Basic Real Analysis
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Lecture no. 29
Connected Sets and Continuity – Part II

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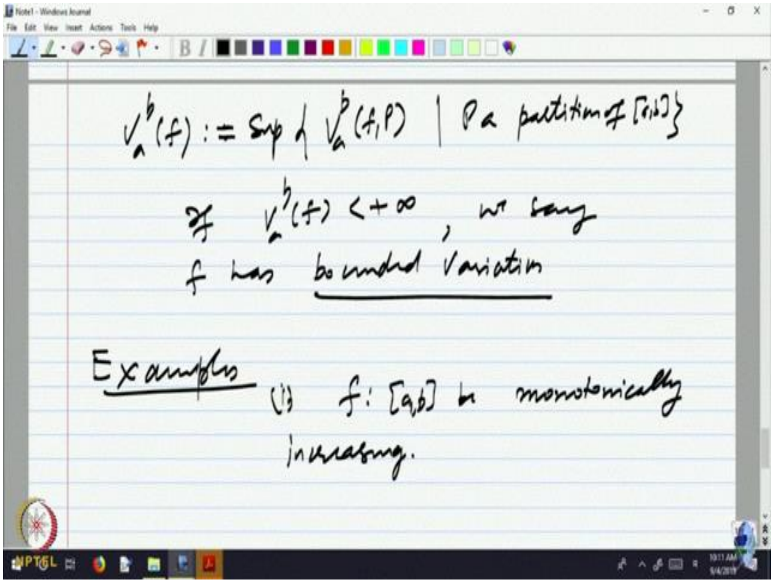

