Basic Real Analysis

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Lecture 30

Connected Set and Continuity Part- III

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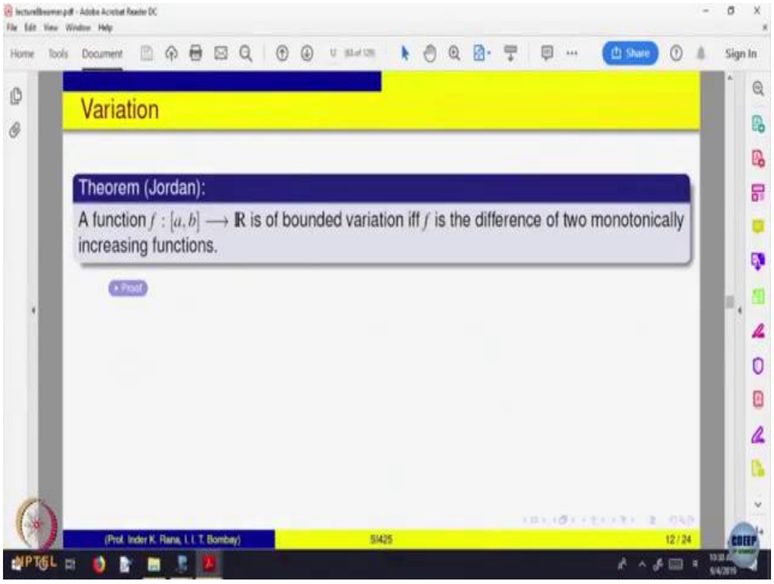

They is a are very, a nice theorem which is a function of bound. We said every bound, monotone function is a bounded variation. It says, if a function is a bounded variation, if and only, if it is a difference of 2 monotonically increasing functions. This is the only way functions of bounded variation can arise. So, this is a theorem due to Jordan. So, it characterizes, what are functions of boundary variation. That is why, so only to state this result, that every monotone function is a bounded variation and conversely bounded variation is a difference of 2 monotone. I brought in all those properties.

So, just understand what is the definition. Some examples, remaining properties just go through once and so that we are exposed to those properties. So, this is about Marton function. I think let us start looking at, so what kind of functions we looked at, we looked at functions defined on a real line or Rn, we looked at the limits of such functions and continuity of this functions. Then, we looked at what are uniformly continuous functions. All these notions were defined on our Rn, you can have continuity, you can have notion of, but tell me about you say something about continuity on Rn.

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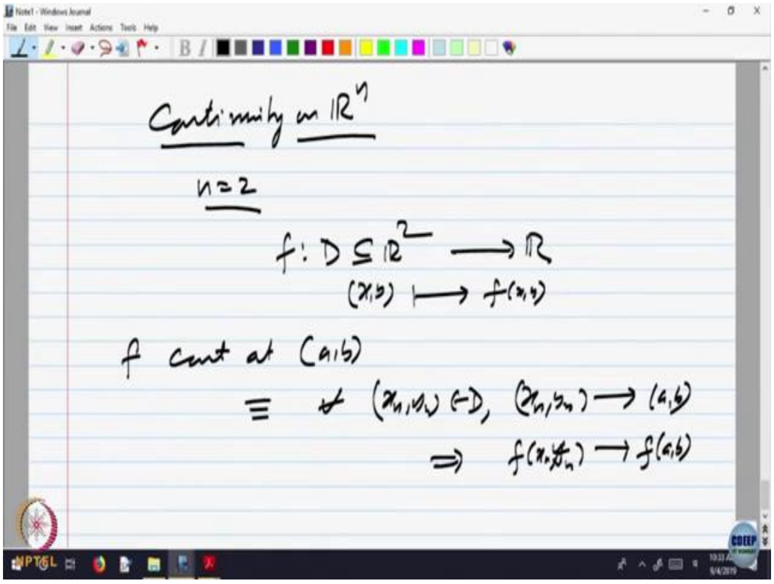

