Basic Real Analysis

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Lecture 24

Continuity and Uniform Continuity-Part 3

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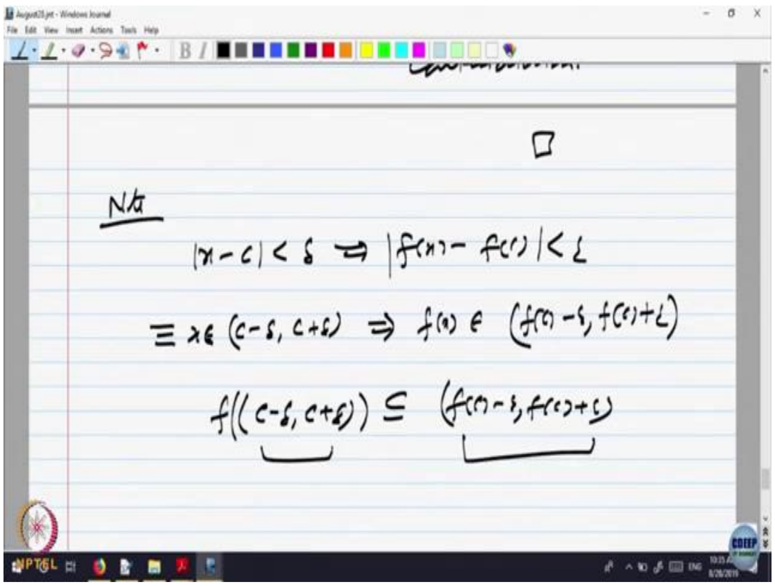

As we how mathematics builds up so let us write, now what is the meaning of note saying x minus C less than delta implies f of x minus f of C less than epsilon is equivalent to saying, this is in term of absolute value I want to write it in terms of sets, it says if I look at the interval so look at the interval C minus delta to C plus delta if x satisfies this property x is in this interval so x belonging to this implies f of x is where? f of C minus epsilon f of C plus epsilon.

That is same as saying if I look at the image of the interval C minus delta to C plus delta that is your subset of f of C minus epsilon to f of C plus epsilon of this interval. I am rewriting everything slowly changing the notations, f of x belongs to that if x is in this then f of x belong that means the image of this is inside that. So, what we are saying is given a neighborhood of f of C given an epsilon neighborhood of f of C there is a delta neighborhood of the point C such that delta neighborhood is mapped into the epsilon neighborhood.

So, now it is going in send of set 3 language, so the reason is in this language it becomes this definition of continuity becomes is still in terms of intervals if I write in terms of neighborhood given a neighborhood of the point f of C there is a neighborhood of the point C which is mapped into it then everything is in term of neighborhoods then it becomes extendable this definition becomes extendable where there are no sequence is nothing but only neighborhoods, you play with neighborhoods.

So, that is the interesting part of it, so I just wanted to keep you make you aware that these two are equivalent ways is only matter of saying what language you are choosing. For example you can write this in this terms of open sets, because what is this? This is the neighborhood, so supposing you are given an open set which includes the point f of C then there will be neighborhood inside it and there will be neighborhood coming from inside this and what it says that if you look at the inverse image of that neighborhood that is also a neighborhood.

That means inverse image of open sets are open that is another equivalent way of saying continuity. So, I am not doing much of this because in your courses probably you will not need it but those who are interested and want to read later on.