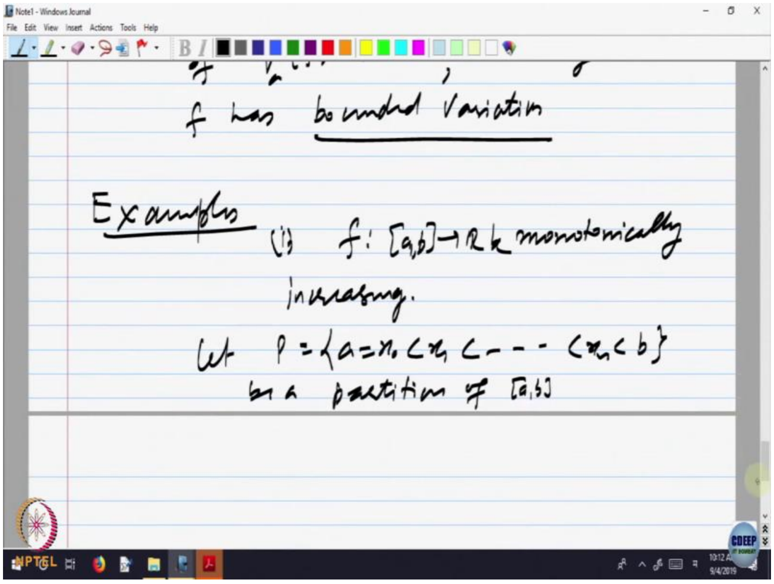

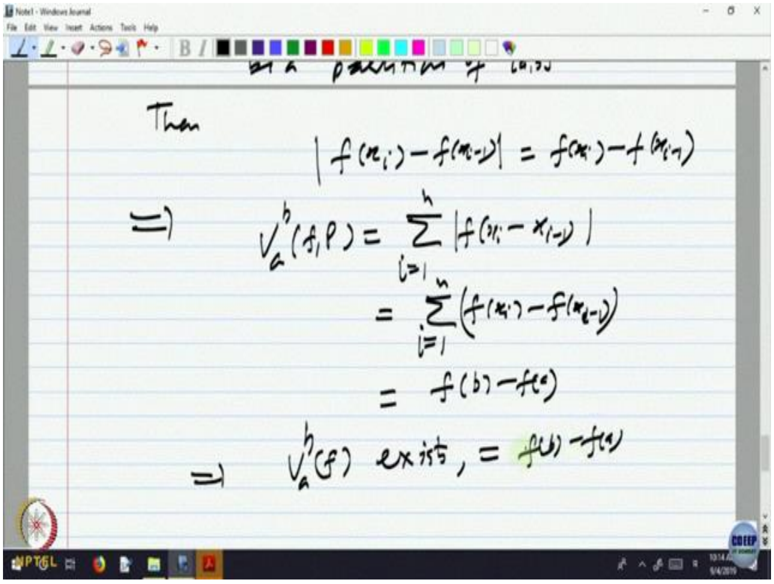

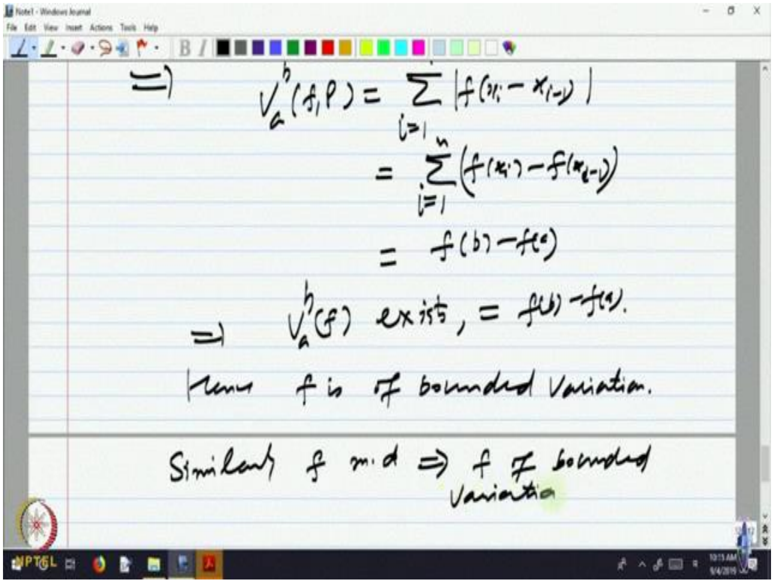

So, let us look at some examples and then that will help us to understand. So, let us look at some examples. Let us look at, why we started all this was, let us take, let a, b to R be monotone, say monotonically increasing. Let us say, it is monotonically increasing. So, let us take a partition. Let, I want to calculate the variation and check whether it is a bounded variation or not. So, let us take apart any general partition xn. So, f a, b to, I did not write a, b, let f a, b to Rb. So, let be a partition a, b.

Then what is f of xi minus f of xi minus 1, absolute value, function is monotonically increasing, xi is on the side of xi minus 1. So, this value is equal to f of xi minus, no need to put absolute value because the function is monotonically increasing. So, implies what is the variation of f with respect to the partition P. So, what will be the variation? So, we said it by definition it was sigma 1 to n f of xi minus xi minus 1 absolute value and that is equal to, there is no need to put absolute value. So, that is sigma i equal to 1 to n, f of xi minus f of xi minus 1 and what is that, now terms cancel out now. So, it is f of b minus f of a.

So, whatever we, so implies Vab of f exists equal to fb minus f of a. Hence, f is of bounded variation, it is a function of bounded variation. If it was decreasing, only difference would have come here, this would have been reversed. So, then it will be fa minus f of b, still. So, same proof works, except for a difference of that sign bounded variation. So, similarly let me write, f monotonically decreasing implies f of bounded variation.