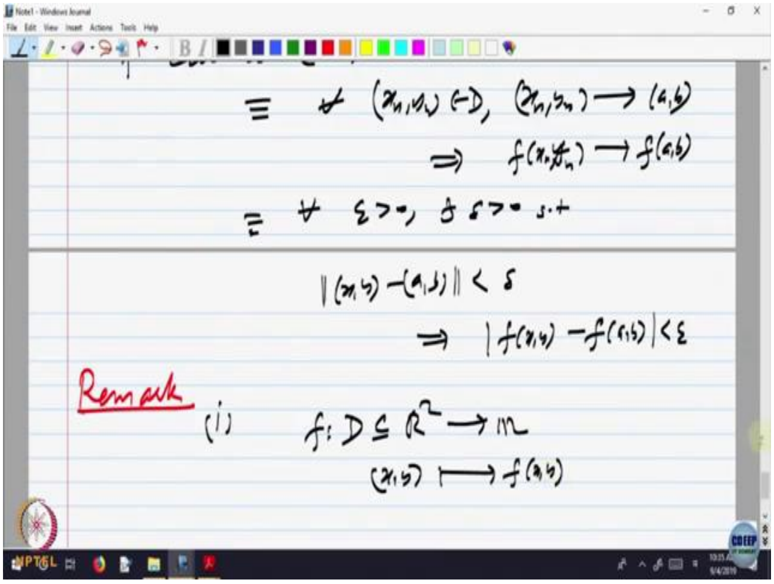

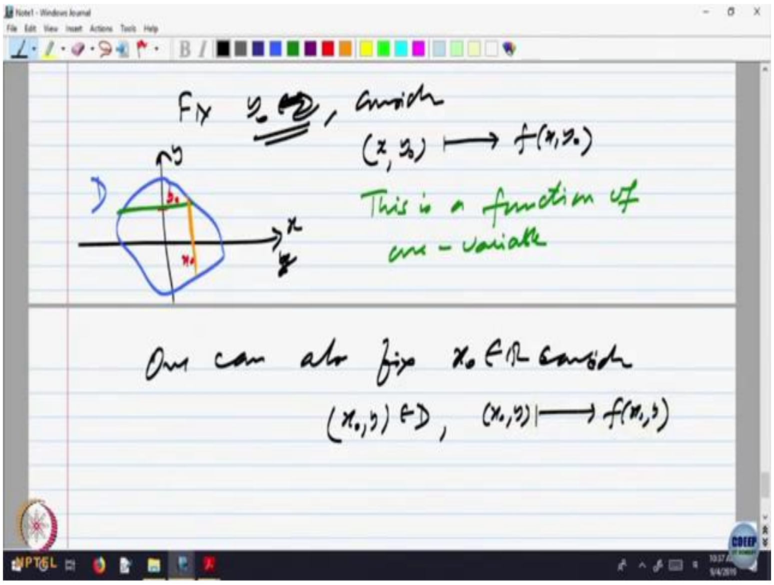

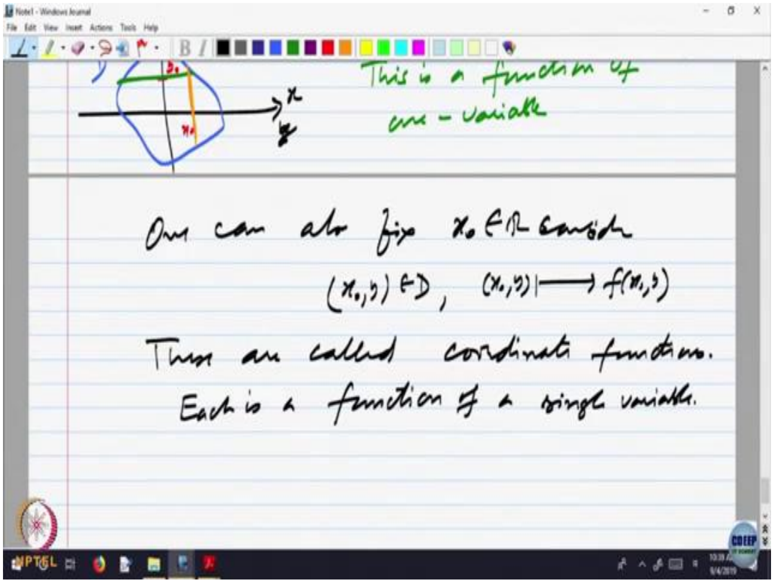

So, continuity, we proved some properties of continuous function. Namely, if f domain is compact then the range also is compact, domain is connected then the range is also connected. But there is some basic differences, or which arise, let us look at that. So, to understand let we take n equal to 2, because that will highlight everything. So, f is a function defined in a domain in R2, taking values in R. So, x comma y goes to f of x comma y.

So, what was the f continuous, at a point say a, b. So, the definition was, for every sequence xn, yn belong to D converging in D xn, yn converging to a, b should imply f of xn and yn, converges to f of a, b. Limit should exist and should be equal to the value of the function. That is what essentially.

In terms of neighbourhoods, this was equivalent to saying for every epsilon bigger than 0, there is a delta bigger then 0, such that whenever the distance between x, y minus a, b is less than delta, that should imply f of x, y minus f of a, b is less than epsilon. That was a neighbourhood epsilon delta definition corresponding. Whenever, x, y are closed by distance delta f of x, y minus f of a, b is closed by epsilon. For every epsilon, there is a delta.

So, this we had all seen this was a continuity and worked very well. Let us, here is a note or a remark, (())(4:48) very often, one word like to f is a function of 2 variables, f D contained in R2 to R, x, y goes to f of x, y.

One can fix, say y naught belonging to D and consider, so y naught is fixed, second coordinate is fix, first coordinate is varying. So, this goes to f of x comma y naught. In the domain, I fixed one of the coordinates and let the other coordinate vary.

