**Basic Real Analysis**

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**Lecture 25**

**Uniform Continuity and Connected Sets Part I**

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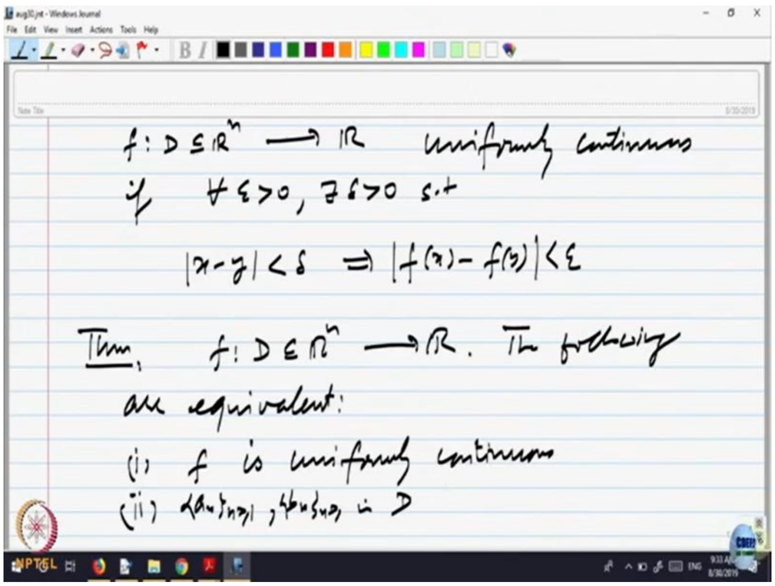


Begin with we are trying to prove the theorem that we defined for a function f on domain D and Rn to R to be uniformly continuous if for every epsilon bigger than 0. There exists a delta bigger than 0 such that whenever two points are close by a distance, delta that implies the distance between f x and f y is less than epsilon.

And we were trying to prove a theorem namely f as before D (())(01:13) and Rn to R the following are equivalent saying that f is uniformly continuous is equivalent to saying whenever we have two sequences A n, bn in the domain D such that bn in the domain D such that they come close to each other.

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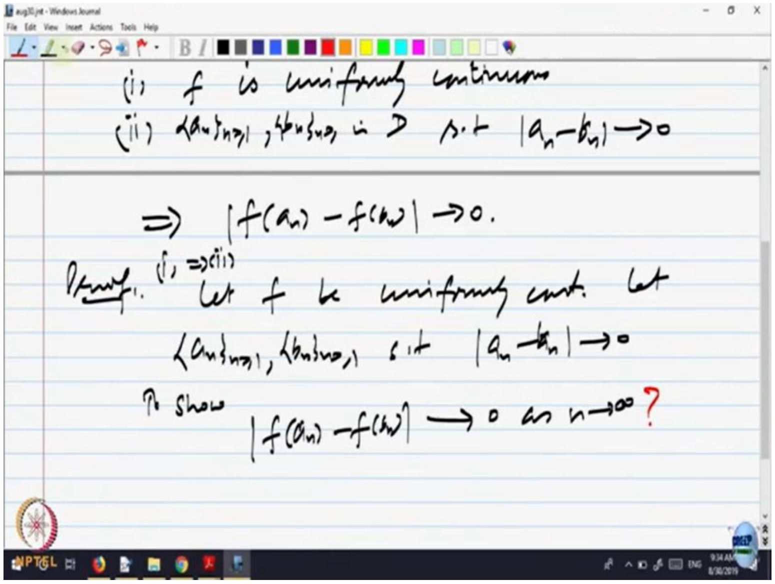

So, this goes to 0, then this implies the images also come closer. Now, goes to 0. So, we had proved one way, but anyway, let us recall. So, suppose f is uniformly continuous, so let f so we are trying to prove one implies two. So, let f be uniformly continuous. Let us take sequences a n and b n say that the distance between a n and b n goes to 0. So, where to show that the image sequences also have the same property as of course n goes to infinity. So, this is what we want to show?