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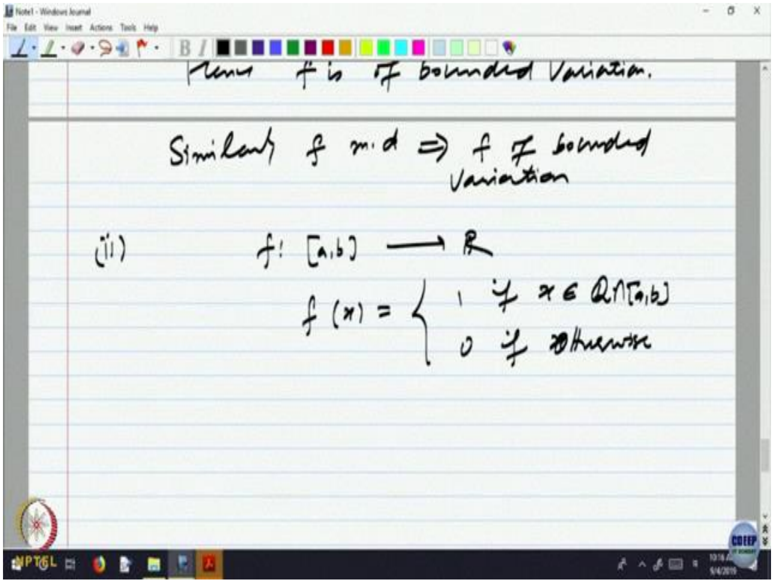