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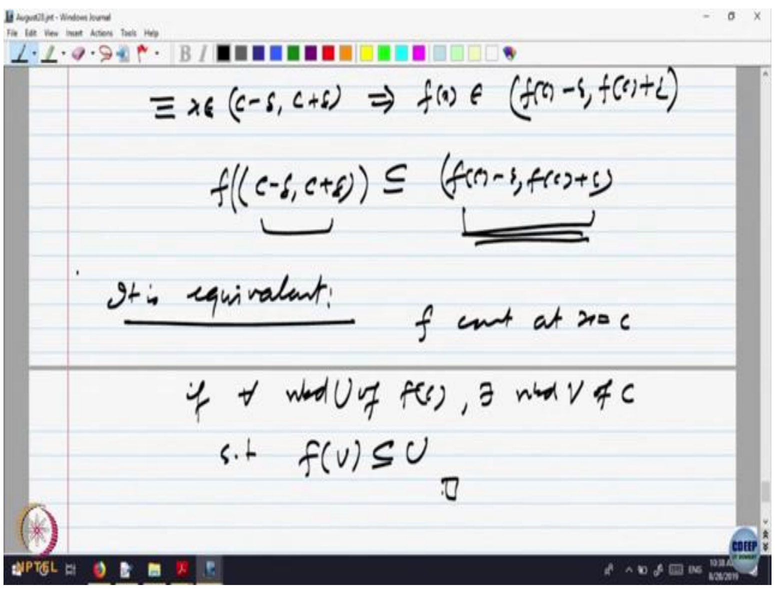

One can say so it is equivalent I am just saying for the sake of completion otherwise not part of this course as for as exam is concerned, so do not bother about it. See, f continues at x is equal to C if for every neighborhood of f of C every neighborhood u of f of C there is a neighborhood v of C say that f of v is inside u and then one can write this in terms of open sets and so on.

So, let us not bother much about it. So, what we have done is, we have looked at limits of functions and then we looked at continuity property, limit is we treated limit at the value that you expect the function to take at points considering estimating the value at a by looking at the values of the function at nearby points. And when it is equal to actually the value of the function you called it continuity and then we look at various properties of continuous functions. Here is something which I think that this is also good.

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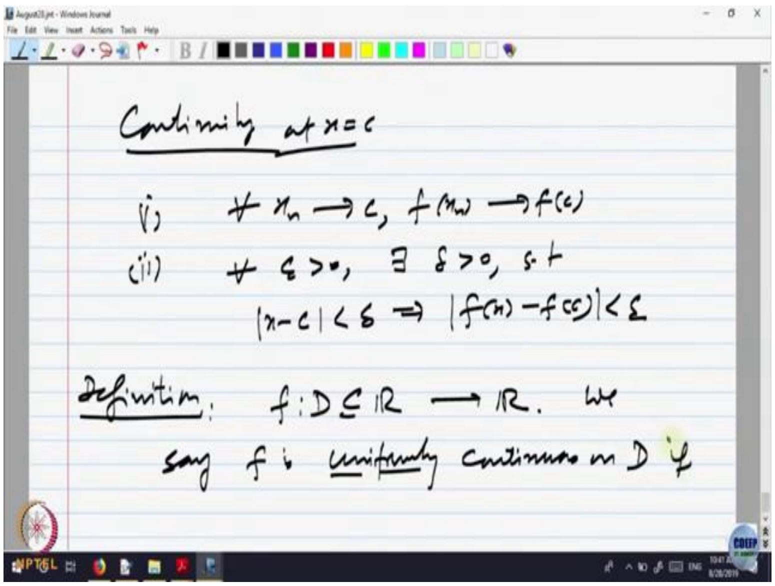