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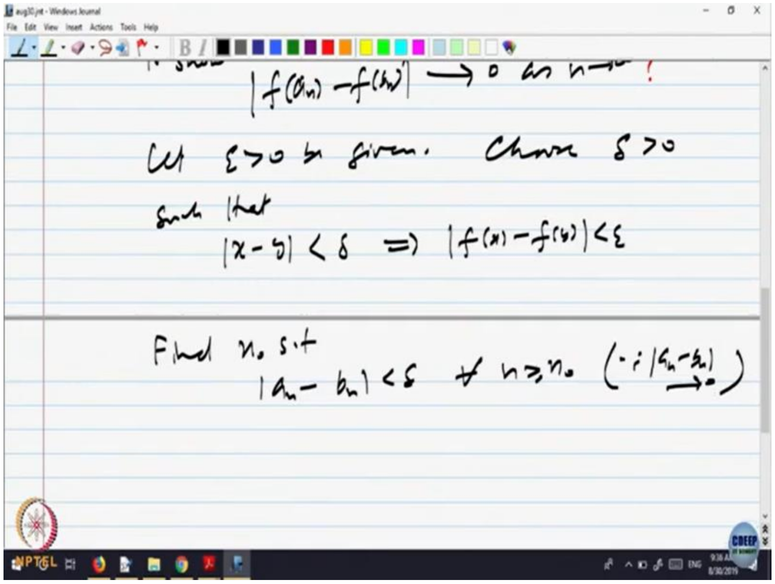

Now, to show that this goes to 0. What one is to show that given epsilon bigger than 0. There is a stage after which everything comes less than epsilon. So, let b given, we want to find a stage and not such that. After that stage this distance between f of a and f of b will be less than epsilon. But we know that will be less than epsilon whenever by uniform continuity if a n and b n are close. And that happens because this goes to 0. So, let us choose so choose, so choose some delta bigger than 0 such that, now.

So, such that x minus y less than delta implies fx minus fy is less than epsilon. That is possible because the function is uniformly continuous. That is given to us. So, whenever x and y are close by a distance delta that will imply that fx and fy are close by distance epsilon. So, if a n and b n come close to delta, then corresponding will be less than epsilon, and that is okay. So, find n naught such that mod of a n minus mod of b n is less than delta for every n bigger than n r, that is possible because we are given that because mod of an minus b n goes to 0.

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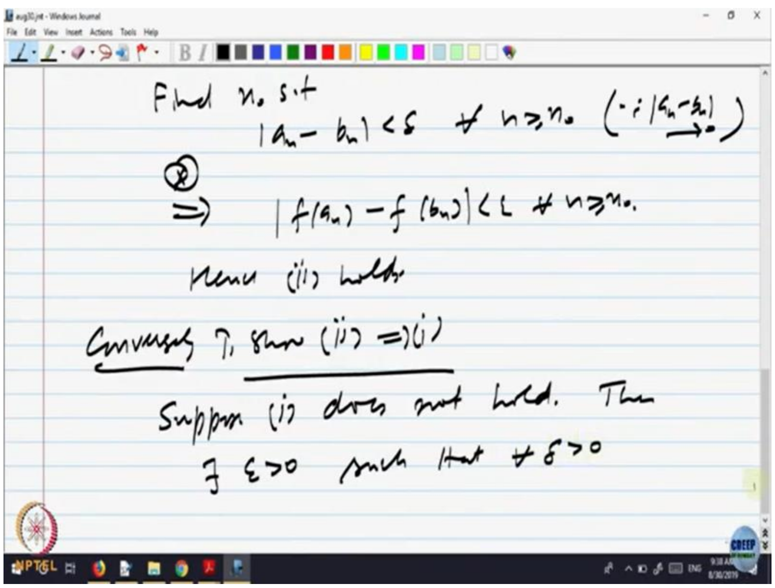


So, given any delta we can find a stage after which they are small? So, implies a n and b n is less than delta, so by the equation star, so star implies, f of a n minus f of bn in less than epsilon n bigger than n naught. So, that is what we wanted to show that if a n and b n are close then f of a n and bn are close. So, that is ends to two holds. So, what we have shown is f is uniformly continuous then this implies this property two conversely let us show two implies.

