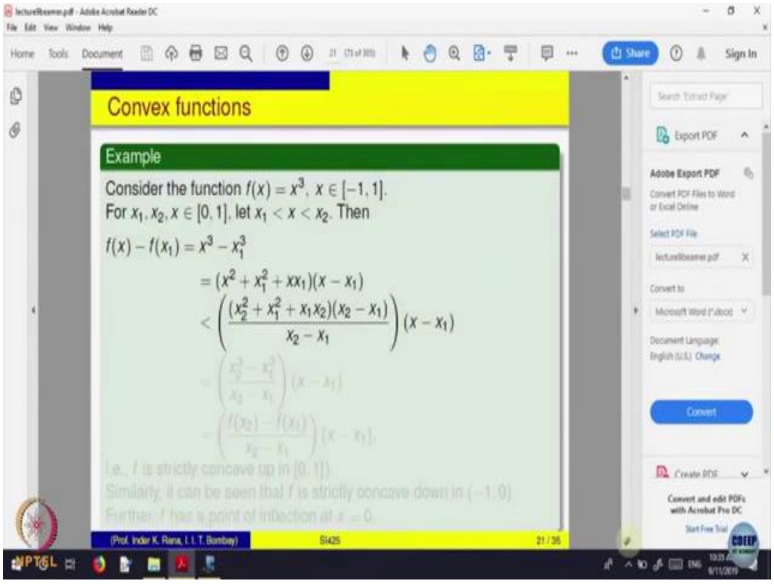

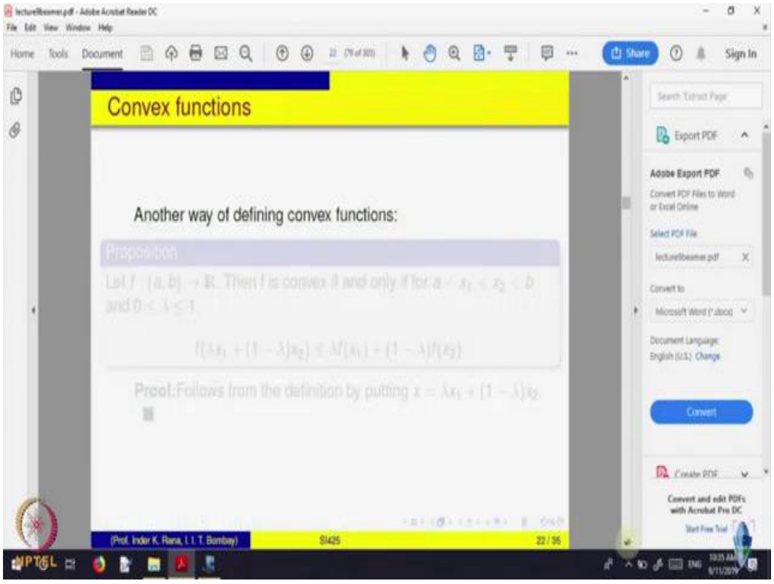

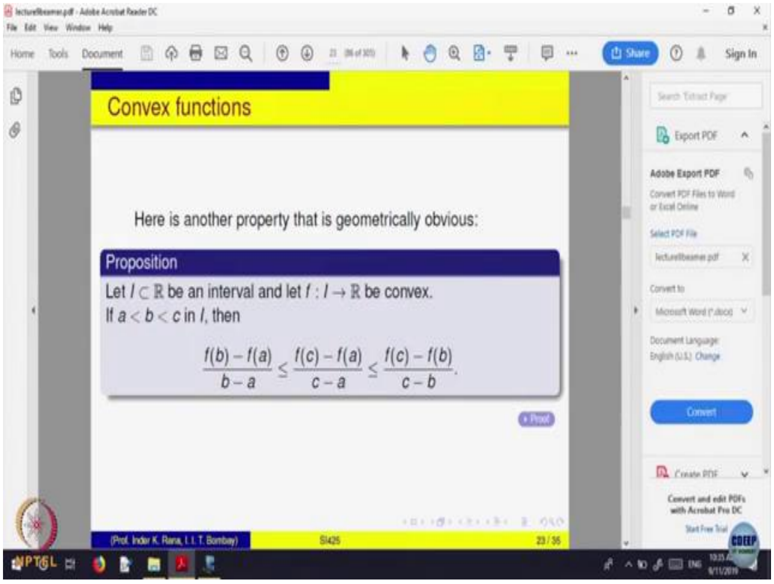

And so, there are examples you can just skip those examples. And important theorems are the following. This is an important that if a function is convex, and you are given three points A, B and C, what is the first one F B minus F of A? That is a slope of the cord joining with B and A. C and A that is a cord joining C and A. And this is the cord joining C with B. So, these are relation between the slopes of these three chords.

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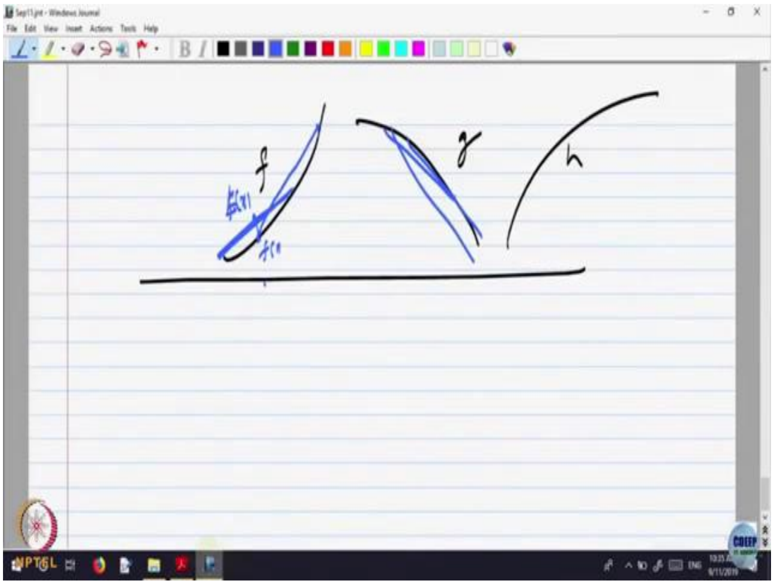

