Basic Real Analysis

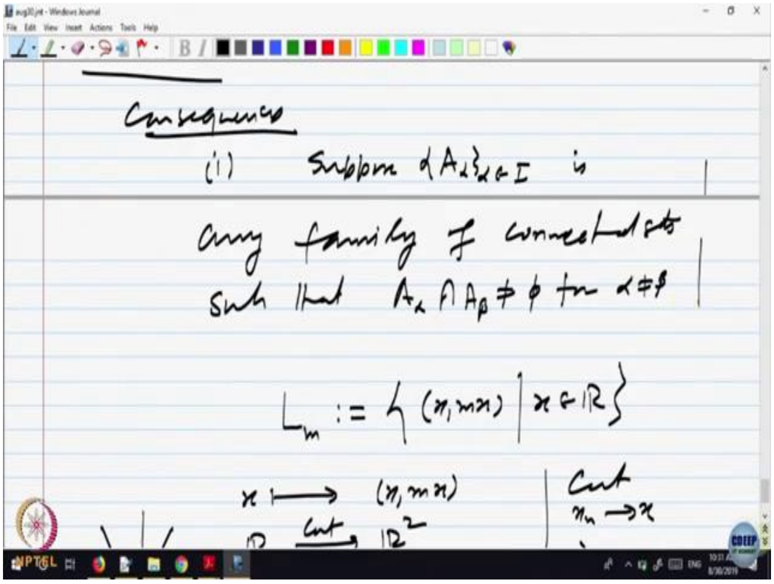
Professor. Inder. K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

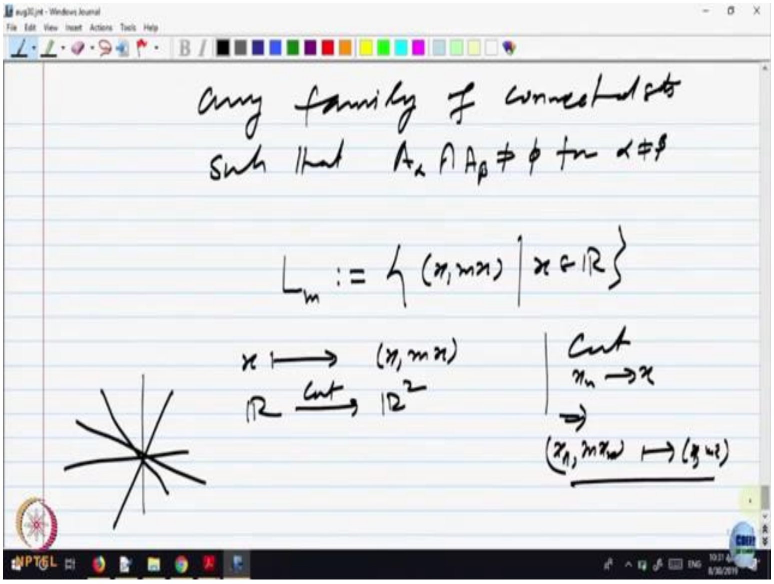
Lecture No. 27

Uniform continuity and connected sets – Part 03

Before proving it, because we are proving too many things, let me give you some consequences of this and then come back to a proof of this.

(Refer Slide Time: 00:28)





So, consequences, so let us look at. Because this theorem is useful in producing many examples of, connected sets. So, suppose A alpha is any family of connected sets, any collection of compact, such that A alpha intersection A beta is not empty for alpha not equal to beta. Any 2 of them intersect at least once somewhere, is a collection of sets.

For example, let us look at R2, look at 2 parallel lines, 1 parallel line other parallel line. They do not intersect. So, we will not be considering that kind of example here. So, look at any 2 lines which are not parallel, they will intersect. Now intuitively, a line in R2 looks like a connected set, in R2, intuitively, I am just saying, a line in R2 looks like a connected set, why it is connected? Why is a line a connected set in R2? Looks like I cannot break it into 2 parts, which are disjoint kind of, which are not, which are separated.

But let us look at, a line is the image of the X axis. So, let us I think I am going in a different direction which I had not thought of. But let me does not matter, it is a nice thing to look at. So, look at, let us look at lines through the origin only for the time being. What is the line through the origin, what is the equation of that line, y equal to m of x. So, a line, so Lm a line with the slope m. I can write it as, points x comma, m of x x belonging to R. That is one way of writing a line through the origin as a subset in R2.