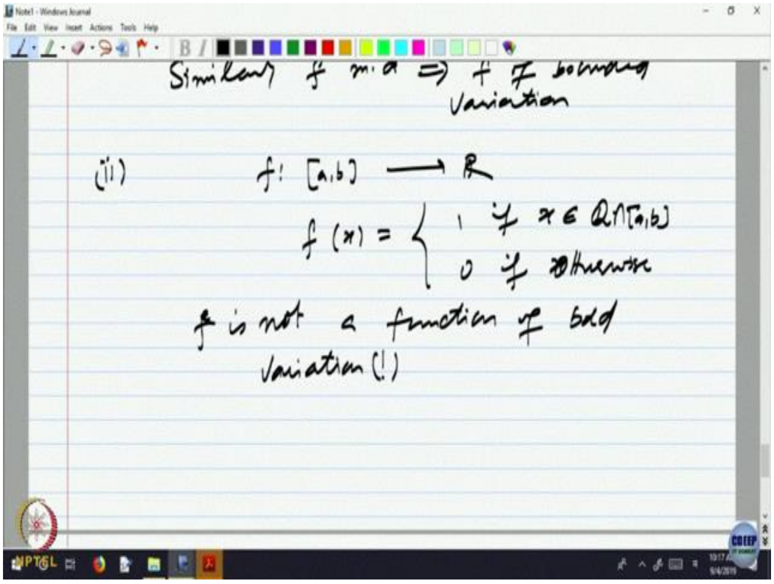

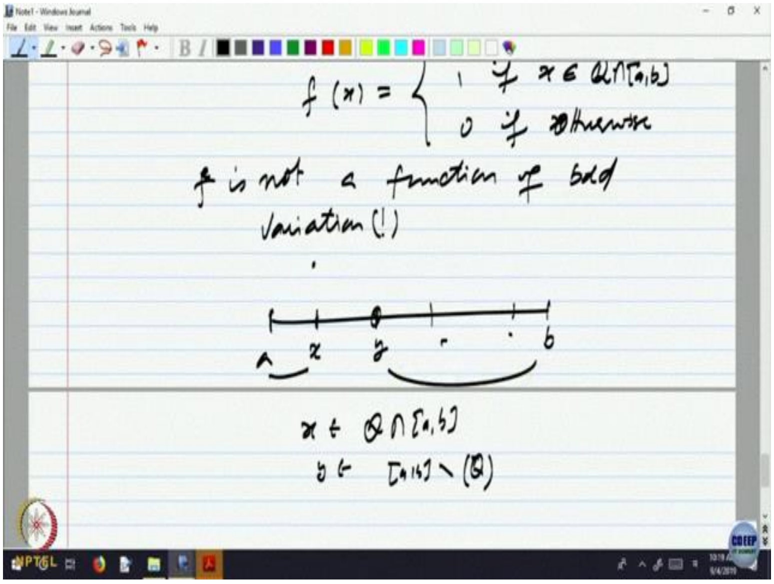

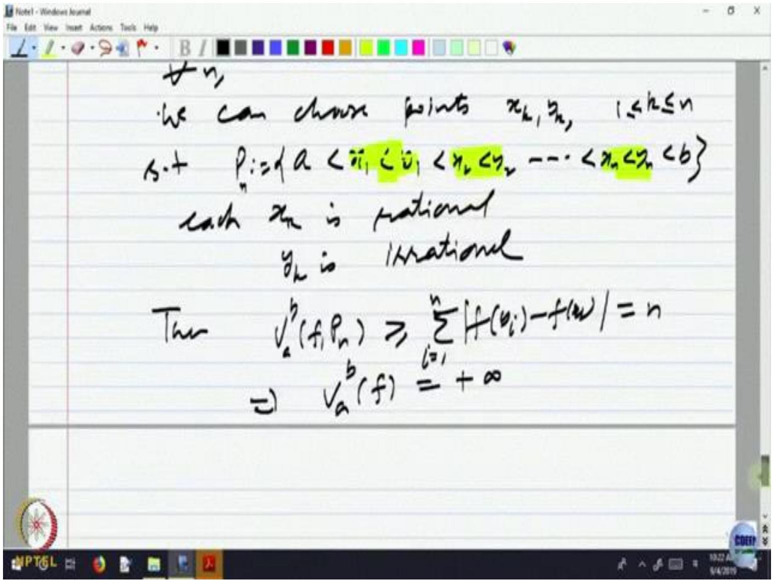

Let us look at, another example. Let us look at the function, just now we looked at indicator function. So, consider f on a, b R so f of x is equal to 1 if x belongs to is a rational inside a, b if it is a rational number, the value is 1, it is 0, if otherwise. So, that means if it is irrational the value is 0, and the rational the value is 1. So, what is it, it is a indicator function of rationals in a, b, that is a indicator function of.

But it takes to only 2 values. So, why do think is a variation of this, it is not monotonically increasing, neither monotonically decreasing.

Professor: So, what do you think is a variation of this function?

Student: (())(5:36)

Looks like, yes looks like 1, no. How much, see variation is how much up and down it sort of goes, that was a. But this function is going up and down very often. At rational points it goes up and irrational point it comes down.

So, it looks like is varying too much. So, it is not a function of bounded variation. So, claim guess is, f is not bounded variation, this is only a guess because it going up and down very frequently.

So, how does one write that, that idea I can visualise it up and down, but I will do a write it. So, the idea is here is a, b and I want to say that the variation, I can find a partition, where the variation is equal to n say, for any n, any natural number. I can find a function, find a partition, so that the variation on that partition is equal to n. Then why take the supremum would be infinite.

So, for example, if I take the point A, let us take a rational here, and irrational here. So, some point make that means x and y, where x is a rational and y is, the complement of that. So, what you want me to write, I will write a, b minus, let us write rational itself. So, at this point, this is a rational point. So, what is the value at the rational, it was 1, at this point the value is 0. So, if I take a partition in which these 2 points come. Then in that summation at least number 1 will appear.

So, I can choose any finite number of such pairs, n number of such pairs. So, the summation of that variation over this partition would be at least n. Because there will be something contributed by this, something contributed by that also. So, it will be bigger than or equal to n.

So, we can, so let me write, so we can choose. For every n, we can choose points xn, let us write a less than x1, less than y1 point. So, let me write xn, yn such that, let me, I want to write xn, yn or xk, yk. Let me just improve it, so write, choose points xk, yk. Where k is between 1 to n such that, a is less than y1, less than x2, less than y2 is less than xn less than yn less than b. So, the pairs, so keep in mind I am looking at the pairs, in between.

So, this is one pair. So, this is one pair, this is another pair and this is the nth pair. Where, what do I want, we want such that, each xi or each xk is rational. Each yk is irrational. Then, if I call that as the partition P, if I call those points as the partition P. Then the variation ab f or call it as Pn, if you like Pn will be what, it will be value at x1 minus the value at a, absolute value plus absolute value y1, x1 plus value at difference between the absolute value f of x2 minus y1.