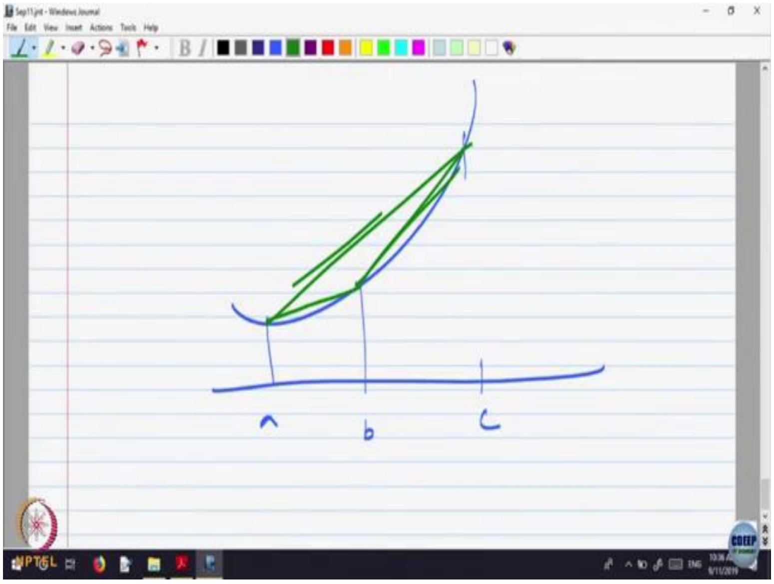

So, let me probably draw a picture that you can see easily, if this is the function and this is A and this is B and this is C, I hope there is a way it is taken A, B. So, look at this cord and look at this cord, what is the ratio between these two slopes? Geometrically you can just look at the slopes, slope of A to C is more than the slope of A to B. And look at this one now. At this point, this slope is and this slope what is the relation? That is less.

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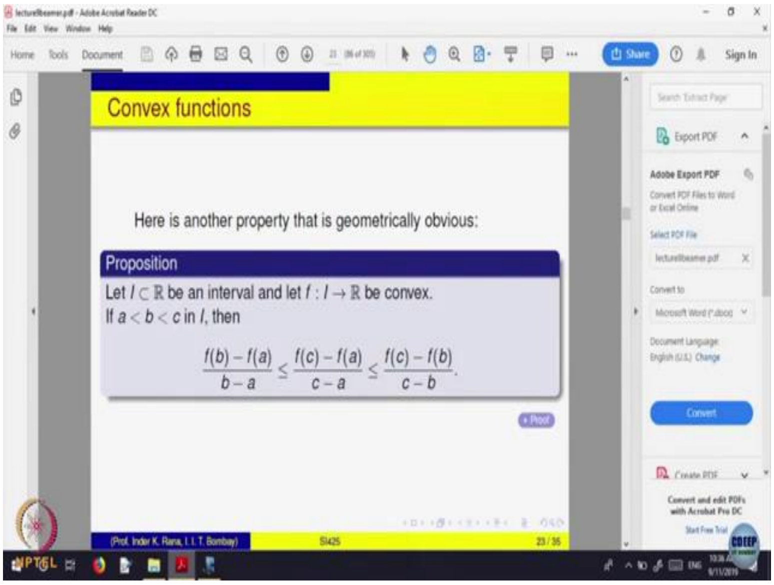

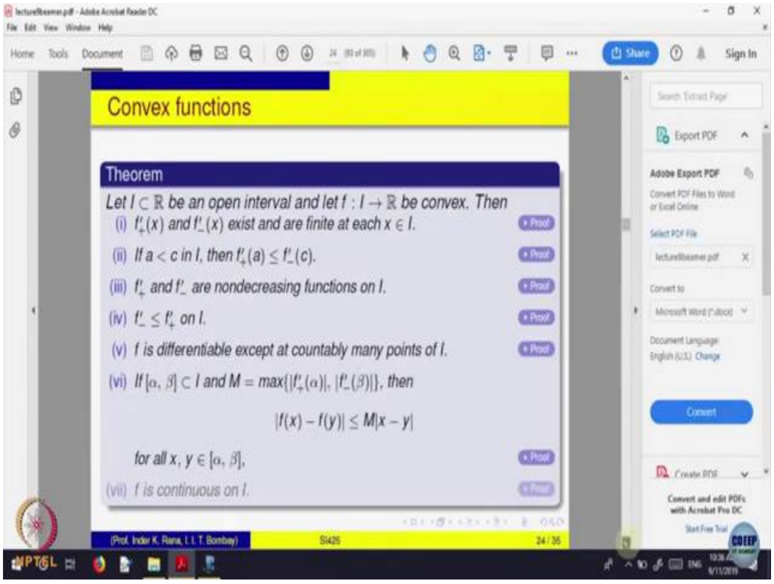

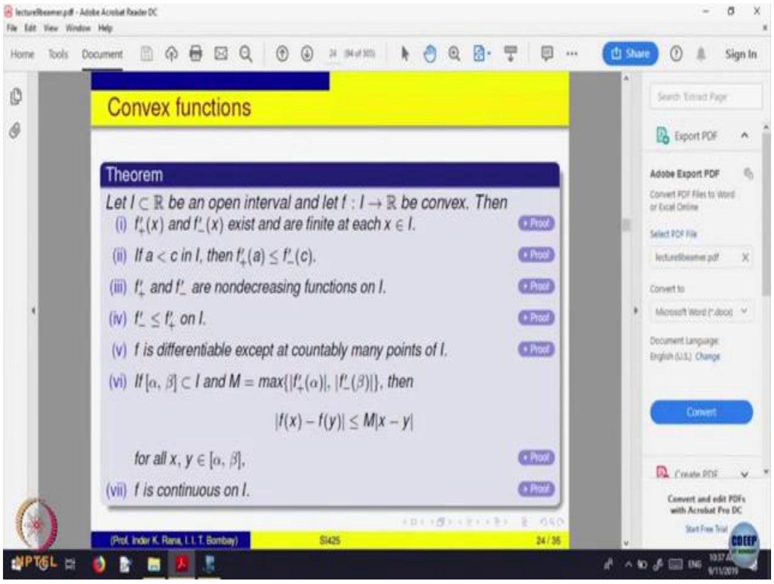

So, that is what this is saying, that the slope between the three chords is as follows, one can prove it mathematically will not do it. As a consequence of this, one proves beautiful theorems about kind of X functions let me. It says at every point you will have a left derivative; you have right derivative and so on. So, there are a lot of the results, so, just go through them once, read them once and forget them, because I will not be asking them in the exam.