So, what it looks like. Let me draw a picture of the domain. So, this is, this is the domain D, that is a subset of R2. So, this is a, this is y and that is x. So, we have fixed y naught. So, that means what? So, here is a y naught, that is fixed, x is varying and f of that is defined, so we are looking at all x comma y naught in the domain.

So, what are all in the picture, what is all such things, so you are looking at all x is that, this is, so this is you are looking at this line. So, that is all x, such that x comma y naught is the domain. And you are looking at the image, so you are looking, this is a function of one variable, one variable. x is fix, y is fix at y0 and you are looking at it as a function of x. And you can also do the other way around, instead of x naught, you can fix y naught, we can fix some x naught. So, you can fix some x naught here, x naught. And look at, what happens when you go around this line.

So, one can also fix x naught, this does not makes sense x naught in D, D is a function of 2 variables. So, fix x naught and fix x naught in R. And consider, x naught comma y belonging to D. So, look at the function x naught comma y goes to f of x naught comma y. So, along that part of the domain, so these are called, you are fixing one of the coordinates. So, for a function of 2 variables, you can fix either of the variable, either of the coordinate. You get a function of one variable, so there are 2 functions of one variable. These are called coordinate functions, are called coordinate functions, fixing one of the coordinates.

So, each is a function of a single variable. Now, you can, one can, ask a question, if I am given a function of 2 variables, which is continuous in the domain D, what are these coordinate functions, they are the same function f but the domain is restricted, along that lines. So, is coordinate functions continuous.