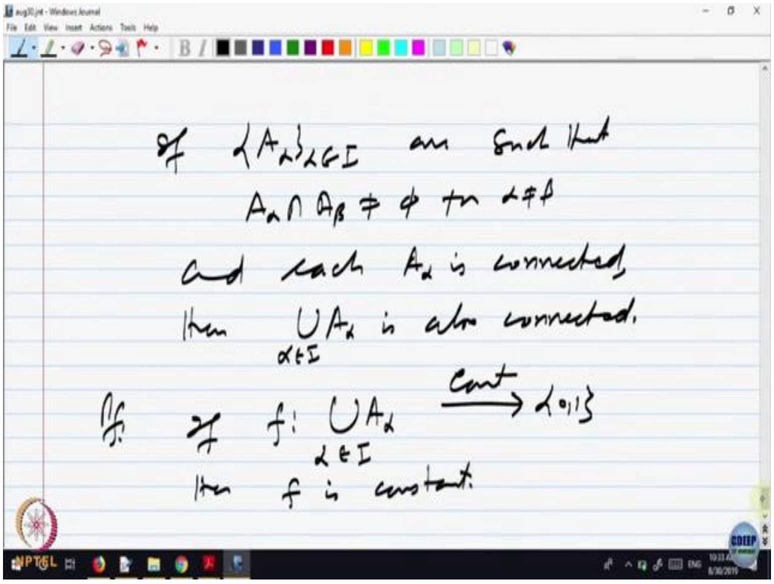
Now, look at the map x going to x, mx. So, this is a function from where to where, there is a function of from real line to R2, is it continuous? Is this function continuous? If x n converges to x, then the first component of this converges, m of x will converge, m of xn will converge to m of x. So, it is continuous because xn converging to x implies xn comma, m xn will converge to x comma, m of x, is a continuous function.

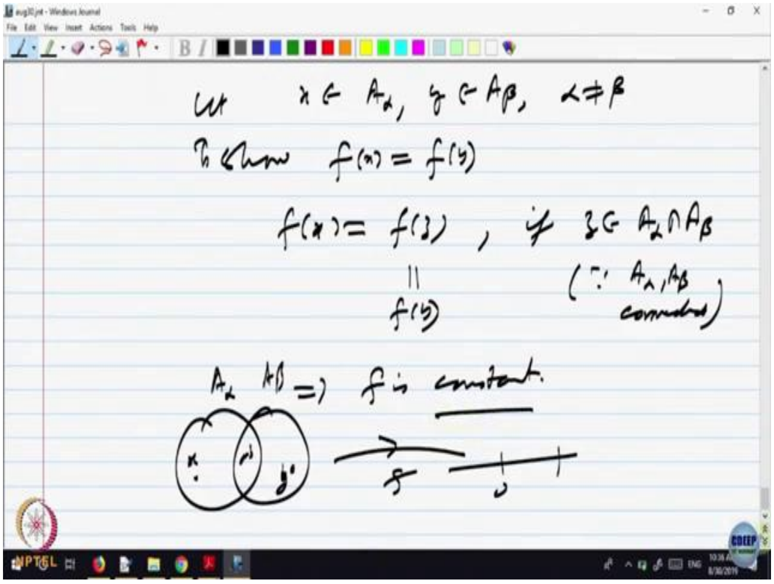
And we proved already, that is a continuous function, how do I prove this function is continuous, in the domain if xn converges to x, the image should converge. That is what we are saying here, precisely, that is what it says. In R2 that sequence will converge, a sequence in R2 will converge if and only if each component converges, same thing.

So, it is continuous and we proved, that if a set is connected, its image is also a connected set. Continuity preserves connectedness, so real line is connected. So, its image, the line through the origin also is connected. So, every line through the origin is connected. So, if you look at all lines in the plane passing through origin. So, this is a family of connected sets in the plane such that any 2 of them intersect.

They intersect at the origin, origin is a common point for all of them. So, this is one example of this kind of a situation. A alpha is a collection of sets in Rn, here example, it was R2 such that any 2 of them intersect. The claim is if each A alpha is connected, then the union also is connected. So, that is a point.

(Refer Slide Time: 05:34)





So, here let me rewrite again, if A alpha are such that A alpha intersection A beta is non-empty. I do not have to write for alpha not equal to beta because alpha equal to beta is non-empty anyway. And each A alpha connected, then union A alpha is also connected. So, arbitrary union of connected sets need not be connected in general.