So, conversely to show two implies one. That means whenever two sequences a n and bn are close, we want to show that f of a n and f of bn are also close. That is given to me. Second we want to show f is uniformly continuous. So, suppose one does not hold. So, what is a meaning of saying that f is not uniformly continuous?

That means uniform continuity meant for every epsilon something happens, not uniformly continuous (())(06:56) hold. Then there exist some epsilon bigger than 0 such that, for existance we had, their registered delta, so such that for every delta bigger than 0.

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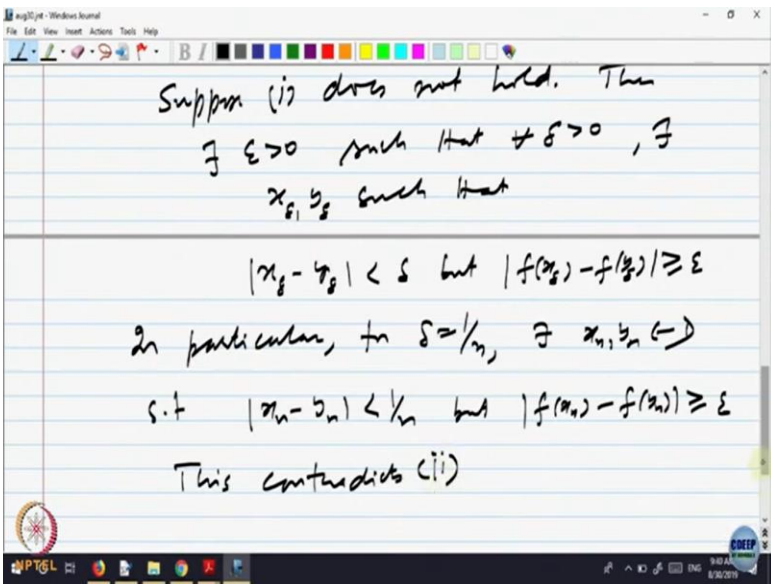


We can find a pair of points there exists two x delta y delta such that the distance between x delta and y delta is less than delta, but the distance between f x delta and f y delta is bigger than epsilon. That is a meaning of saying f not uniformly continuous. Something similar we have done when we were trying to prove that continuity implies a n converges to a implies f of a n. So, same proof basically. Negation is important.

So, we want a sequence now. So, in particular for delta equal to 1 over n there exists xn yn in the domain such that xn minus yn is less than 1 over n. We are specializing delta to be equal to 1 over n but f of xn f of yn is bigger than or equal to epsilon. But that contradicts the given hypothesis two, which is whenever sequences are, so this contradicts.

So, that is what was given to us, two implies one and what was two to us. Whenever a sequence is close images must be close. So, we found a sequence xn yn they are close. So, this goes to 0, but f of xn minus yn the distance always remains bigger than epsilon. That is a contradiction. So hence two implies one.

So, uniform continuity can also be expressed as in terms of sequences, namely whenever two sequences are closed in the domain, the image sequences must be also close. So, that is in terms of sequences. So, let us look at some applications of this or some consequences of this so that, they normally you approve two statements are equivalent not just because we want to prove mathematically something nice, but also in some situations one is useful, one criteria is useful in some other situations. Some other criteria is useful, but both are equivalent.

So, for example, saying that a sequence is convergent if and only if it is Cauchy the two statements are equivalent as equals it is convergent is equivalent to saying a sequences Cauchy. Sometimes if you want the limit then you have to use the first one. That sequence is convergent by finding the limit of it. But if you to do just want to prove that a sequence is convergent. You are not in really interested in knowing what is a limit. Then proving Cauchy as is good enough, elements are coming close to each other. So for example, here, let us look at some example.