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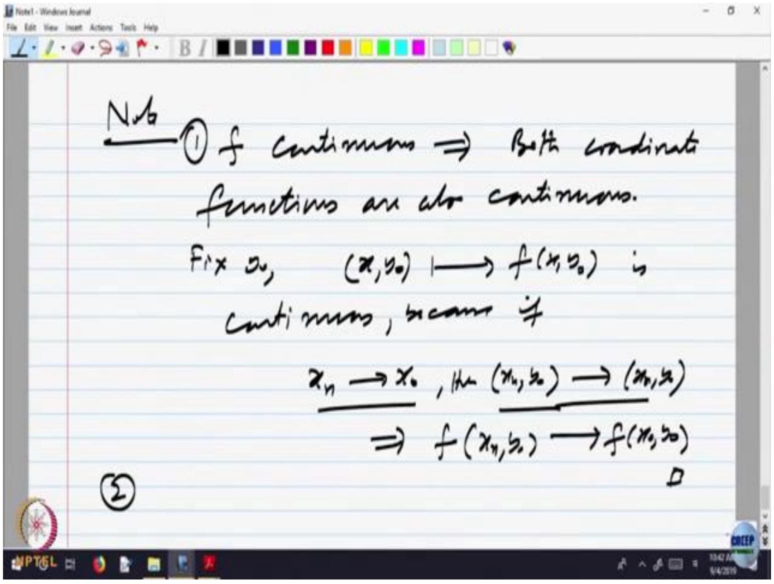

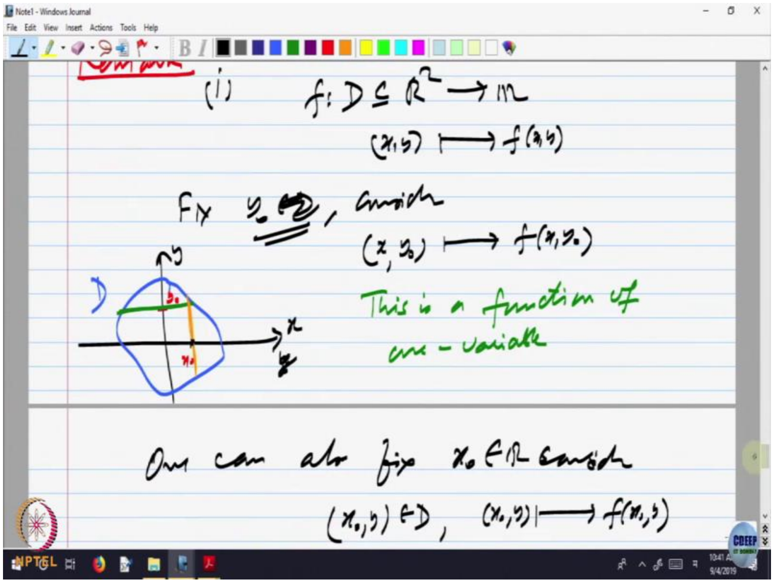

So, note f continuous implies both coordinate functions are also continuous, both coordinate functions are also continuous. Why it is so obvious, because, so let us consider one. So, fix y naught, then we want to look at x comma, y naught going to f of x comma x comma, y naught. I want to check whether this function is continuous or not, is continuous because how do I check, I can do it by sequences.

If xn converging to x naught, then what happens to xn comma y naught 0, that converges to xn comma y 0 as of sequence in R2, which implies by continuity of f, f of xn, y0 converges to f of x0, y0. So, it converges to x0, y0 and that converges to (())(11:11) So hence, so that is okay. So, the coordinate functions are continuous. So, in fact what we are saying is the following.

See if, if you want to look at the picture. So, this is the point x0, what, what we fix, we fix y0, so this is y0 fix here. So, if I take a sequence, xn converging to x0, then that corresponding sequence R2 will converge to x0, y0. So, the function is continuous, so it will give f of xn, y0 converges to, so that is what we are saying here. So, this is, this is one coordinate, this goes to in the domain now, this is in the domain. xn, y0 goes to xn, y0 in the domain, f is continuous. So, that implies this. So, this is note one. Can I say the converse is true? f is a function, so that each coordinate function is continuous. Can I say the function of 2 variables itself is continuous?