But you should understand what this implies? For example, it says that plus these are the derivative from the right side, F dash plus, plus is on the right side and the left side derivative exists at every point, that is a beautiful thing. And the right derivative is always less than the left derivative you can because of that slopes actually basically taking limits of them.

And eventually all of them need so I say, accepted. So, here is something interesting a convex function is differentiable everywhere except at countably many points. It need not be increasing, it would be like, in some parts increasing, some parts are decreasing, but still it has that properties. It has to be and of course, differentiable implies and it is continuous everywhere. So, very nice properties of convex functions. And you may come across these kinds of things in your courses in probability and statistics. So, that is why exposure is a good idea.

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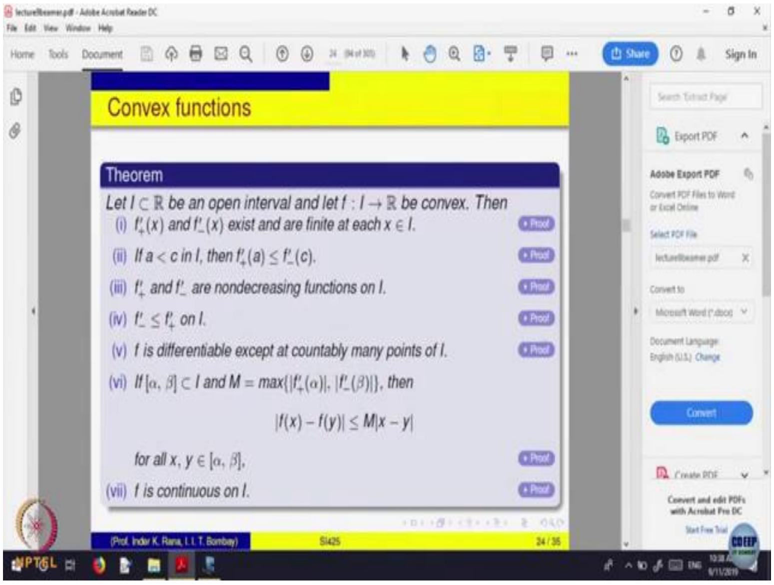

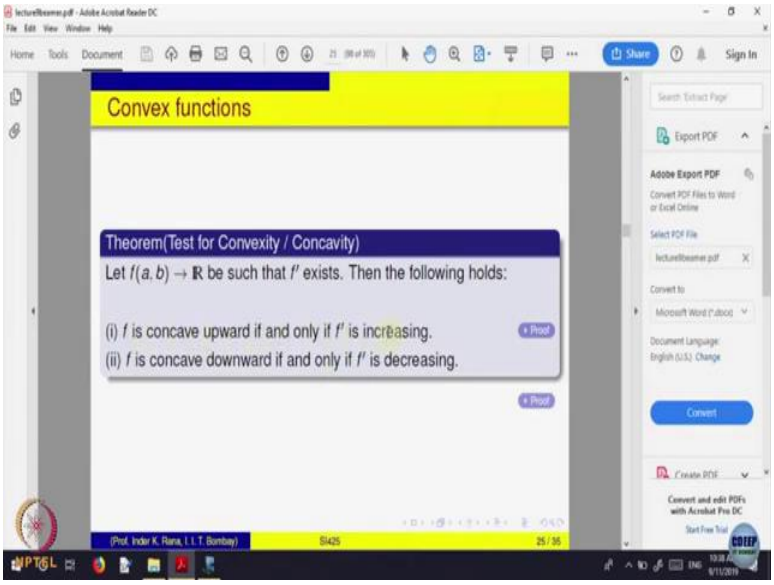

One can state results sufficient conditions for a function like derivative gives you the nature of the function increasing or decreasing. Similarly, nature of the second derivative gives you whether it is a convex or concave. So, essentially this is the what if F is concave upward if and only if derivative is F dash, see look at the slope of X square right at 0, negative becomes 0 and then started increasing so, that is F dash is increasing.

And if the second derivative is here, when is secondary derivative in a function has second derivative and F dash is a function which is increasing what is the nature of the second derivative? F dash is increasing. So, what can you say about the second derivative? Just now we said increasing size derivative exists derivative should be bigger than or equal to 0. So it says second derivative should be bigger than or equal to 0. So, that is the second derivative test for convexity, concavity. Function may not have second derivative, if it has then that should be the property. So, that is the first derivative test.

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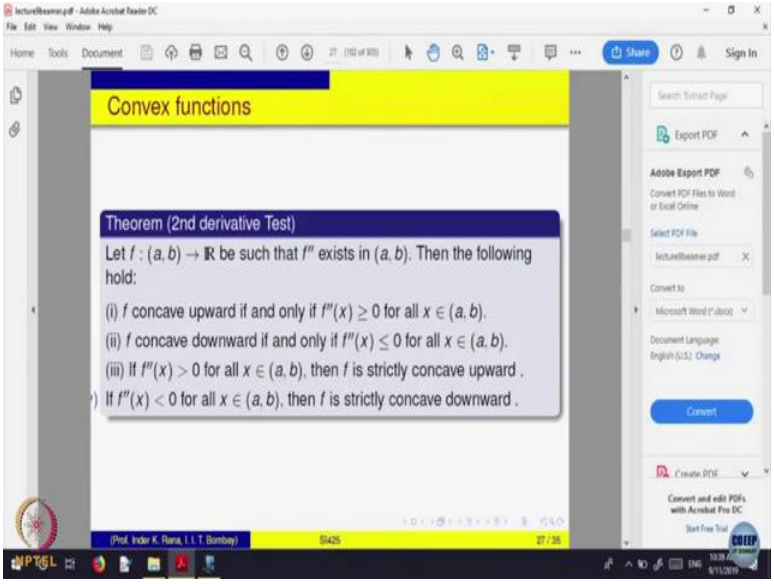

And the second derivative test is like, F concave upward if and only if second derivative is bigger than or equal to 0 and other all, right. So, all this is because we know the function is differentiable once or twice and we know the property of the derivative or the second derivative, it gives you back the knowledge about the function that is an important thing. So, that is convexity and concavity.