So, I am just saying that this is bigger than or equal to sigma f at yi minus f at xi. Other terms I just dropped, only keep the values difference. So, which is equal to n, because xi is rational, yi is irrational. Other terms I have just dropped, so instead of equality is bigger than or equal to now. So, implies Vab of f is not finite, so on right set as plus infinity. So, this function very nice, very simple is not a function of bounded variation.

Many examples you can construct of such things. So, probably I think, let me just say, what is it I want to know. So, let me, because we do not want to prove too many things for the functions of bounded variation, from linear higher course, you will come across these things. Let me guess, because I am not writing, so let us show you, I think this is okay.

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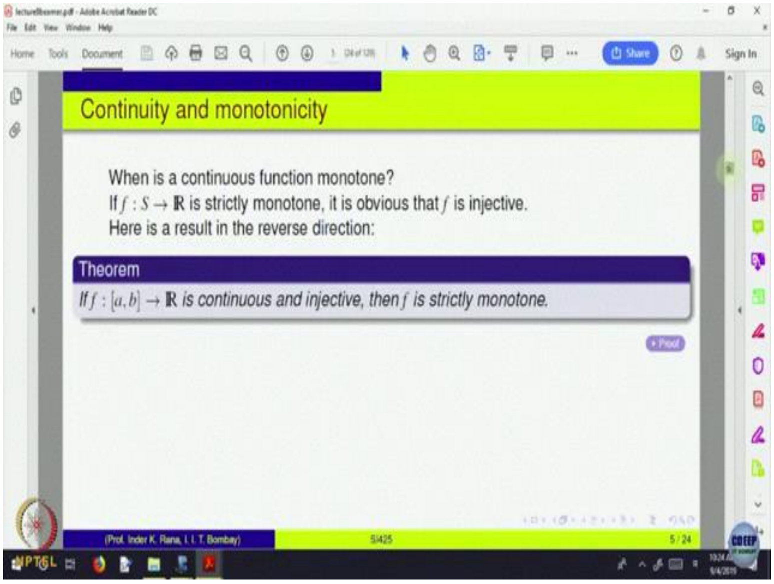

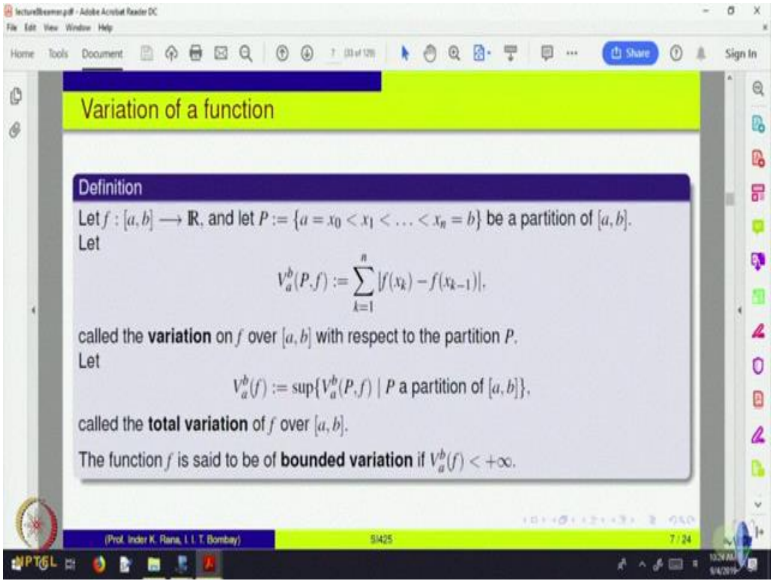

We had already analysed many things about monotone functions, so let us. So, it has jumped discontinuities, they are accountable in many and so on. We already had looked at continuous function which is 1, 1 strictly monotone those properties, we already looked at, in other. So, here is, so let me anyway revise, so saying that a function a, b to R, this is a partition. Then this is called the variation of f with respect to the partition P. And look at the supremum that is called, normally called the total variation and if it is finite, we say a function is a bounded variation.