For example, in the real line, take 1 interval 0 to 1, other interval 2 to 3, both are connected, but their union is one interval here, one interval there that is not connected. But if they intersect, so if any 2 of them intersect, then the union also becomes a, so is a kind of connected you can from one you can go to another through that connecting point, intersection point. You can think of that way also, and that is what we are going to do.

So, proof, if I want to prove this is connected, let us take a function f and union A alpha, alpha belonging to I into 0, 1. I am going to use that theorem that we are not yet proved, but I am here is giving a illustration of it, how useful that will be so and continuous. I should show this is constant, claim if then f is constant. So, how do I prove, for any 2 points, the value should be same, that is what we want to say. If they both belong to A alpha then we are through anyway because A alpha is connected.

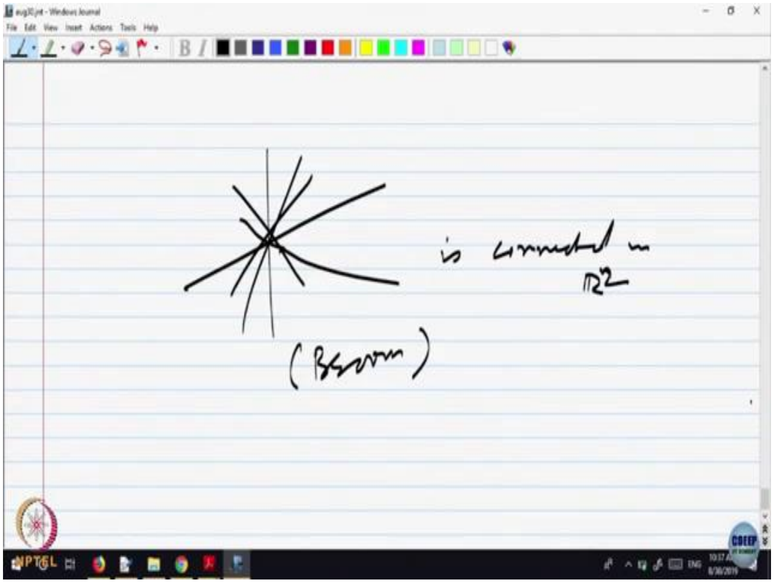
So, on each A alpha, f is going to be, on each alpha because A alpha is connected, restricted to A alpha cannot take 2 values either 0 or one, it will be constrained by that theorem. So, let us take one element in A alpha, other element in A beta. So, let x belong to A alpha and y belong to A beta, alpha not equal to beta. I want to show f x is equal to f of y, any 2 points should take the same value, then it is a constant function.

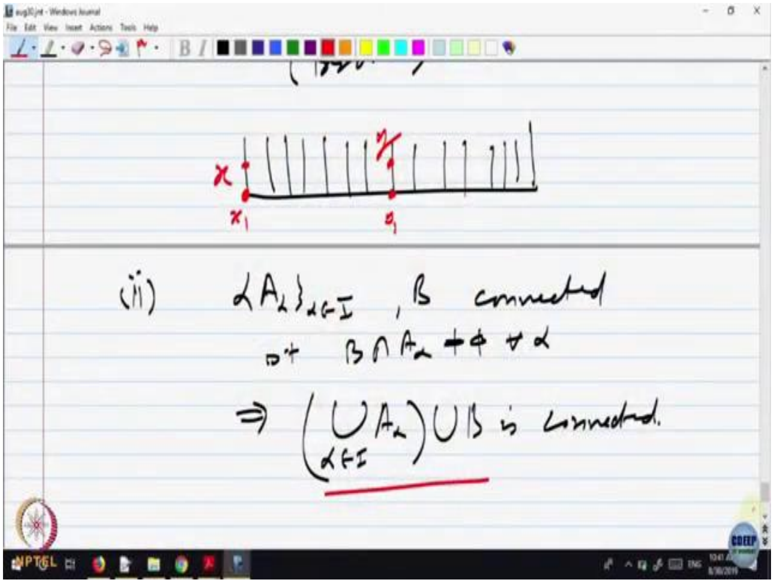
But what is f of x is equal to? From A alpha I want to go to A beta and I am provided a route that there is a point of intersection between the 2, A alpha intersects. So, let us go there and then come to other point, is equal to f z, if z belong to A alpha intersection A beta and that I know is there. At least there is one point z in the intersection and that is equal to f of y because A beta is connected.

f of x is equal to A z, because z belongs to both A alpha and A beta. So, f of x must be equal to A z because x and z both belong to A alpha, A alpha is connected. And z and y belong to A beta. So, they should be equal, A alpha, A beta connected. So, we use the fact that both are connected and intersection is non-empty.