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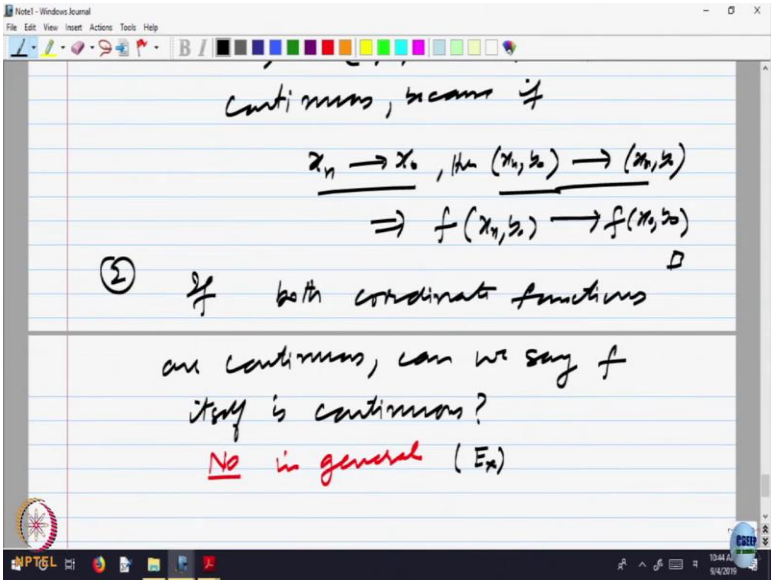

So, let us ask the question. If both coordinate functions are continuous, can we say f itself is continuous, the answer is no, no in general. That means what, you can construct functions such that, their coordinate functions are continuous. But the function is not continuous. So, try, let me for the timing, leave it as exercise. In 2 variables, so that when you fix one of the variables, the function is continuous. Because other variables, still the function is continuous, but jointly this function is not continuous.

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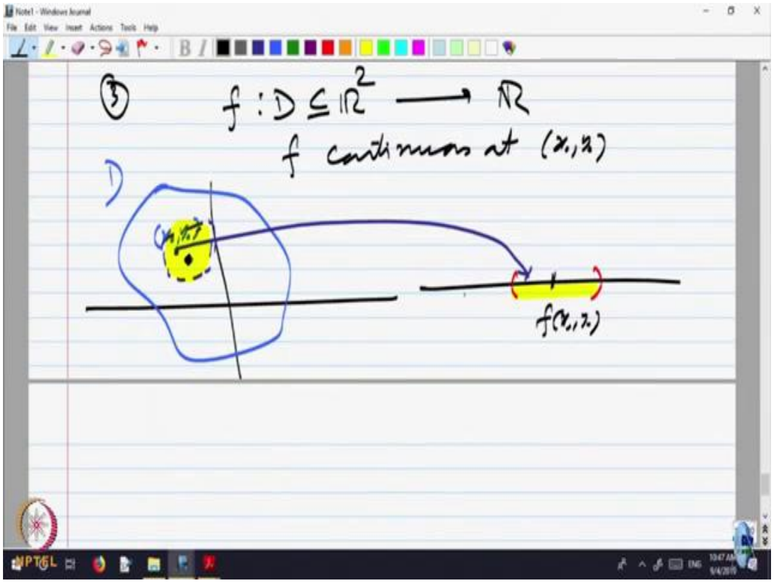

So, for that, let me also give you some more input so that you are able to do, so let me write 3. So, let us take a function f in a domain D in R2, to R f is continuous. Of course, continuity, is continuity at a point, we are not looking at continuity of the whole domain. We are saying, if f is continuous at a point, then both coordinate functions are also continuous at that point. Conversely, we have said no. The function, both coordinate functions maybe continuous at a point x0 y0, when x0 y0 are fix independently. But jointly the function may not be continuous at that point, x0. So, let us say continuous at a x0, y0.