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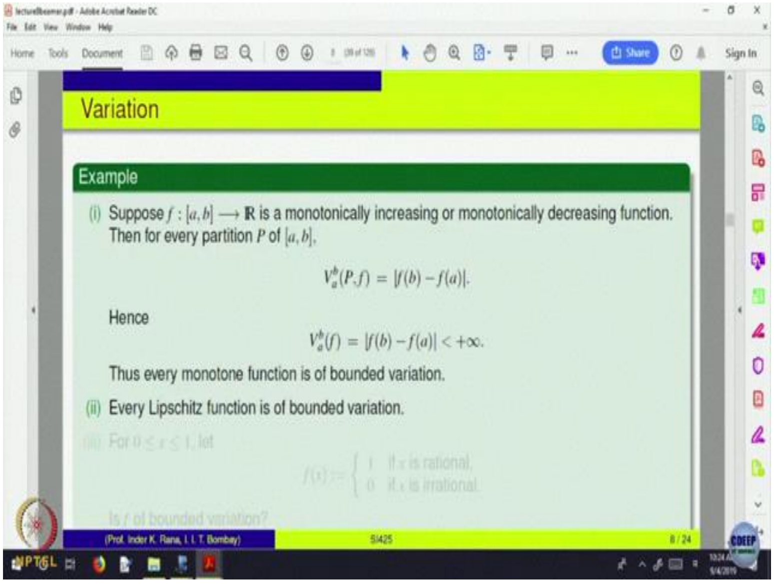

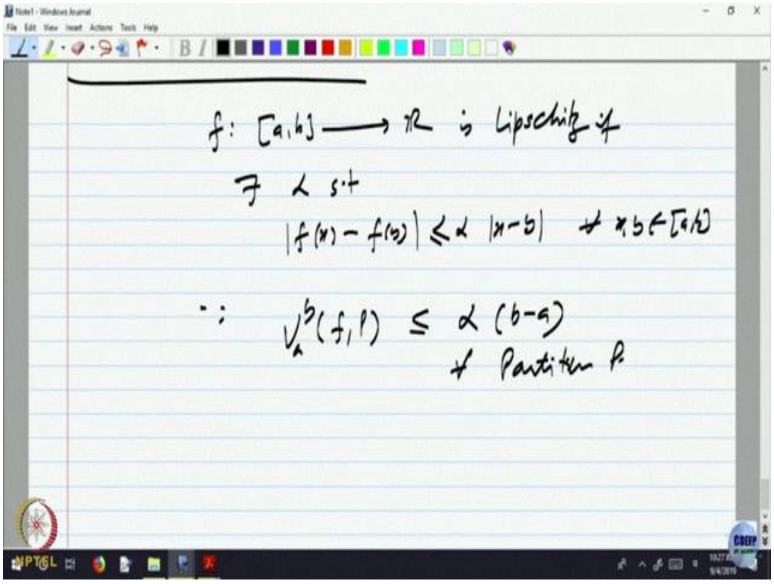

So, here is every monotone function is a bounded variation that we just now saw, is a, b. Here is another one, if f is Lipschitz function, remember what are the Lipschitz function? Let me just say what was the Lipschitz function. Another example of, so f on any domain actually. Let us write it on a, b to R, we said it was a Lipchitz, it is a German mathematician, Lipchitz. We defined it first, if there is some alpha such that fx minus fy is less than or equal to alpha times x minus y for every xy belonging to a, b.

So, how much change in f comes with respect to the change in, it is a directly proportional kind of a thing, less than or equal to alpha times, change is less than. We, this we had come across actually, when we said that every Lipschitz function is uniformly continuous. So, alpha times, epsilon delta or you can write sequence, whenever a sequence xn, yn goes to 0 that will imply f of xn, yn also goes to 0, so uniform continuity.