So, that implies f is constant. Is it clear what we were saying, here is A alpha, here is A beta just for the sake of illustration, A is A beta, here is z, here is x, here is y, f is a function either taking 0 or 1 on x. f of x has to be equal to f of z because A alpha is connected and z and y, so z and y also and the same A beta. So, that f continuous connected implies they should be same. So, that implies it is a constant function.

(Refer Slide Time: 10:46)





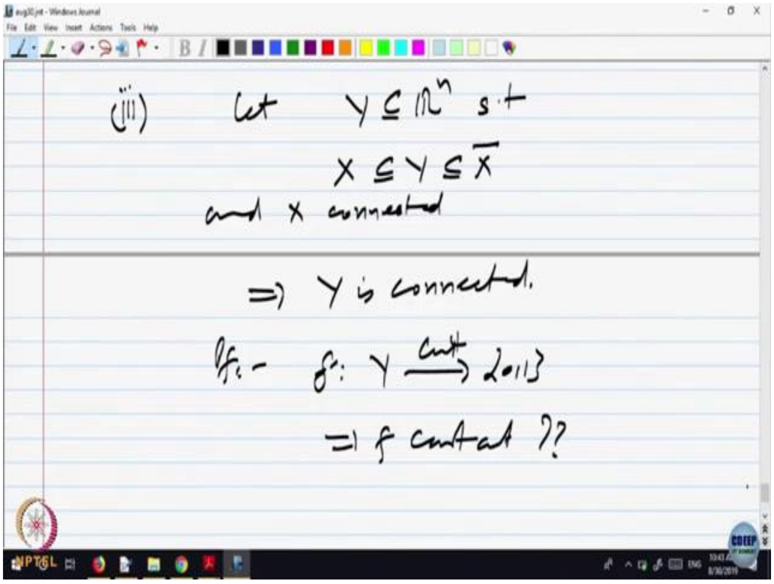
So, as a consequence of this, you can think of this kind of a set, this kind of a set is connected in Rn, in R2, here it is in R2, this is what is called a Broom, a very common object in the household, a broom. So, that is a connected set. Another illustration of the same, let us give me an, now, what do you think of this kind of a thing? Do you think intuitively it is a connected set, in R2? I am just looking at pictures in R2, is very common thing again in household, what is that, is a comb, is comb your connected set?

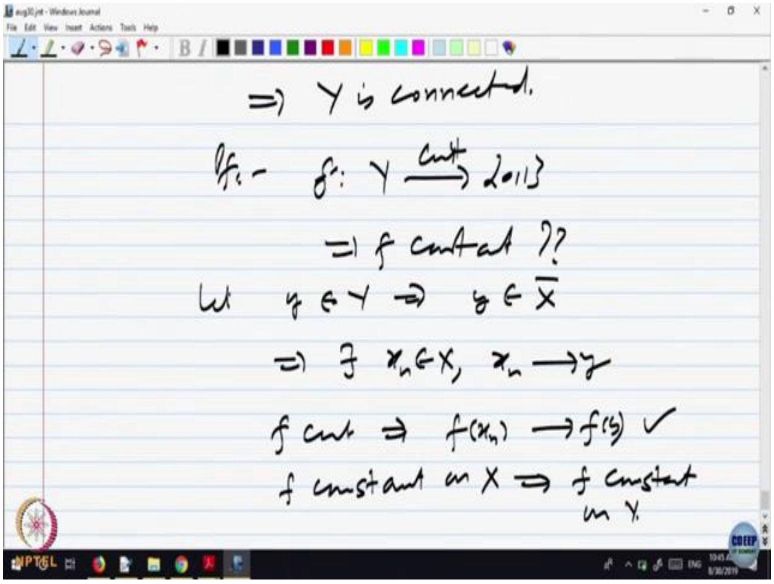
We can think of each vertical line as a set collection A alpha and bottom is a line call it B, and B intersects each one of them and claim that is also connected. So, let me write second, A alpha be connected such that B intersection A alpha is non-empty for every alpha. B intersects each A alpha somewhere then that implies union of A alpha along with B, I should write union then implies union of A alpha, alpha belonging to I, union B is connected. Again, the idea of the proof is straight forward, given any 2 points, so let me keep that picture one point is here, other point is here. I want to say that value at this point x is same as the value at the point y, for any continuous function in this union the value is same.