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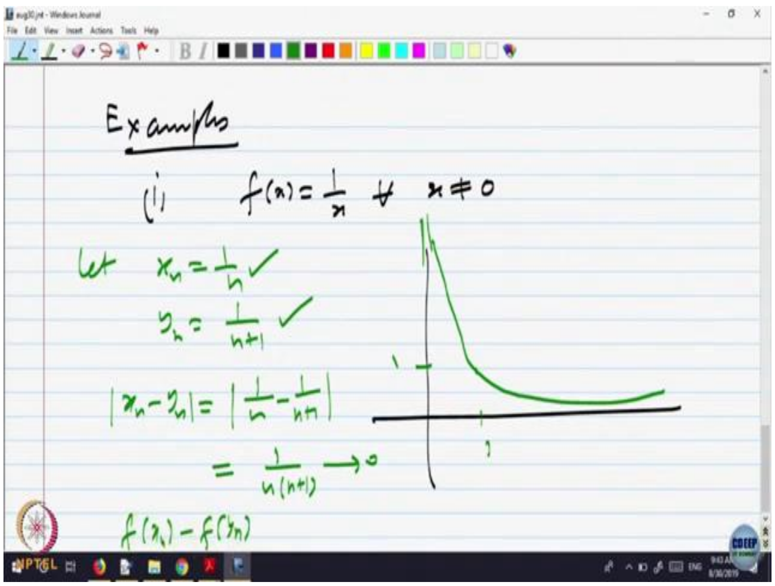

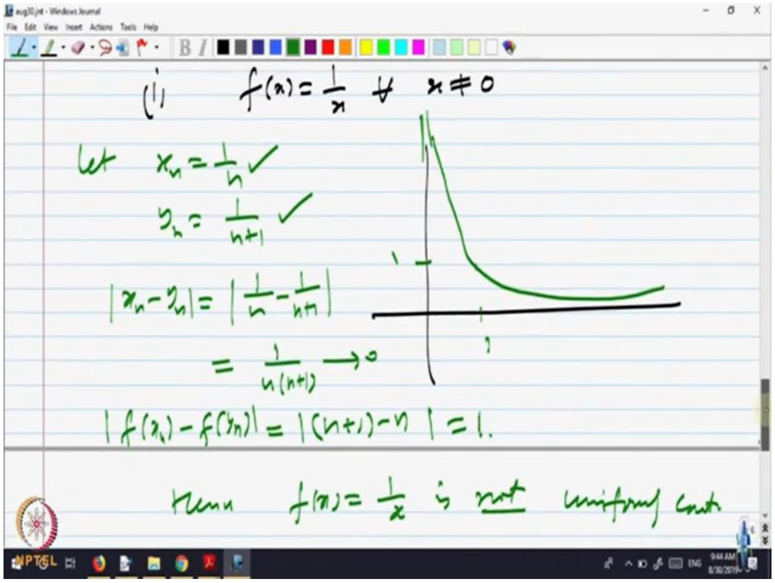


Okay, so here it is. Let us look at f of x is equal to one over x, for every x not equal to 0. So, look at the function, f of x is equal to 1 over x. So, if you look at the graph of this function geometrically, as you come closer to 0, your graph blows up, is increasing at is increasing much faster. So, it looks like, so at value 1 the value is 1. So, here it goes like this, it is coming closer and closer to 0. But if you see that between 0 and 1 is becoming large very fast.

So, same notion of closeness near one will not work near 0, you might require much bigger kind of thing. But if we want to look at sequences, so let us look at, let us look at, let xn be equal to 1 over n and yn be equal to 1 over n plus 1. What is that difference between the two? So, what is that there is 1 over n into and that goes to 0. So, in the domain, I have got two sequences. One is one over n other is this? And what happens to f of xn.

What is the difference between these two? So, that is n okay. Absolute value that is equal to n plus 1 minus n absolute value that has equal to 1. So, I got a pair of sequences in the domain xn yn such that the distance between them goes to 0, but the distance between the image does not go to 0. So, that there is one way of saying hence f of x equal to 1 over x is not uniformly continuous. Okay so that is one example of how sequences are useful in proving these kinds of things. Let us look at some more examples.