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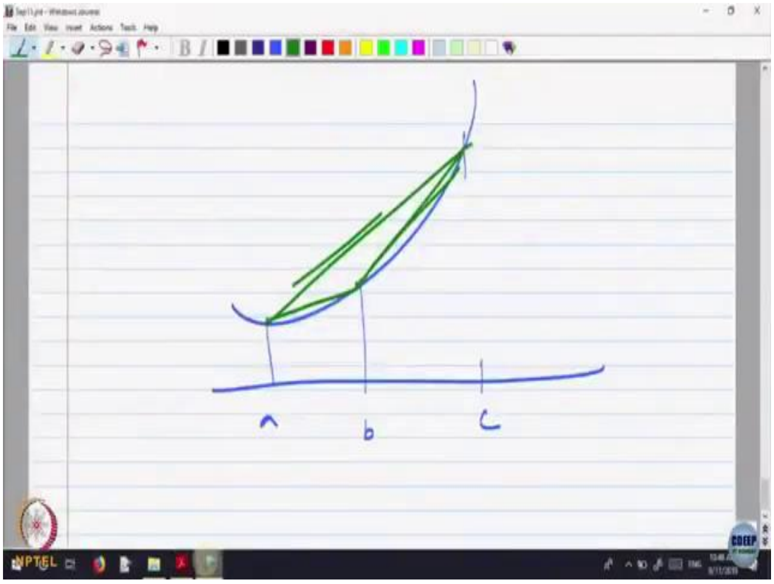

Let us start looking at, the we have time. So, let us start looking at. So, in basically in one variable what have tried to give you a field for what is derivative? Why we consider derivative, the function of one variable? The important property of derivative is that it gives you some kind of a smoothness of the function that is saying at every point you can draw a tangent.

And the derivative gives you the slope of the tangent, derivative does not give you the tangent quite often is a misunderstood, misstated that saying function is differentiable, derivative is a tangent, derivative is not the tangent. Derivative is the slope of the tangent at that point, tangent is a line which you have to draw, once you know the slope, you know the point, you know the tangent line.

So, and the important algebra of differentiable functions that we just went through, assume that you have all gone through these kinds of theorem before. Derivative of addition, subtraction, product tool, quotient rule, change rule and so on. Important thing of derivative function mean differentiable also is that you can approximate the values nearby with the values at that point and the derivative that is important way of saying what is differentiability.

And then we saw that if you have, if you know what is the derivative? You know some property of it, then that gives you back some properties of the function and the main term is called Lagrange’s Mean Value Theorem which gives you the applications.

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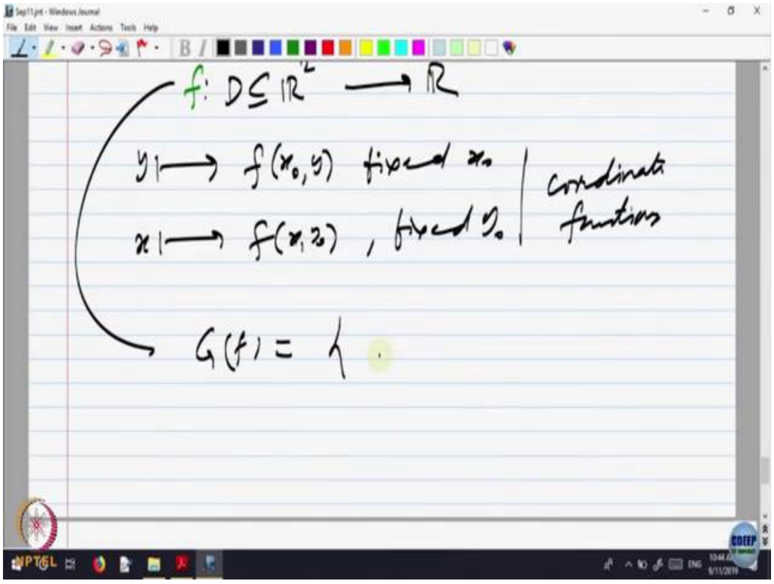

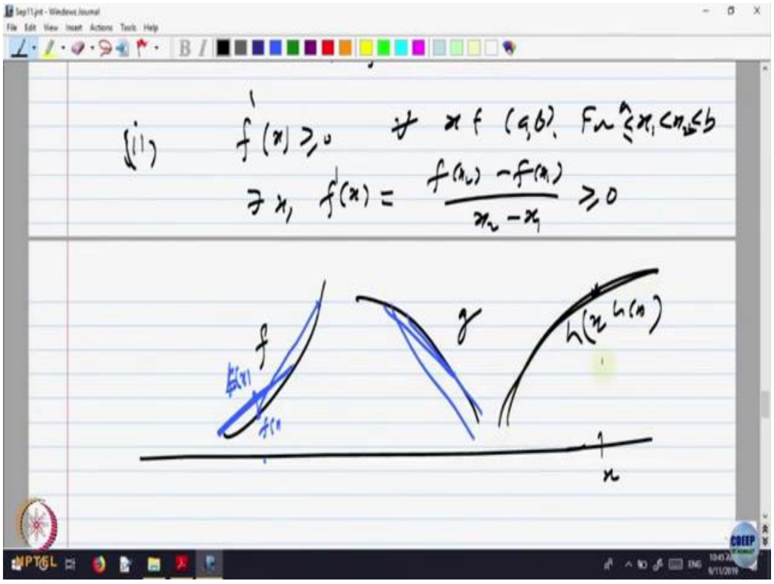

So, let us now look at a function of several variables. F is a function defined on intervals, so let us look at domain D will do it in R2, same analysis works in R3 and so on and RN. We have already analysed continuity of functions of several variables, in the previous lectures. And we try to look at given this function, we if I fixed X naught and Y. So, Y going to or you will fake for fixed X naught and X going to F of X, Y naught, fixed Y naught, so they were the coordinate functions.