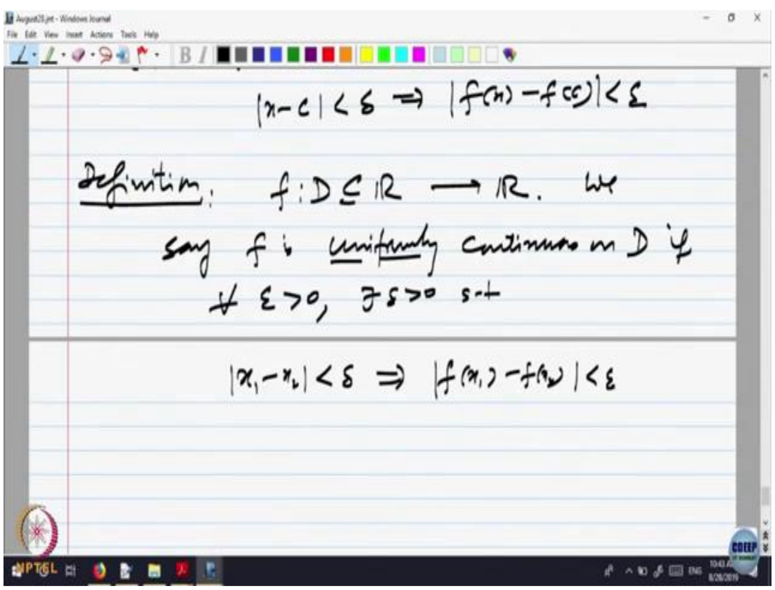

So, we have two ways of defining continuity at x is equal to C, one was for every xn converging to C f of xn converges to f of C and second for every epsilon bigger than 0 there is a neighborhood delta be greater than 0 such that x minus C less than delta implies f of x minus f of C is less than epsilon.

Now, again I am trusting the point that continuity is the property of the function at a point, is a local property. What I mean by local property? Because… xn converging to C, so you specializing what we happening to even here, if for the same epsilon supposing there are two different points of continuity C1 and C2 for this, for the point C1 given epsilon some delta may work that may way not work for other places.

So, neighborhoods may change as you points change, given neighborhood existence of neighborhoods may change at one point a bigger neighborhood may be, at other point you may need to go for a smaller neighborhood. But there is a notion of continuity which says irrespective of where you are it works, so let us define what is call definition, f and now this is going to be a property which is a not a local but more of a global property.

f is defined in domain D2 R we say f is domain D does not actually I should write a interval is uniformly continuous on D if, it something like continuity but it does not depend on the point if for every epsilon neighborhood there is a delta such that whenever x1 and x2 are any two points less than at distance delta should imply f of x1 minus f of x2 is less than epsilon.

So, now it says take any two points x1 and x2 in the domain, if x1 is closed to x2 by distance delta does not matter where they are then f of x1 is closed to f of x2 by the distance epsilon. Say if supposing a fixed x2 and is a point of continuity then given epsilon there is a delta such that this will happen. If x 2 is a point of continuity then for every x1 there is a distance at the most delta from x2 f of x1 minus f of x2 will be less than, but this choice of delta there exists a delta may depend upon what is the point x2.

Supposing I reverse the roles, I say x1 is a point of continuity then again given epsilon there will be a delta say that x1 minus x2 less than will imply that but that delta may be different from the earlier one when you are using continuity at the point x2. So, let us I will give examples to illustrate that, so in a sense it looks like that given epsilon there is a delta does not seemed to depend on the point where you are looking at continuity.

Anywhere if two points are close then the values are close continuity says if the points are closed to that given point then the values are closed to the values of the of that point. So, that specializes on.

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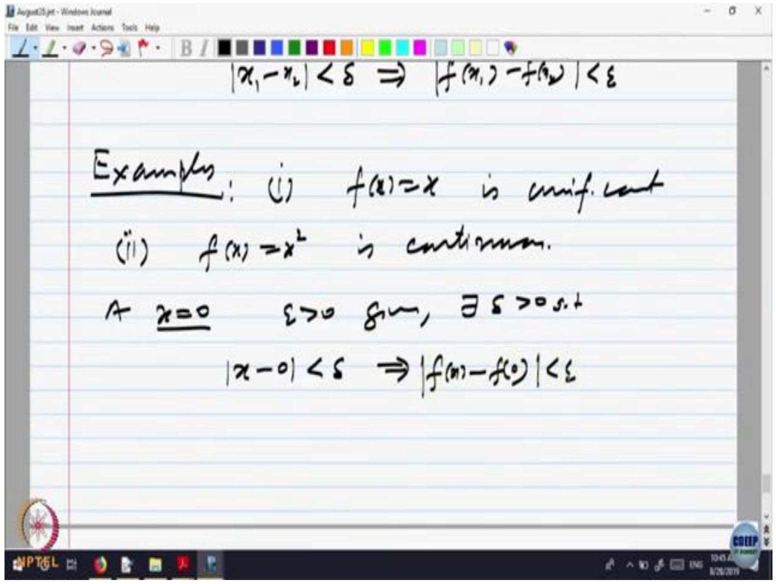