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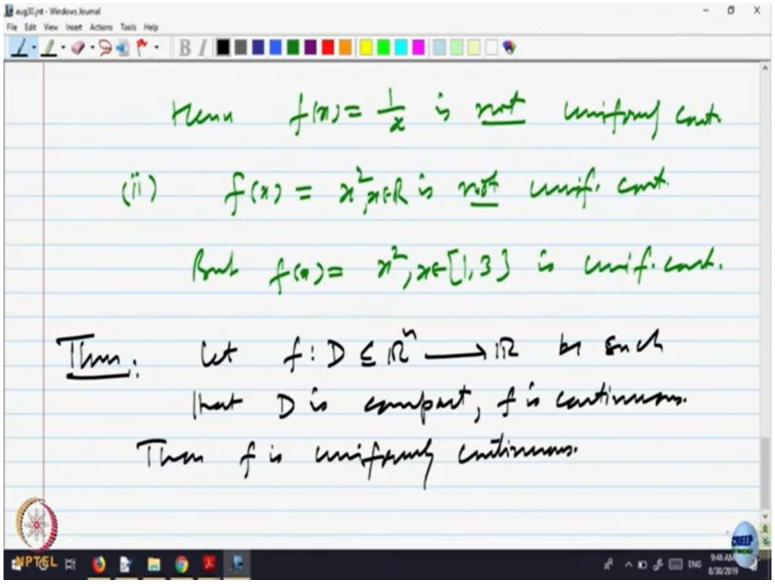


We had already shown that f of x is equal to say for example, that x squared we showed is not where x belonging to R is not uniformly continuous, but if I restrict the domain equal to so x square between say anything between say 1, 2, 3 x belonging to is uniformly continuous. That we proved that example we showed that if you restrict the domain then it becomes by using epsilon delta definition. In fact, so here is a theorem.

Let f be from D contained in Rn to R be such that D is a compact. That means equivalently closed and bonded and f is continuous. Then that is the kind of thing happening here in f of x is equal to x square. There is continuous and the domain is a close bounded interval 1, 2, 3, and we were saying whenever such a thing happens, f is uniformly continuous and f becomes uniformly continuous.

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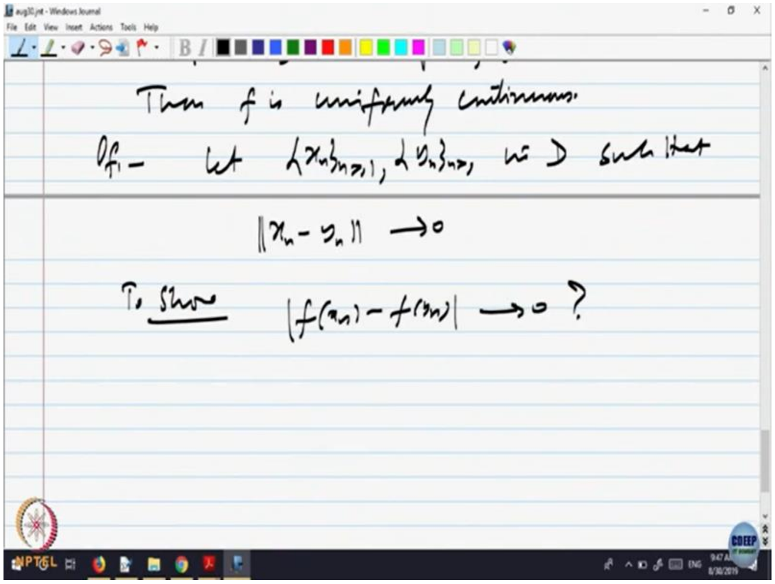


Okay, so Let us see, why is that? So, let us use sequence criteria here? So, let xn yn in D such that is. Then Rn so I will just write absolute value, meaning that in Rn it is a norm. If you like, you can write this is norm of xn minus yn goes to 0. Everywhere whenever we are a (())(16:05) Rn, instead of absolute value. You can think of absolute value of vector as the norm or whatever it is, is it okay?

Is as metro mutation for a vector in Rn, the norm, I sometimes call it an absolute value also of that vector is same as sigma Ai square by square root that thing the notion of distance in Rn. So, it goes to 0 to show. So, what we have to show f of xn minus f of yn we are in the real line, so that goes to 0. So, that is to be shown? So, now look at xn.