So, a function of two variables once you fix all but one coordinate that gives you the coordinate functions, and we saw that continuity of a function is much more than saying each coordinate function is continuous. So, now, let us look at differentiability we want to come to a notion of differentiability of functions of two variables.

And we would like to say that differentiability should enable us to give something what we have done in one variable and what are these things? Notion of tangent I should be able to define in one variable, and I should be able to say it is smooth and smoothness was saying the tangent is possible. Now, what is the meaning of saying corresponding thing if you want to translate for functions of two variables. In one variable smoothness was we are able to draw a tangent at every point to the graph of the function.

What should be the corresponding thing for functions of two variables? So, let us look at the graph of function of two variables, what is the graph function of two variables? So, F is a function. So, for this function what is the graph of the function? Before you want to draw something like tangent, we should know what is a graph? What is a graph of a function of two variables? Or what are the graph a function of one variable. All the points in the plane X comma F of X every point on the graph is F, X comma F of X. So here, this is X and this H of X. So, what is this point? Coordinate of this point X comma H of X. So, all these points are H comma, X comma H of X.

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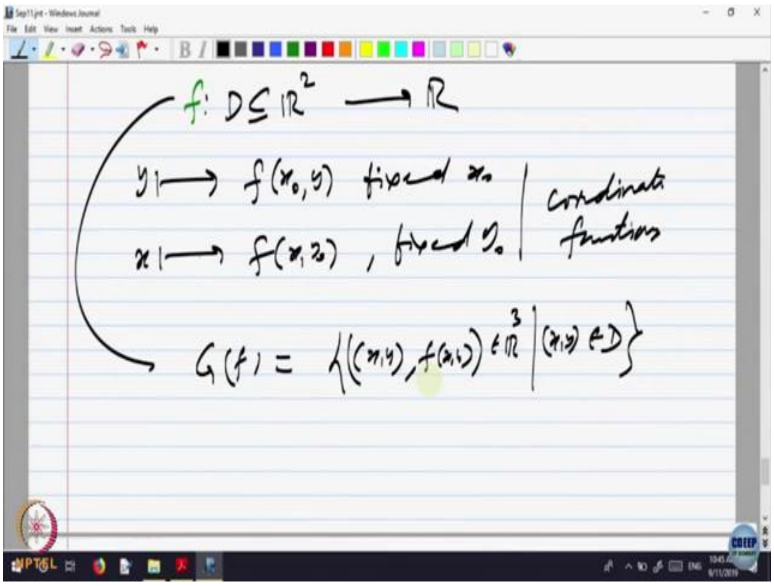

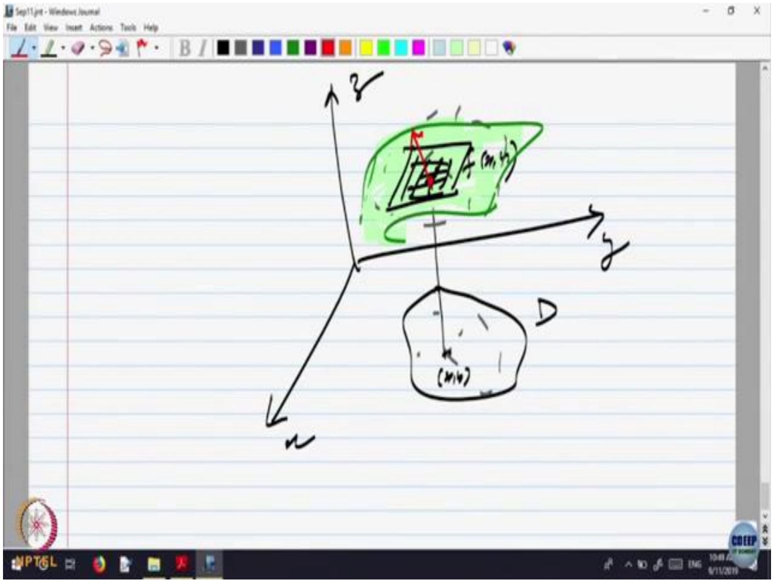

So, so, here what is the domain? Domain is X, Y two points and what is the value? F of X, Y. So, what is this point? If I know this thing, I know the function, for every point X, Y belong to the domain, if I know this people, I know the function at X, Y the value is F of X, Yso, this is the point in R3, this is a point in R3, X, Y. So, what does it look like?

X, Y, and here is a Z coordinate. So, here is a domain D, you take a point X, Y, we have done it, when we define what is the function of two variables? And what is F? It takes a point here and gives you the value F of X, Y, so that is that height. So at every point in the domain, you look at those points in space. So, that gives you something like a surface in R3. So, graph of a function of two variables is a surface in R3. For two variables, it is a curve in R2, it is, now it becomes a surface in R3.

So, what I want to do is for this surface in R3, so this is my surface. At every point, at any point, at any point here, I want to draw something like a tangent. I want like this as smooth surface. So, when do you say, when physically what would you say this kind of thing is smooth? When you would want to say that this is something smooth?

I am not asking mathematical, I am just saying, when we will say, this plane is very smooth. If there is a stone lying here, then you cannot just glide over that thing, you have to sort of the obstacle, look at a hill with stone inside it. So, there is not, there are obstructions, you cannot glide kind of a thing.

So, saying something is smooth means you should be able to, at this point you should be able to draw a plane which should touch the surface only at one point, that you will call as the corresponding thing of tangent line for one variable function. The graphical function of two variable at every point I should be able to draw plane which touches the graph only at one point that point. So, how do I capture that thing, problem?

Now, a plane is known, see there a tangent line was known, once I know the point and the slope. Now, if I want to find out the equation of a plane geometrically, I know the point through which it is going to pass, this is the point I know that at this point I know the plane. If I know a normal to the plane, I know the equation of the plane, if I know that at this point this is a normal.