It should be a constant function, any function defined on this set is a constant that is what we want to say. So take, in the picture it looks like this kind of a set. Each A alpha is the vertical one, V is the horizontal one they at least intersect at one point. So, value here is same as the value here, value at x is same as the value at this point. So, call it x 1, x 1 belong to the horizontal line. Now, that value is same as the value at y 1 because that is in connected set and then I can go up to y, the value should be same, same idea basically, that is also connected.

So, intuitively, you can, if you want to write, you can write take any function from this set into 0, 1. For any 2 points x and y, if x belongs to, x and y both belong to same then no problem at all, if you belong to different one, then why are this kind of a route because of non empty intersection, you can pick up points and go. So, that is the basic idea. So, this gives you a lot of examples.

(Refer Slide Time: 14:54)





Let me, prove one more before giving you the abstract proof of that theorem. Let us look at 3, let us, let y be contained in Rn or X containing Rn such that, X contained in Y contained in X closure and X connected. So, what I am saying, Y is a set which is trapped between a set X and the closure of the same set X. If X is connected, this implies Y is connected.

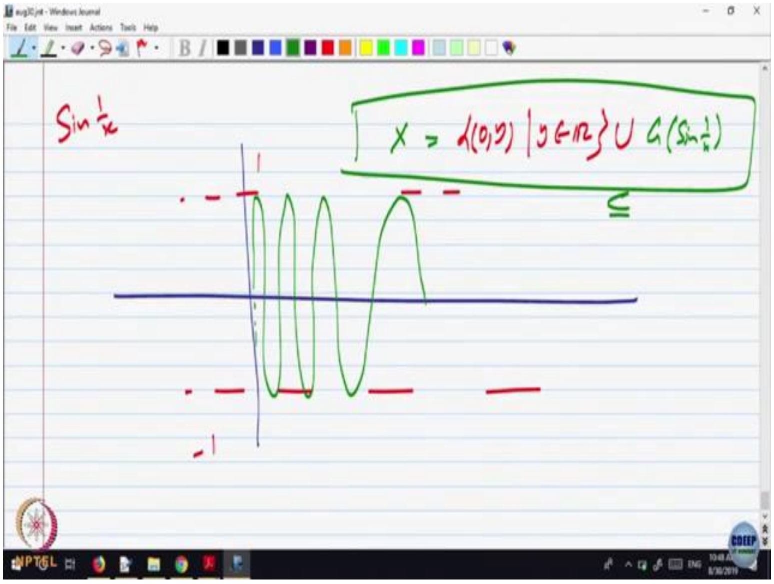
If something is trapped between the connected set and its closure, a connected set need not be closed. For example, open interval, 0 to 1 is an open set which is connected, it need not be closed. So, again what will be a proof of this, same consequence of the same theorem. If I take any, if I take any function on Y, then it should take only, it should be a constant function. So, f Y to 0, 1 continuous implies f constant. So, that is what we want to show.

Then Y will be connected? if any function on Y 2, 0 1, 2 point set 0 and 1 is continuous and we are able to prove it is constant then y is connected by that theorem. So, let us take 2 points or let us take any either way it is, let us take a point Y in Y.

So, let implies y belongs where? y belongs to y in X closure, in X closure implies what? There is a sequence xn belonging to X, xn converging to y, Because we just now said, we have this situation, X is inside, Y is in X closure. Take a point Y in Y that belongs to X closure. So, it must be a limit of a sequence in X. So, that means what? f continuous implies what? f of x n must converge to f of y, f is a continuous function on y. So, it must converge but what kind of sequence f of x n is where are x n's? They are in X and X is connected and f is a function taking values in 0, 1.

So, either it can take everywhere the value 0 or it can take only the value 1, it has to be constant function. So, if f on x is the constant function 0 and each f of x n is equal to 0. So, constant sequence, so f of y is 0. For every point y, f of y is 0. If is other one f of f restricted to x is constant function 1, then each one is equal to 1 then the value on y is 1. So, claim f constant on implies f constant on y because of this, whatever value on x it takes, if it is 0, then each of f of x n is 0. So, that is f of y, so the constant is same whatever it was on x and on y

(Refer Slide Time: 19:43)



And here is the interesting thing about this, example, let us look at this example. Look at what is the equation of the Y axis that is x component is 0 comma y, y belonging to R. So, that is the y axis. Let me write union of this with intuitively, what I am trying to do is, look at the function sine 1, sine of 1 over x, what is the graph of sine 1 over x? Have you come across, it is like sine x. So, it is wavy, but 1 over x that means near 0 it is getting cramped. Height is minus 1 to 1, so height is minus 1 to 1. So, that is 1, that is minus 1 and the graph looks like it goes like this.