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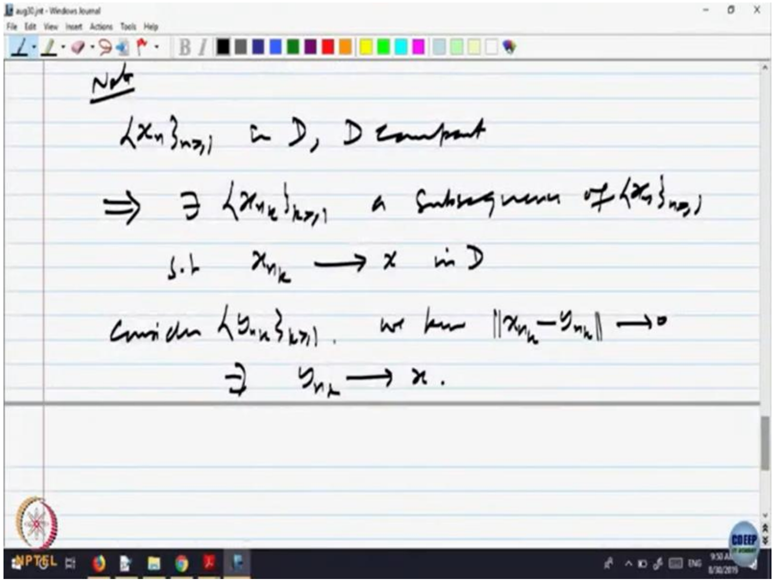


So, let us note, so let us write and then probably explain why I am doing that. So, xn in D D compact, what does that imply? It should have a convergent subsequence implies there exists xnk a subsequence of xn such that xnk converges to something let us call it as converges to x in D.

Now, xnk converges, so consider ynk the corresponding charts of the subsequence ynk. So, what do we know? We know xnk and ynk are coming closer goes to 0, because xn yn goes to 0 subsequences is corresponding will also have the same property and xnk is converging to x. So, where does, ynk converge? So, implies ynk converges to x, so what does that imply?

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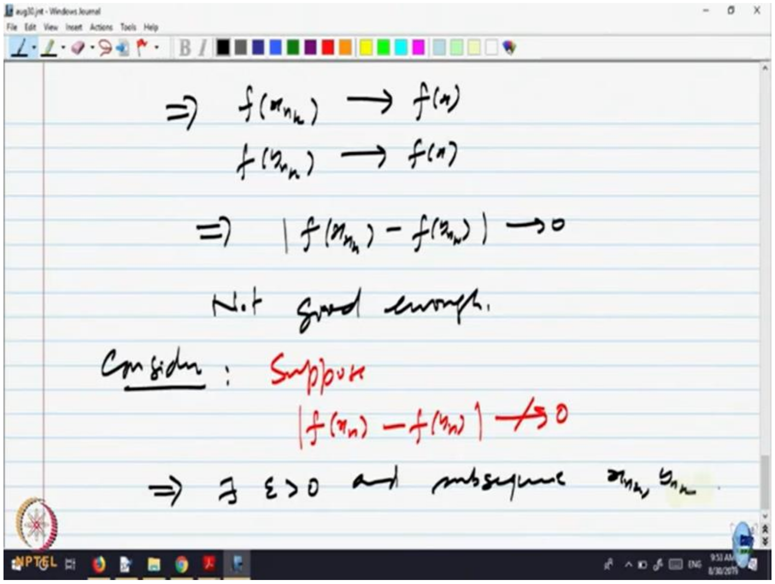

So, what, what is given to us f is continuous. So, f continuous so implies f of xnk converges to f of x and f of ynk also converges to yes? f of x because continuity. So, that implies that both are converging to f of x same limit. So, f of xnk minus f of ynk goes to 0. So, what we are showing? Are we showing what we wanted, we wanted that f of xn should and yn should converge the difference should become, what do I shown is there is a subsequence for which that is happening.

That is not good enough. That is not we have only shown for subsequence. We want to show it for the original sequence. So, let us, so this, direct kind of argument does not work. So, let us go to, we want to avoid that. Going to a particular subsequence. So, let us assume that this statement is not true. Assume so let us assume that, so not good enough. So, consider so we need to modify the proof.