So, given a line, given a vector which you like to call as the normal there is only one plane, there is only one object in the plane, object in R3, which you can call as a plane to which this will be the normal. And that you would like to call as a tangent plane. So, here the problem would be first to define what is a normal and the plane which is perpendicular to that normal will be called as a tangent plane. So, that is how we are going to go about functions of two variables.

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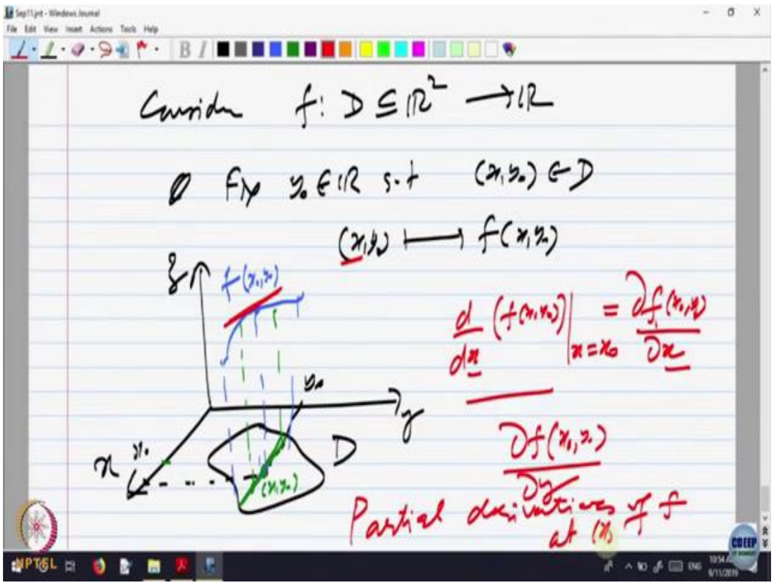

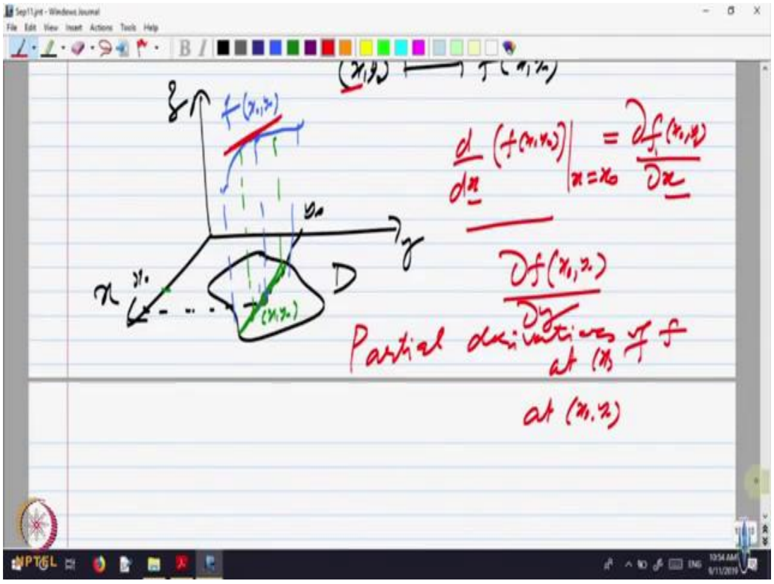

So, to draw that normal, what we do is the following. So, some elementary things first consider. So, our function is F, D in R2 to R, will assume D is a nice like an interval, like is a ball which is the function is defined. So, consider fix say Y naught, belonging Y naught belonging to R, such that X, Y naught belongs to D for, and look at X, Y naught going to F of the coordinate function, look at the coordinate function.

Let me draw a picture here on the side, here is the domain X and Y. So, this is the point we are fixing, this is Y naught and this is X naught. So, this is the point we are looking at. So, we have fixed Y naught, I am varying X, so where I am moving? So, in the domain, I am shifting X. So, I am moving along this line, which is, does not look like parallel to X, so let me do it slightly better.

So, this itself look like, so I am going to like this line. So, what is this line green one, these are points X for a point X, X comma Y naught, so that is a point here. So, this point is X comma Y naught, at this point look at the value of the function F of X comma Y naught. So, what will that give me? As a vary X, so that will give me a function of one variable, so they will give me a curve. So, at every point, so this is the point of this, so this is X naught, Y naught, F at X naught Y. So, at every point.

Now, we can ask the question for the coordinate curve, does it have a tangent at that point? That means I am asking does it have a tangent at this point? If it has, then will say the function has partial derivative at the point X naught, Y naught in the direction of X. So, look at D by DX of F X, Y naught at the point X is equal to X naught. If it exists, then you will write it as F dash X naught, oh sorry, I should not write F dash because it is only with respect to X naught, Y naught partial derivative of F with respect to.

So, instead of writing D by DX, you start writing del, delta. So, this is, what is this quantity? This is derivative of the coordinate function Y naught fixed; X is varying, so X is varying. Similarly, you can also ask for whether the other coordinate function is differentiable or not at that point. So, you will have partial derivative of F, X naught, Y naught with respect to the variable Y. They may not exist like functional one variable, why should derivative exist, but if it exists.

So, these are called the partial derivatives of F at the point X naught, Y naught with respect to the two variables, the function of two variables. So, we are looking at a function of three variables you will have partial derivative with respect to the third variable also Z. So, function.