So, let me draw a picture again here. And here is the domain, and here is the point x0, y0. We are saying the function is continuous at this point. So, what does continuity say, it says, so here is, where it is going. So, here is the value at f at x0, y0. Now, what does continuity say, continuity says given a neighbourhood of, epsilon neighbourhood of the point, value.

There is a delta neighbourhood of, so there is a delta neighbourhood of this, such that whenever I take a point inside, we take a point inside here, it values goes inside here. So, you can think of this domain, this is the and the value goes here. So, that is continuity. If you look at the sequential thing, whenever a sequence converges to this point x0, y0, so if a sequence xn, yn is converging to x0, y0 it is going to come inside that domain.

And hence, it will come inside this. That is a equivalence of the sequential definition and the epsilon delta. But so, the point is how does the sequence x0, y0 approach this point x0, y0. How does sequence xn, yn approach, in the real line you can approach a point from the left or from the right. But in R2 for example, you can approach y, you can go along any line through that point or are you going to go around zig zag path, going through it. So, it should mean whichever way you approach the point, the corresponding values should come closer.

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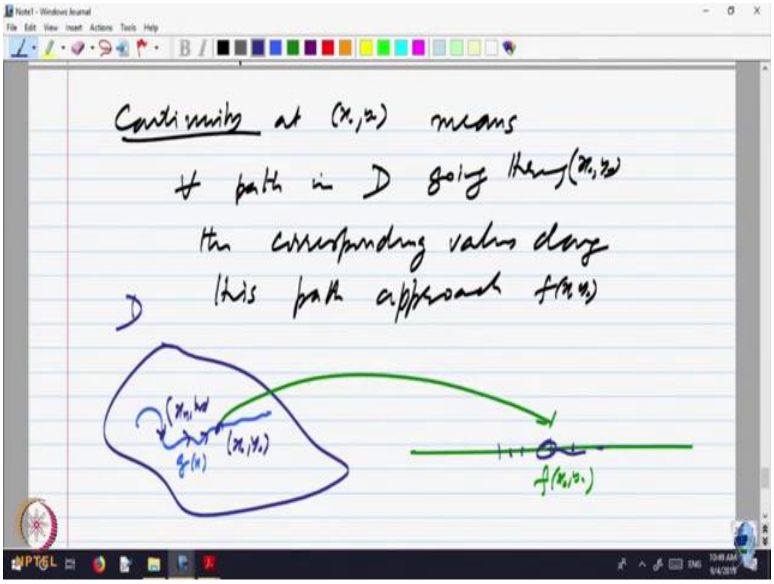

So, another way of saying that would be, so let me just write. Continuity means what, implies continuity at x0, y0, means I can make it more, more precise means for every path in D, for every path in D going through x0, y0, the corresponding values along this path, there is so much of English but was a very easy to understand, along this path approach f of x0, y0. Let me just draw a picture of this and bigger picture so that you understand what I am saying. So, here is the domain and here is the point x0, y0 and this is a value at a point of f of x0, y0.

So, continuity says, suppose I approach this point x0, y0 along this path. So, this is a path, you can call it as g, keep on the, the picture. So, what I am saying is, I am going to approach this point along this path. In the real line, you can approach only from left and from right. So here, I can approach this point along this path. So, I will take, I can take a sequence if you like going along this path to this point. So, xn, yn. So, this is a, so I am going to this, then the corresponding values should come closer to the value f of x naught, y naught. So, let us, why this thing is useful.