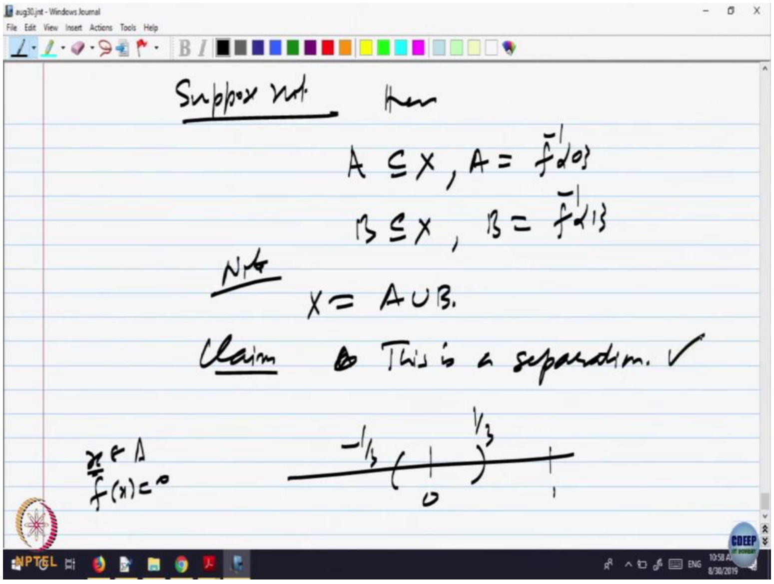
So, I am looking at the graph of sine 1 over x, union with the, the part minus 1 to 1 of ix is that is connected, intuitively clear. The line is connected that part is connected. What about a graph of sine 1 over x? It is a curve kind of a thing. Let us intuitively assume that it also is connected. Or you can think it as a continuous image of R 1 over x there is a continuous function accept at the point 0, sine 1 over x, if we do not look at.

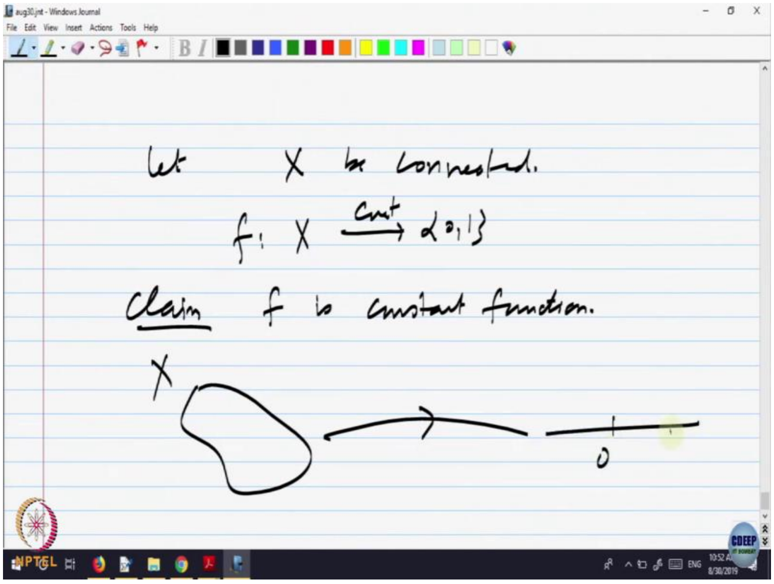
So, let me write graph of sine. I am just trying to give you some interesting examples. What do you think is a closure of this set? Call this as X. What do you think is the closure of this set? The graph of sine 1 over x and the interval 1, 2 minus 1 to 1 on the Y axis, see this is coming, it will be something, you cannot give a better geometric interpretation of this.