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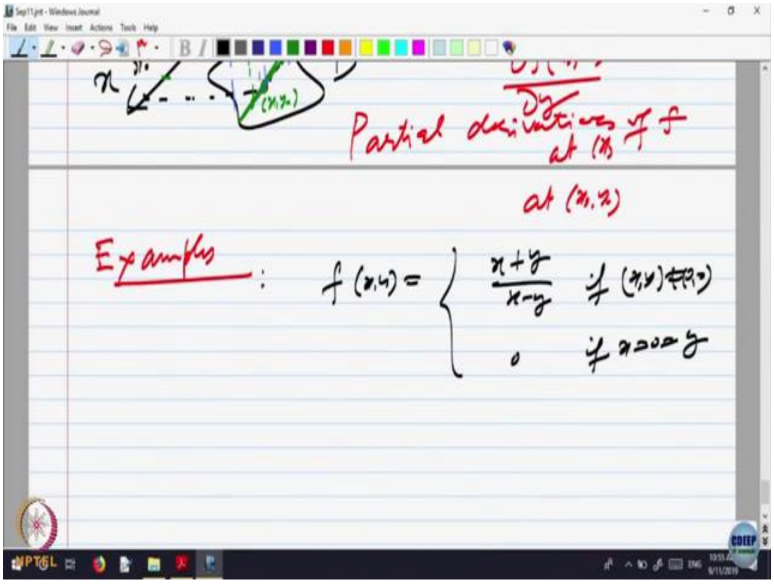

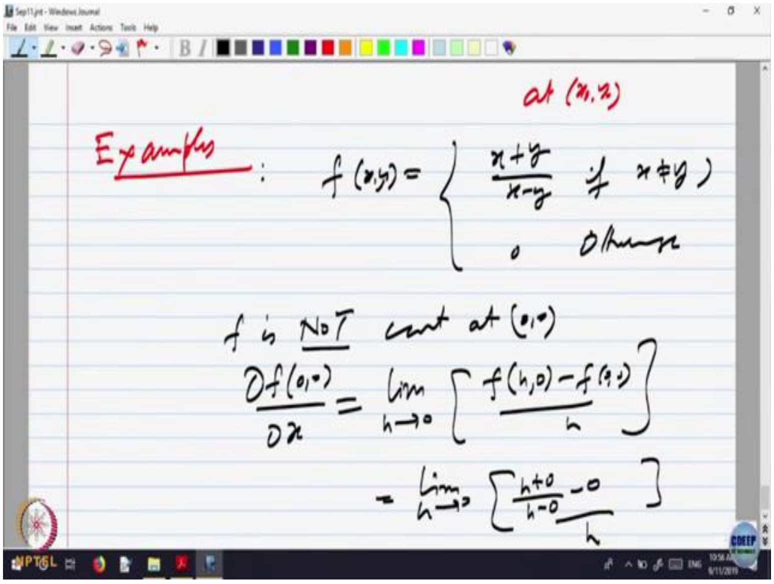

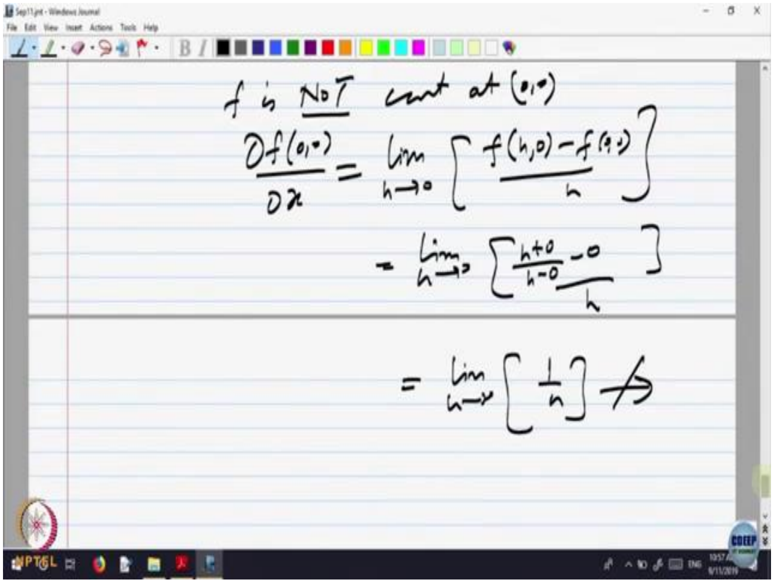

So, let us look at some probably examples before we go further, I think we looked at that example for continuity of function F of X, Y. So, let me write as X plus Y divided by X minus Y. If X, Y naught equal to 0, 0 and 0 if X is equal to 0 equal to Y, oh sorry not, where does X minus Y. So, I should write a bit more carefully because the domain should not have X is equal to. If X is not equal to Y and 0 otherwise let us write.

So, domain of the function is R2 minus the line Y equal to X. And we saw this function F is not continuous at 0, 0. We saw that this function is not continuous at 0, 0. Let us try to look at the partial derivative of this function at the point 0, 0 with respect to X, whether it exists or not. So, what is this partial derivative? It is a limit H going to 0, F of with respect to X. So, H comma 0 minus F at 0, 0 divided by H.

So, what is this equal to? Limit H going to 0, what is H 0? What is the value of the function? That is equal to H plus 0 divided by H minus 0 minus F at 0, 0, 0 divided by H. So, what is this? Is the limit H going to 0, this cancels out one over H that does not exist, H goes to 0, we have to look at limit H going to zero. So, the partial derivative of this function does not exist with respect to X. Similarly, you can analyse with respect to Y. So, this function is neither continence nor the partial derivative of this function exists.

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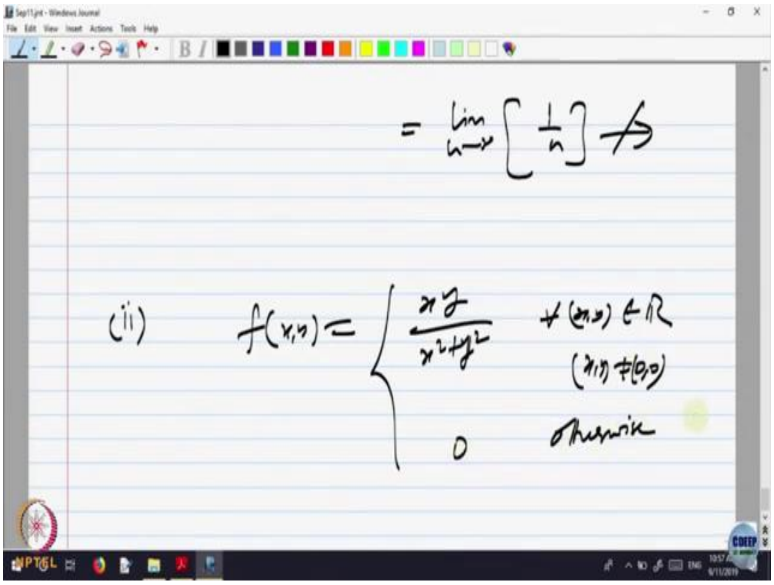

Let us just look at one more example before we conclude or stop the lecture today. Let us look at X, Y divided by X square plus 1 say. Now let me write X square plus Y square. So, then I have to modify X, Y not equal to 0, 0 equal to 0 otherwise.