So, consider assumptions suppose minus f of yn does not go to 0, but we still have the faith that xn yn are going to 0. That is already given to us anyway. This does not go to 0. So, implies what? And something, a sequence of numbers, the sequence of numbers that does not go to 0. That means what negation of the statement, implies there exists some epsilon bigger than 0 and subsequence xnk, ynk.

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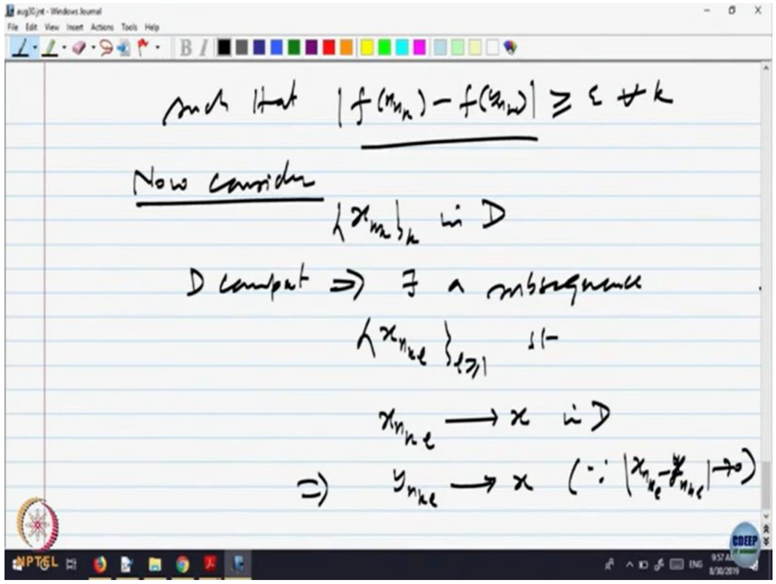


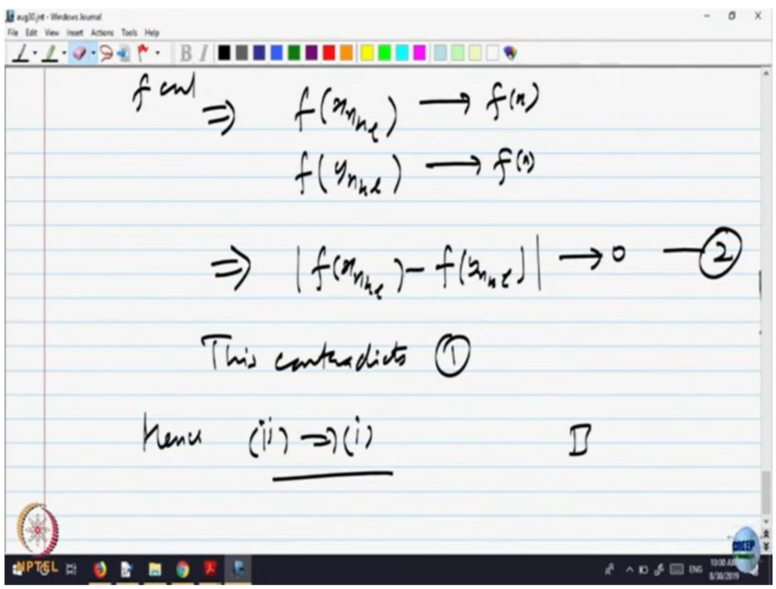


Such that mod of xnk minus f of ynk remains bigger than or equal to epsilon for every k, is that okay? Negation of the statement that does not go to 0 so it should remain away from 0 for at least a subsequence. Now, I can work with the subsequence. So, now consider xnk in D, which is compact. D compact will imply what? Will, will (())(23:10) same track xnk as. Will get a subsequence, which is a convergent. So, that will be convergent. I think we are landing up at a same blockade in our proof.

I think I was thinking a sequence proof should be good enough, but you know, the problem will be reaching at the same place because you get a subsequence for which this contradiction will be there? But for every ynk no that should be okay probably, for every k yeah I think maybe. Let me just write and see whether that is good enough. If not, will change the argument, D compact implies there exists for this. There is a subsequence so same argument as before, but for the subsequence, there exist a matter of writing.