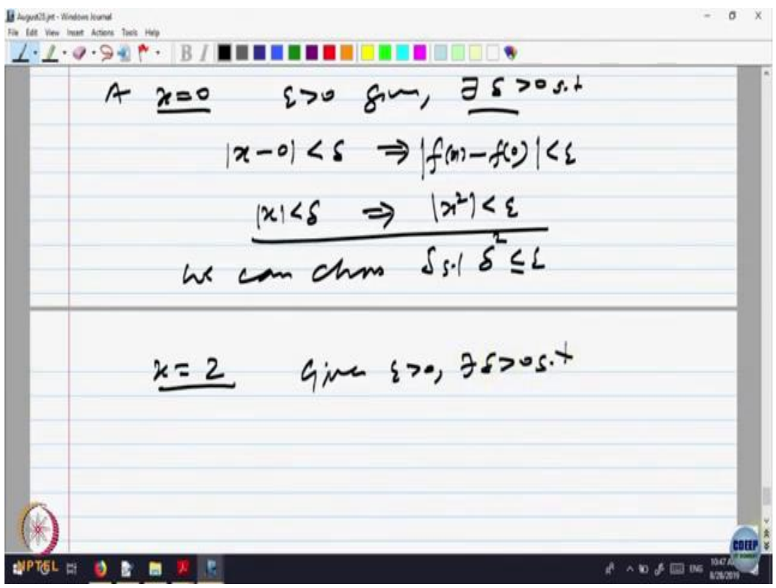

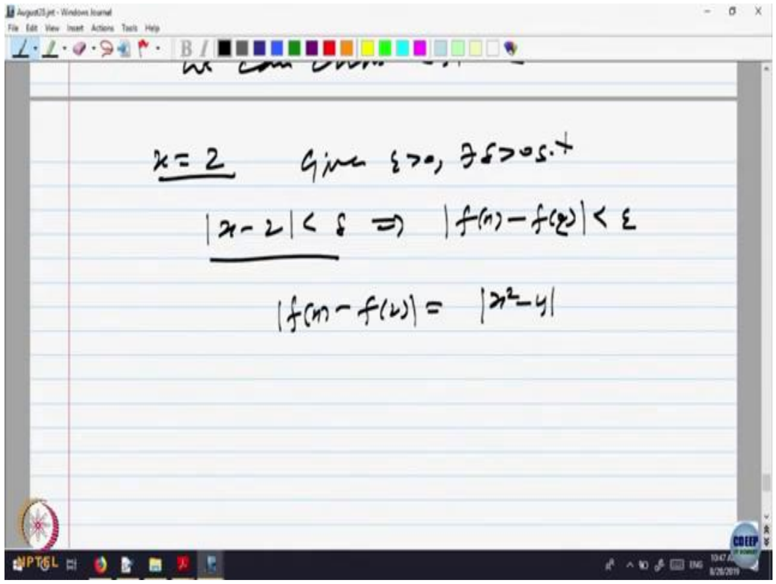

So, let us look at probably some examples so that, the simplest example is if you look at f of x equal to x is uniformly continuous, there is nothing because the function is not doing anything, is not changing at all. If two points are close f of x is x itself so nothing, but let us go a step further let us we got f of x is equal to x square, we know it is continuous at every point, f of x is continuous because f of x is continuous product of continuous function so f of x square is continuous, if you want to look at it but every polynomial function is continuous by that limit theorems, is continuous.

Let us try to write x is equal to 0 continuity, continuity at the point x is equal to 0, so what does it mean? Epsilon given I have to choose there is delta bigger than 0 such that x minus 0 less than delta should imply f x minus f0 should be less than epsilon, that is continuity. So, let us just elaborate this for our setting that means mod x less than delta should imply x square at 0 value is 0 is less than epsilon.

Because f of x is x square, so what is the best possible choice of there exists a delta, so what is the best possible selection of delta largest possible I can make, so we can chose delta what would you like to choose? Delta such that delta square chose delta such that delta square is less than or equal to epsilon. Your epsilon is given to you, so choose obviously from this equation.