Now saying, this a function of bounded variation also, because when I look at f, that variation f of xi minus f of xi minus 1, that will be as the alpha times, xi minus xi minus 1. So, that is summation terms will cancel out, what we will left is b minus a. So, the variation with respect to any partition is less than or equal to alpha times b minus a.

There is something like monotone function, what even if it is not monotone, it is still true. So, every Lipschitz function is a bounded variation. Because variation a to b of f with respect to P will be less than equal to alpha times b minus a or every partition P. So, because of this inequality. So, it is of bounded variation.

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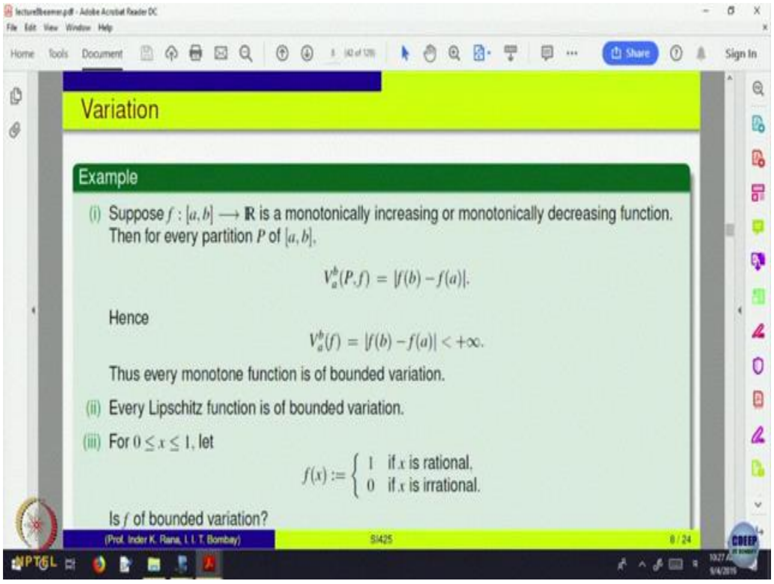

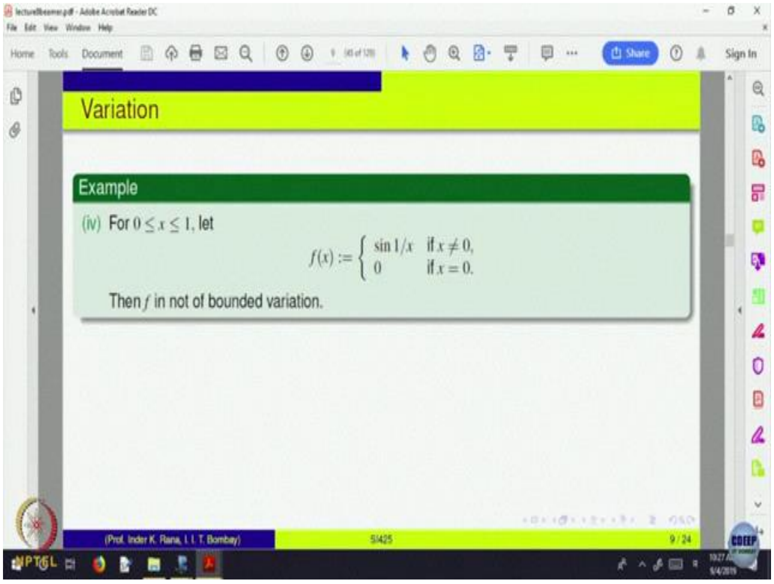

Just now, we saw that this function is not a function of bounded variation, it is not of bounded variation. So, we can give some more examples. Here is one example, this function, this also we had come across while trying to do something about connected subsets of R2. So, as you come near 0, the graph goes up and down very fast. So, again you can see that the variation of this function will be infinite. Because you can always pick up points where the value is 1 and minus 1, points in 0, 1. So, this function is not a bounded variation.