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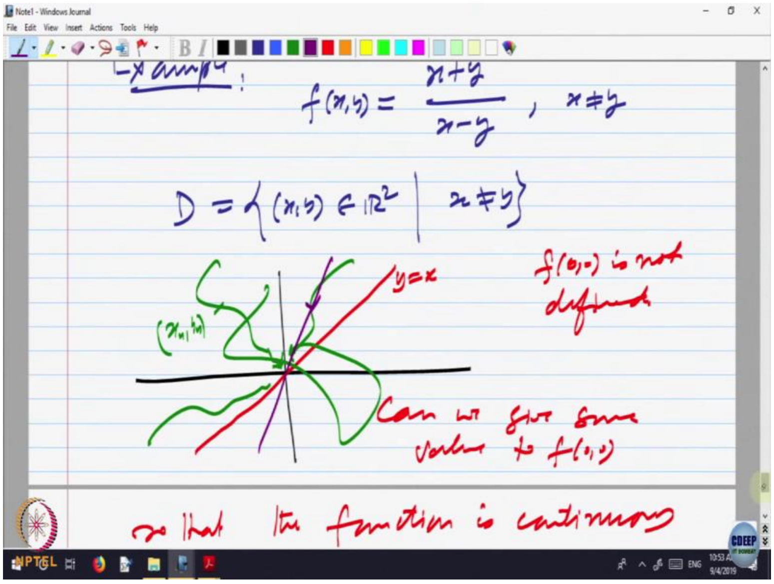

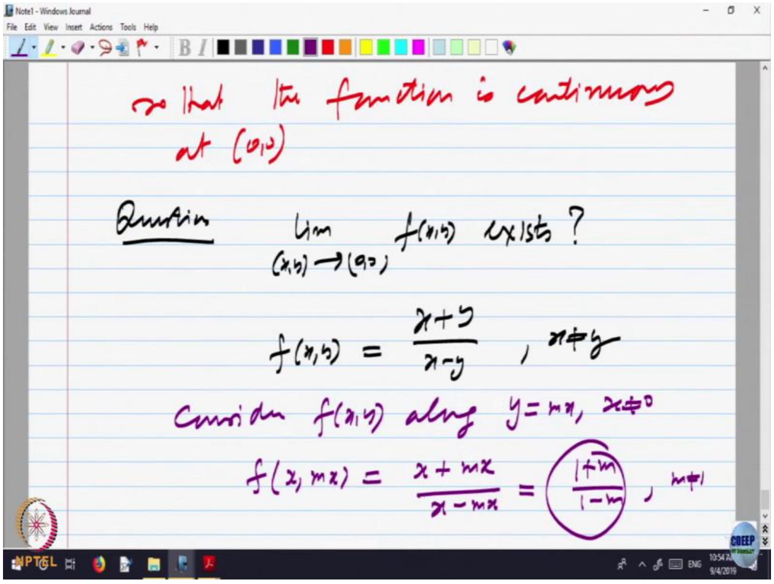

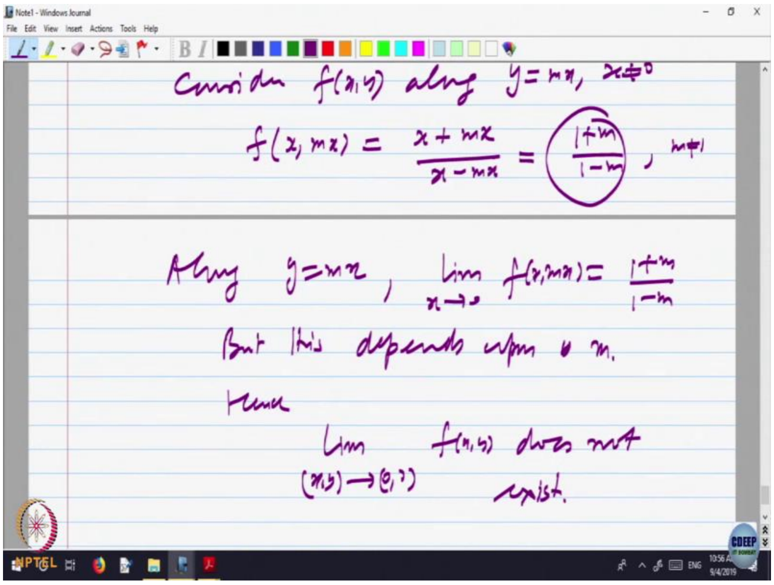

So, let me look at an example, probably to illustrate this. So, let us look at the function f of x, y is defined as, let me write x plus y divided by x minus y. Now, x minus y is in the denominator. So, I should remove x not equal y.