Now, let us look at x is equal to 2, what is happening? At x is equal to 2, right, so I want the same thing given epsilon bigger than 0 there is a delta bigger than 0 such that x minus 2 less than delta should imply f x minus f 2 less than epsilon. So, what is f x so that is mod f x minus f2 what is that equal to x square minus f 2 the value is 4, so this less than epsilon. So, epsilon is given to me I have to choose a delta. So, what delta you choose? How do you choose delta?

If you have not done this kind of thing earlier so here is the way, see I want x square minus 4 to be made small what I know is x minus 2 is small so somehow I had to bring in x minus 2 in this so what we do is x square minus 4 I can write it as x minus 2 into x plus 2 mod of that. Then whatever my delta is going to be.

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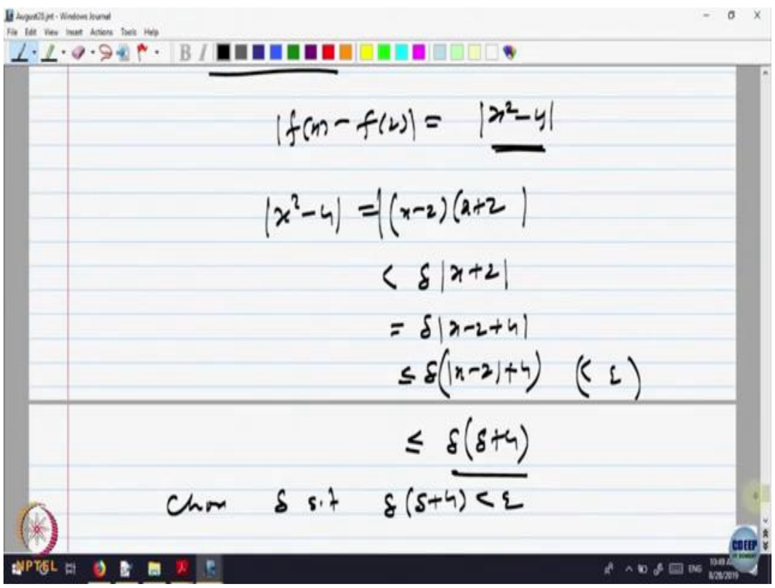

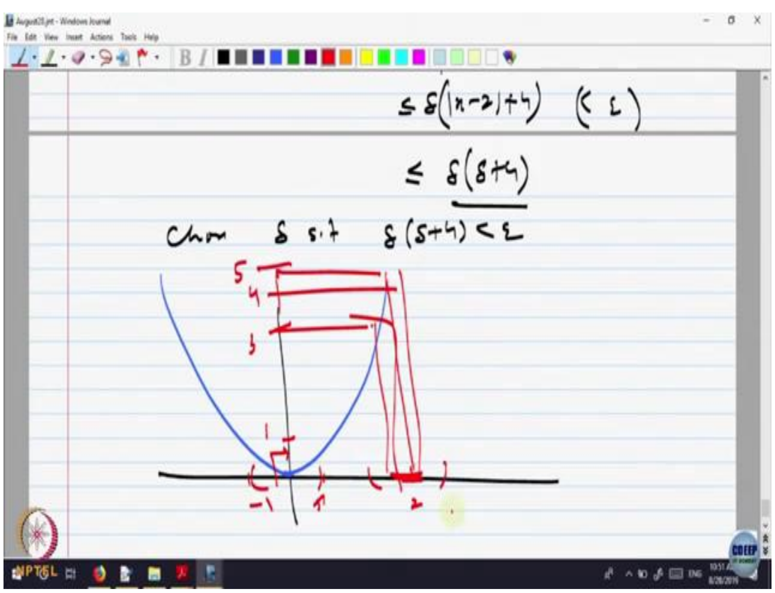

So, this is going to be less than it is only what we want to do, less than delta into mod of x plus 2. If you still want now still x is hanging around I do not want that x 2, so what you can do is you can write this there are many ways one can proceed x minus 2 plus 4 because everything I want in terms of x minus 2, so which is less than or equal to delta mod of x minus 2 plus 4, so that is less than or equal to delta sorry now bracket here delta this is less, going to be less than delta plus 4.