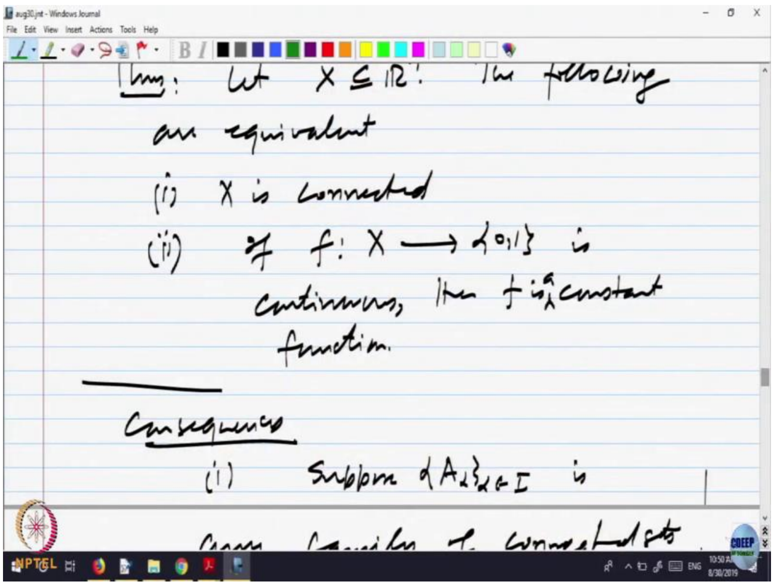
But what I am saying if this is contained in, what is the closure of that? What will be the closure of that? That will be union of, so what I want to say is this set is connected, graph of sine 1 over x, and that, they look like separate parts, does not look like they are connected.

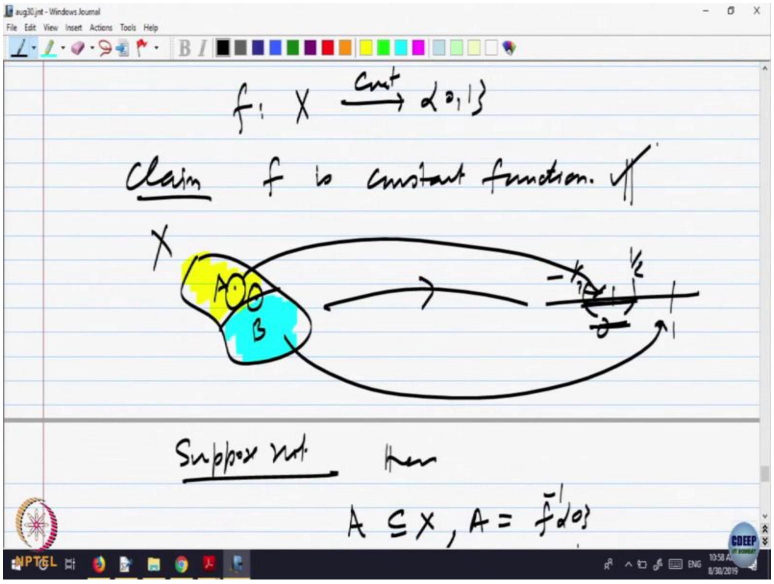
But what I am saying is, if you look at the closure of that set, the graph of sine 1 over x is coming closer and closer to minus 1 to 1. So, union of that is a connected set, so this is a connected set. If you are able to visualize it, if not I am just giving you some examples of connected sets in R 2. Do not worry about whether this will be asked in the exam or not, try to understand it. Institutively this is also a connected, so very exotic kind of things you can produce by using that theorem alone. So, I think got some time to prove that.

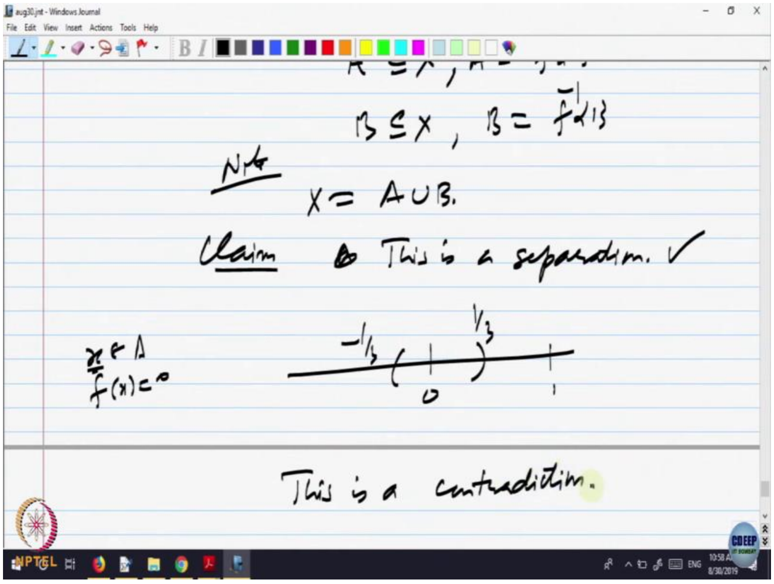
(Refer Slide Time: 23:45)