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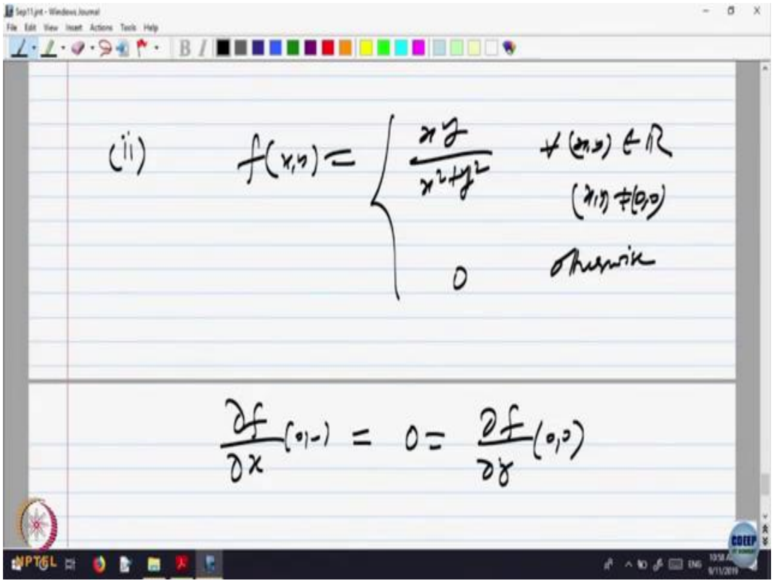

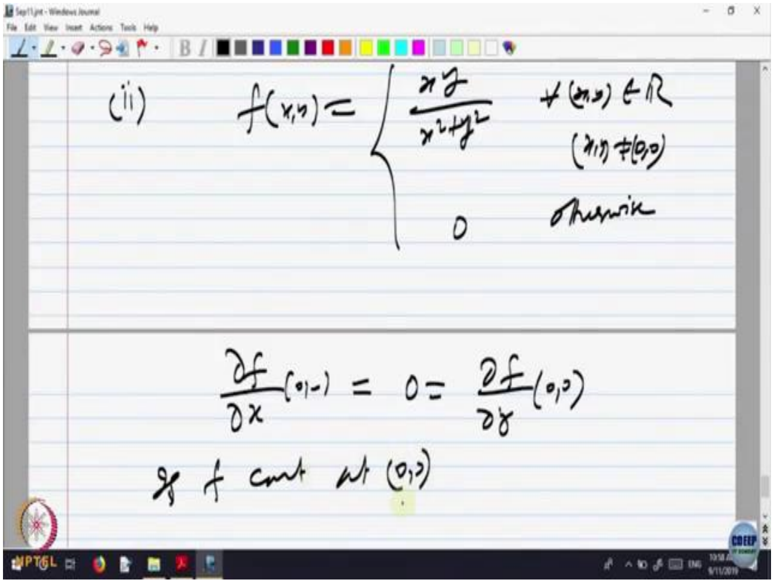

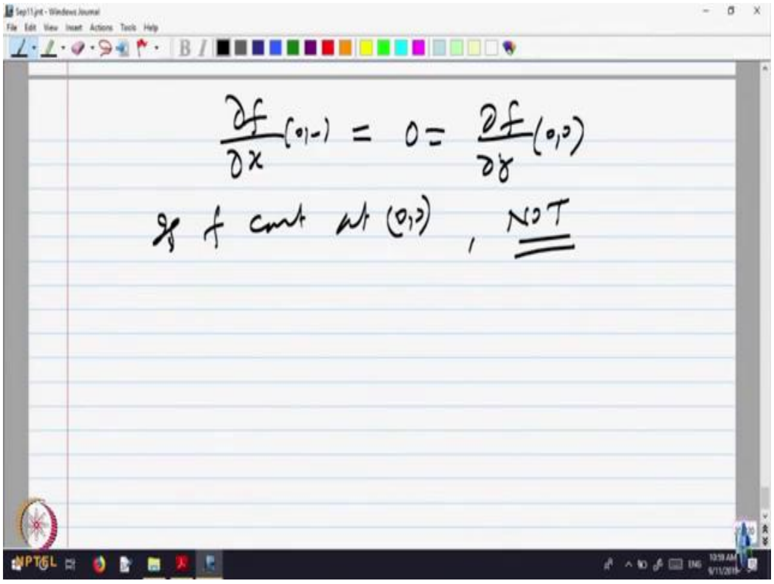

Now look at the function partial derivative of F with respect to X at 0, 0, I want to point that. So, I should look at the value of the function when Y is, X is varying so Y is 0, when Y is 0 this function is 0, constant function. So, equal to 0 that is same as a partial derivative with respect to Y at 0, 0. When I fix one of the variables X at 0 or Y as 0 the function is constant function 0, so derivative is it this, is it continuous? is F continuous at 0, 0?

So, one of the ways of looking at it, look at a along a curve, how does the limit look? So, look at Y equal to MX. So, what will the function look like around the line Y equal to M of X line passing through origin, the paths should through the origin, we saw last time. So, that will be equal to MX square divided by MX square plus M square X square, X square will cancel. So, limit is M square, M divided by 1 plus M square. So, that depends upon the slope of the line. So, along each line the slope is different, so, the limit is different. So, the function is not continuous at 0, 0, not continuous.

So, existence of partial derivatives at a point need not imply the function is continuous. So, analysing differentiability of one variable at a time is not good enough to say that the function is differentiable. We need something more than saying both the partial derivative, like continuity also continuity of function in each variable in not imply continually jointly. So, we will see what is that we required to say that the function is differentiable as a function of two variables. So, we stop here.