So, if I want x square minus 4 to be less than epsilon I know this is going to be less than this if x minus 2 is going to be less than delta so choose delta such that delta into delta plus 4 is less than epsilon, then this quantity will become less than epsilon, so everything will be okay. But now you see what is a difference coming earlier given epsilon I could choose delta square less than epsilon and now I have to make delta square plus 4 delta to be less than same delta is not working, I have to make some more modifications.

Geometrically if you want to look at here is the geometric way of looking at it, my function is y equal to x square, so let us look at the graph of this function, this is the graph. So, at x is equal to 0 given epsilon so let us say this is epsilon is equal to 1, then what is my delta square? Delta square should be less than or equal to 1, so that will work so, so between this and this minus 1 to plus 1 everything will be going there.

But if I look at some point here, so I had to get 2 and I look at same epsilon equal to 1 that means what, this is 4, this is 3 and this is 5 then that big thing will not work, I have to make my interval smaller so that the values go inside it, so what will value square root 3 other way around, 2 square root of 5, so that interval I should take then the values will go inside when you square it.

So, it becomes different delta for the same epsilon different delta is required, so I am just trying to illustrate with some example. But what it is saying uniform continuity says I should not bother about where the point is if two points are closed then the distance between them is closed, that is called uniform continuity.

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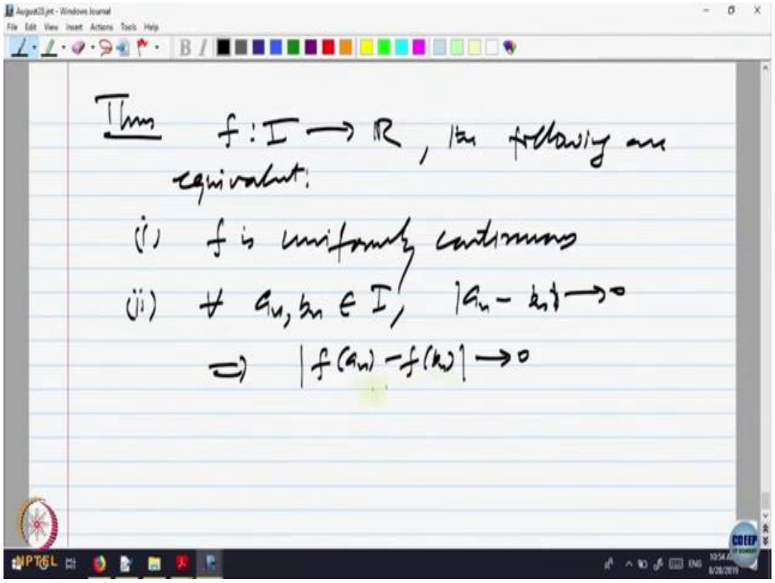

Let us formulate this in terms of one can ask can I characterize this in terms of sequences like there is a more like epsilon delta kind of a definition, sequences are much easier sometimes to handle so here is a theorem, so f I to R the following R equivalent f is uniformly continuous and second, see in continuity you got A n converges to C then f of A n converges to f of C but if you take two different sequences converging to C then the limit also is converging.

But if I want to remove that point C then what is the property? An converging to C, Bn converging to C then what is happening to An and Bn? They are going to 0 and what should happen, f of A n minus f of B n should also go to 0, so that makes it independent of the point. So, what you saying is continuous is equivalent to saying for every A n, B n belonging to I mod of A n minus B n going to 0 should imply f of A n minus f of B n goes to 0.

For example, if I take B n all equal to as a constant sequence C then A n converging to C implies f of is it okay, so obviously says continue, f of continuity implies continuity it is stronger than continuity, is that okay that uniform continuity is stronger than continuity. For example, if this theorem is true you can look at this way or epsilon delta either way. I can take all the B n to be constant sequence C then A n converging to C imply f of A n minus f of C goes to 0 that is continuity at the point C.