So, what is the domain of this function, where is the function defined, it is defined at all points x, y belonging to R2 such that, x is not equal y. So, what x is equal to y, that is a line. So, the domain of this, so I should remove that line. So, this line is not part of the y equal to x, on this line, except on this line, everything else function is defined. I want to know, can I, so this function is not defined at 0, 0. So, f 0, 0 is not defined. It is not defined because the line y equals x passes through it.

Question is, can we define some value to f at 0, 0 so that the function becomes, is continuous at 0, 0, so that the function becomes continuous. At 0, 0, it is not defined. Function, if I want to say a function is defined at 0, 0 also and continuous, so what should I do, I should look at the limit of the function at that point. And if the limit exists, that should be the value of the function at that point.

So, the question is, does this function have a limit? So, the question, limit x, y going to 0, 0 f x, y exists or not? So, what is f of x, y, so that is x minus y, x plus y divided by x minus y, x not equal y. I am looking at limit of this at 0, 0. Now, if the limit at this point is to exist at 0, 0 then what should happen, then whatever path I take going to 0, 0. If I take a sequence xn, yn going to 0, 0 then f of that should have some value.

Whichever the path, along this path or all this path, around this path or whichever way I want to go. So, let us try to test it for some nice path. Let us take a path like this, which is a line through the origin. So, let us approach the point 0, 0 along a line, other than y equal to x because at y equal to x that is not defined. So, what, so let us make a line. So, consider f x, y along.

So, what is question of some other line, y equal to m of x, x not equal to 0 of course, 0 is not defined. Then what is the value of the function? f of x comma, along this line, what is the function, y is m of x so, x plus m of x divided by x minus m of x. And what is that, so that is 1 plus m divided by 1 minus m. So, let us assume m is not equal to 1. So, I am taking the line where m is not equal to 1 of course. That is the line y equal m of x, anyway. So, along this line, the function always is a constant. So, limit will exist. So, along every line the limit exists.

But the, so let us write that first, what is the observation. So along, y equal to mx, limit x going to 0 f of x comma, m of x is equal to 1 plus m divided by one minus m. There is the constant function. Now, if the limit at 0, 0 has to exist, that limit should be independent of the path. Left limit in real line should be same as the right limit. So, here it depends on m. So, this limit exists along every path, every line passing through 0, 0 but limit is different. So, limit at 0, 0 does not exist.

But this depends on m. Hence, limit x, y going to 0, 0 f of x, y does not exist. So, for this function, the limit along 0, 0. So, I can not define this function in any way to make it continuous. So, the important thing is the path in R2, R3, Rn, there are more than one. In real line only left and right possible things are there. But in R2, Rn, there are infinite number of ways, so it should not depend upon.

So, how is this useful, to prove that the function is not continuous. If we can show it along 2 different paths, the values are different than the function will not be continuous. So, that is continuity in R2, or similarly in R3. So, coordinate function continuous implies. So, what are coordinate functions, when you are fixing one coordinate you are going vertically or horizontally, that is a path you are going.

When you are fixing, say y naught, your y naught is fixed now. So, you are going horizontally, y coordinate is fixed. So, you are moving horizontally, x coordinate vertically only, but not only that, it should be along every path. So, no wonder that coordinate functions, continuous need not imply continuity or the function jointly of the 2 variables. So, many times we tend to push the problem on one variable, in the sense that to check the function is continuous in 2 variable.

If we can show it is not continuous in one variable fixed, then it is not continuous jointly also. So, these are necessary conditions, continuity in each variable is necessary for a function to be continuous but not sufficient. But necessarily things are always useful, proving something is not that. So, let us, I think only 2 minutes left. So, let me stop that here. So, next time we will start looking at the differentiability of functions.