There exist is a subsequence xkn is already a subsequence. What you would you write to write so let write xknl. Such that xn xnkl, not kn but I just is subsequence of a subsequence xnkl, xnkl converges do some x in D, So, let us write, so implies that is good enough, I think. Yeah, it is greater than that is, so implies what implies, if I look at the corresponding subsequence, ynkl that also converges to x because xnkl minus ynkl the distance goes to 0.

Are you that subsequences of subsequence I will all you all explain again what I am saying? So, let us negation means this thing that is saying that this does not converge to 0, that means there is at least one subsequence where the distance remains bigger. Now, look at this sequence xnk which is actually subsequence of the given xn. But look at that as a sequence that should have a convergent subsequence because D is compact as before.

So, it is a matter of writing xnkl that means it is a subsequence of xnk so internal also a subsequence of xn. Subsequence of the subsequence that is a subsequence of the original also. So, let us so that converges to x by compactness, but the corresponding y subsequence because xns and yns are coming closer. So, ynk also will converge to x because they are coming closer. So, that implies what? So, that implies by continuity.

f continuous that f of xnkl converges to fx, also f of ynkl also converges to f of x. So, that should imply what there are two sequences which can converging to same limit f of x. The sequences must come close to each other now, so implies mod of xnkl minus f of ynkl that goes to 0. The same proof as before but only for subsequences. But now look at this statement, call it two and call that earlier statement as one where is that this statement as one.

So, look at for the original subsequence xnk the distance remains bigger than or equal to epsilon right? For every element. So far, the subsequence also of this sequence, the distance will remain bigger than epsilon, so xnk and ynk are sub sequences whose distance should remain bigger than or equal to epsilon by 1. But by second it says that should go to 0. So, that is a contradiction. So, this contradicts 1.

And what was that 1? So, just to remind you again, because of our assumption the distance between the corresponding elements of the subsequence must always remain bigger than epsilon. But here we have got an two subsequences of the given same sequences so either the distance goes to 0. So, that is a contradiction. So, hence, so that assumption must be wrong and that means, so hence mod xn minus yn going to so that, so what was the, what we have proved?

We assumed it is not, so suppose it is not true then that must happen. So, that proves, so we are proving two implies one? Hence two implies one is okay. So, continue, now what is two implies one? I am sorry, we are not proving two implies one, we are proving if a function is continuous on a compact domain, then it is uniformly continuous.

So, hence that was earlier one hence f is uniformly. Let me just, because I change things too much. So, let me revise this. You are given f is a function on D Rn to R D is compact, f is continuous you want to prove that f is uniformly continuous. So, we have to prove whenever two sequences x and y in a such that the distance goes to 0, the corresponding distance between the image must go to 0. So, the basic idea is if this does not happen, I will have a sequence xnk ynk in the domain.

Where the distance of xnk and ynk goes to 0, but this distance does not go to 0. It remains bigger than epsilon that is the negation. So, that is the negation we wrote, to show. So, this was now, this was not the correct one here. So, suppose this is does not go to 0. So, given epsilon bigger than 0 there is, so this is what is the assumption? So, if this does not go to 0, then for any there is a epsilon (())(30:56) the subsequence so the distance remains bigger. If it does not go to 0, there is at least one subsequence for which the distance should be remains bigger than 0.

That is bigger than epsilon. Now, look at this xnk in the domain, xnk and ynk are two sequences in the domain. Look at xnk. It should have a convergent subsequence so xnk converges to some x for some subsequence corresponding subsequence of ynk, should also converge to x because xnk and ynk are close to each other. That is given to me, so I have got a subsequence of xnk. I have got a subsequence of ynk both converging to same point x by continuity, the image of this subsequence will also converge to same point f of x.

But that means what? If two sequences are converging to same point and they should come close to each other. So, that implies for the given sequence I got a subsequence, ynkl and so on, such that their distance goes to 0. So, that is what we got here. But that cannot happen because corresponding terms of the original sequence always remain bigger than epsilon. So, this is a contradiction. So, that proves that every a continuous function on a compact set is uniformly continuous. So, as a consequence one could have said that, in that example, this is uniformly continuous, but we proved it by using definition alone.