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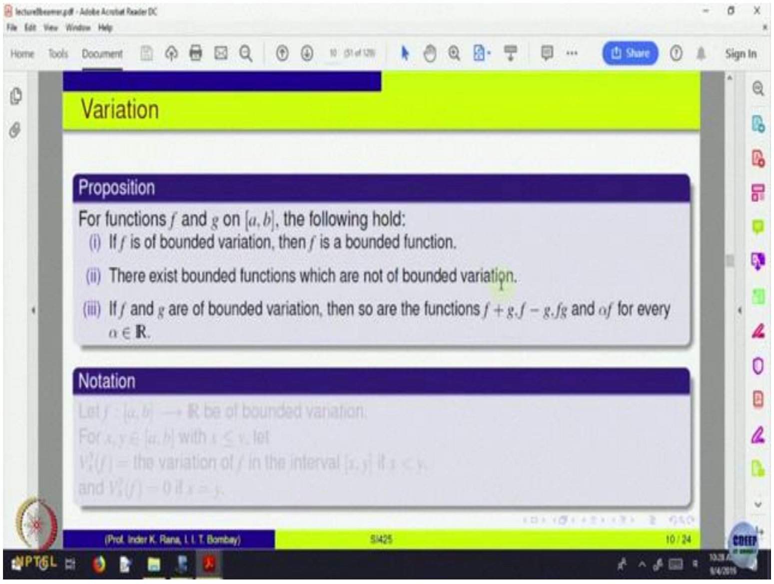

So, here are some properties which will not prove, but just try to understand why. You will come across these things probably in a higher course. If a function is of bounded variation, then it should be bounded function. Because if it is unbounded, that means you can go on increasing the values. So, you can enclose those points in the partition if you like or you can simply prove also mod f of x is less than or equal to mod of x minus a plus mod. So, trivial partitions you can choose.

There exits, functions which are bounded but not of bounded variation, just now, we example that 0, 1 rational, so is a bounded function it takes only 2 values, it is not a bounded variation. Bounded variation functions have nice properties. So, this is called algebra or functions of bounded variation f and g are bounded variation. Then f plus g, f minus g, fg alpha times f all are function of bounded variation.

Every monotone function is of bounded variation, one can define many things. Like you can define the variation instead of full interval a, b only in the part a to x or x in a, b and so many properties.

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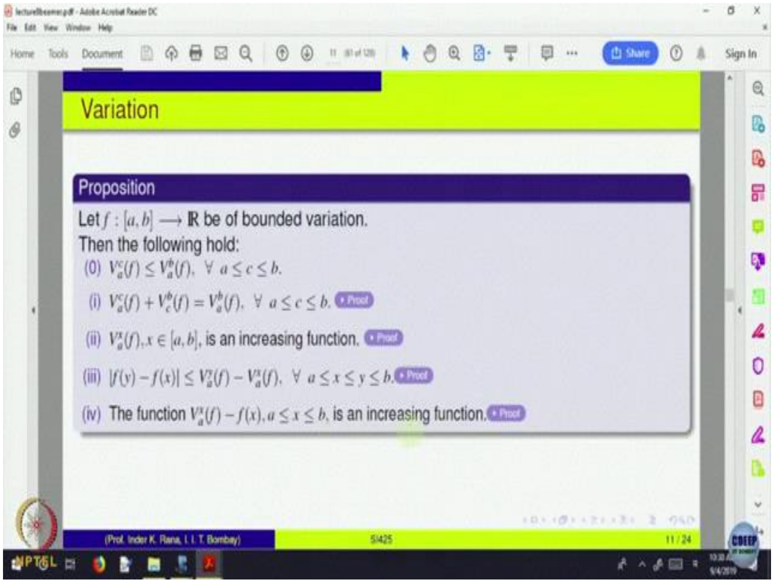

So, just go through these properties once, but do not look at the proof because we are not going to ask you. We basically, that if c is a point in between, then the variation over a to c is less than or equal to variation over the whole interval, that is obvious. And it says, the variation adds up, variation a to c plus c to be same as variation a to b, they are function of bounded variation. So, these are the properties. And this a to x, x belonging to a, b. This itself is an increasing function. And using these properties one can prove this is also increasing function.