So, I want to prove this theorem, x is connected and 2 point set 0, 1, a continuous, it should be a constant function. So, let us assume one of them and prove the other. So let us write once again the theorem. So, here is the, where is a theorem, here is a theorem, so I want to prove it again, I think I made some, where is that happening. So, let x be connected, f x to 0, 1 continuous, claim f is constant. So, for the just sake, for the sake of visualization it is good to, this is x, this is the 2 point set 0, 1.

So let me just draw a picture, this is 0, this is 1. Suppose not, not of this, it is not a constant function. That means there are points where the value 0 is taken and there are also points where the value 1 is taken, we do not know what is where. So, look at then, look at the set A contained in x and what is A? That is f inverse of 0 and B in x, what is B? That is f inverse of.

So, collect all the points where the value 1 is taken, collect all the points where the value 0 is taken.

So, intuitively, you will get this kind of a picture. So, this is kind of A and this is kind of B, so this is A, this is B. So, every point in A goes to 0 and every point in B goes to one. Now look at the set A, so X is equal to A union B, so note A union B, every point of X has to go either in 0 or 1, so inverse a measure both.

Now claim, this is a separation of, I have written X as union of 2 sets A and B. And A and B are separated from each other. Because I have look at any point of A, if I look at any point of A and look at, can you find a ball, so that no point goes to the value 0?

See at every point, let me look at the picture here at every point of A I can have a ball. Even if you think this way, I will have a ball, this is a ball in A, the only the part in A is to be considered. So that will not intersect with any point in B. Or if you like, here is another way of looking at it, f inverse of singleton.

Can you say Singleton is a open or a closed set? Singleton set in R n, a singleton point is open or closed?

Student: (())(28:11)

Professor: What is the compliment of it?

Student: (())(28:17)

Professor: Is open, so singleton is closed, every singleton is closed. Inverse image of a closed set is closed. If you believe in continuity that definition equivalent way of saying that or look at, so basically what we are saying is that f inverse of 0 and f inverse of 1, both are open as well as closed in A, in X, in X. Both are closed and open in X or the 2 are separated from each other, whichever way you it helps you to visualize, that every point of a value is 0 and between 0 and 1 there is a distance.

So, I can take open ball around 0, take the universe image that will not intersect anywhere in B, no one of that should go in be. You can also think it this way, if you like take the open interval around it, which does not include 1. So, this is a neighborhood of 0, f is continuous. So, I should have a neighborhood of the inverse image. So, neighborhood of whatever point I am taking that will be in A. And that should not intersect with 1, because that is going to be completely contained there.

So, every point in A will have a neighborhood which does not intersect in B, so A is separated from B, is that convincing for you? Yes, take a neighborhood of the point 0 which does not include, say for example you can take 1 by 3 and minus 1 by 3 open interval. So, let me just for example here it is 0, minus 1 by 3 and 1 by 3, this is a neighborhood of 0 and let us take a point x belonging to A then what is f of x? That is 0.

So, x is a point, the value is 0. So, if I take this neighborhood of 0, what does continuity say? There should be a neighborhood of in the domain which is mapped into the, inside it. So, there is a neighborhood of the point x which is mapped into minus 1 to 1 by 3 and no point of that because that is inside this.

So, no point of that neighborhood will go to 1. That means that neighborhood does not intersect the set B. So, that is another way of saying this is a separation. Similarly, from B to A, so every point of A is separated from B, every point of B is separated from A. And that is a contradiction because x is connected.

So, there is a separation, this is a contradiction because x is given to be connected. And we are producing a separation, if it is not a constant function, so it must be a constant function. So, we approved one way. If x is connected, then the image, then every function on x to 0, 1 taking 2 values at the most, it should have only take one value if x is connected.

Every continuous function on x into the 2 points set 0, 1 must be a constant function. The converse is also true, but I think we do not have time to prove the converse. So, we will do it next time. Namely, if x is such that it has properties that every function is a constant function to point, then it is it a connected sets. So, we will prove the converse next time.