Basic Real Analysis
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Lecture 23
Continuity and Uniform Continuity- Part II

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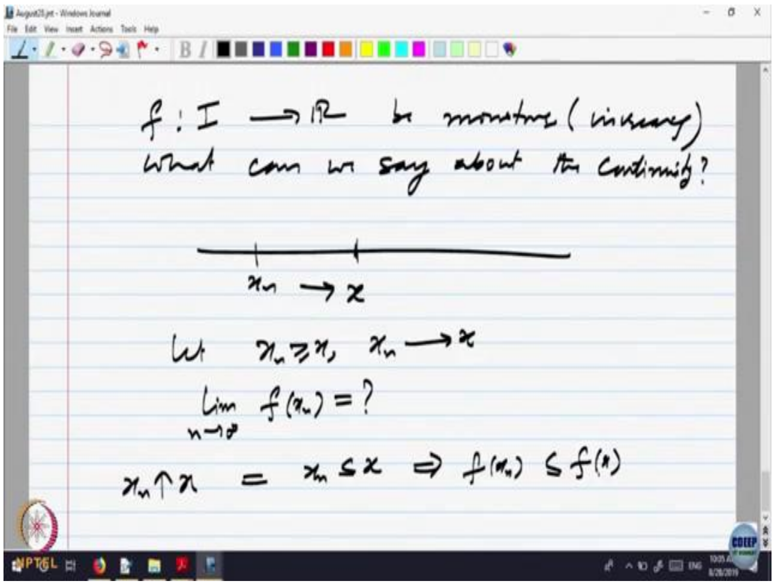

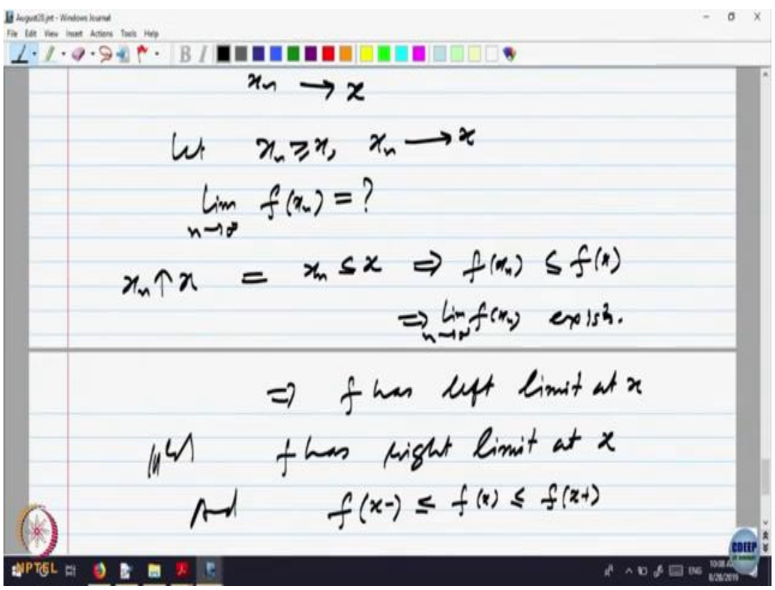

So, another property of monotone functions, f let us say on a interval I to R be monotone, let it be monotone. We are trying to analyze now more-more we have already said it is strictly monotone then it is to be 1-1 anyway. What more properties we can say about this, so here is let us analyze what can be one say about the continuity. What can you say about the continuity of this function?

Alright, so function is monotone and monotonicity is a property of a function over a part of the domain, it is not at a point, so let us say it is monotonically increasing. Otherwise what I can say, I can break up the domain into parts where the functions is increasing and where the function is decreasing, analyze continuity in all those parts separately, no problem. So let us assume it is monotonically increasing.

So, let us take any point x in the domain, I want to analyze whether the function is continuous at this point or not. The function need not be monotone function need not be continuous obviously, you can have a step function, going up and up everywhere, discontinuous. So, at this point I want to analyze continuity that means what? I should look at limits of sequences converging to x and look at the limits of the (())(02:29) sequences.

So, let us take a sequence xn which is converging to x from the left side, so let xn is bigger than or equal to x, x n converging to x. We want to look at limit of f of xn, if it is equal to f of x then f will be continuous at the point. So, we have to analyze the image f of x n limit of that. Does it exists or not? If it exists what is the value? But xn is increasing to x, so implies so xn increasing to x implies xn is less than or equal to x that we already taken anyway.

So, implies by monotonically increasing f of x n is less than or equal to f of x that is given to us. Because function is monotonically increasing and what are we interested in? Limit of the sequence f of xn all the f of xn are bounded by f of x and it is the increasing sequence, x n is increasing so f of xn also is increasing, it is monotonically increasing function. So, everything remains on the left hand side, so what should happen?

Is a monotonically increasing sequence which is bonded above it must converge by the completeness property. So, implies f of xn limit n going to infinity exists, at least the limit exists, so what we are saying is if f is monotonically increasing at every point the left limit exists, left limit exists. What about the so implies f has left limit, what about the right limit? A function if a sequence is decreasing then f of xn also will be decreasing because the function is monotonically increasing.

Is it okay? Same analysis applies when if you go to the right side, still the function is monotonically increasing, sequence is decreasing, the mi sequence also must decrease, the again limit will exists, so similarly so let me write similarly f has right limit at x and so how do you write the left limit? F of x minus that is normally written as the left limit at x is less than or equal to f of x because every point is and that is less than or equal to f of x plus.

So, f of x plus is the right limit at the point x plus you are approaching x from the right side, x minus you are approaching f from the left side, so they exists and this is what is happening, continuity when both are when the left limit is equal to the right limit, in general it may not happen. So, there will be possibility of a jump is there in the graph of a function.

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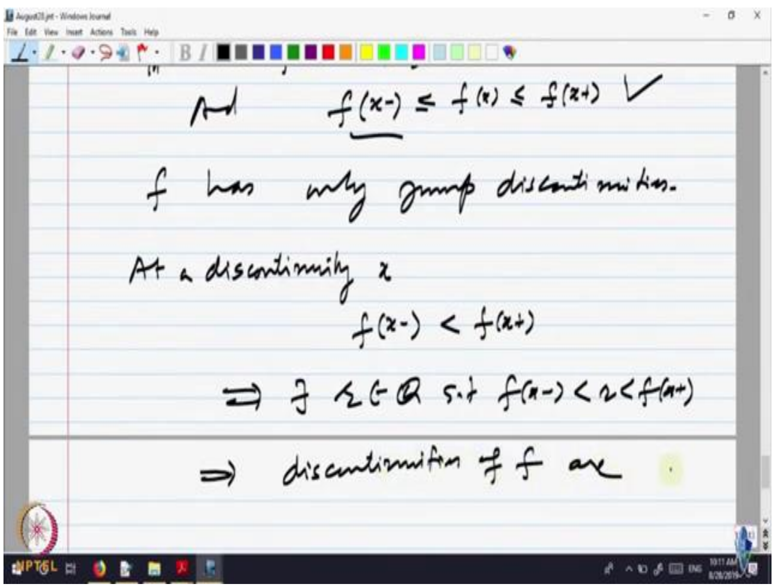

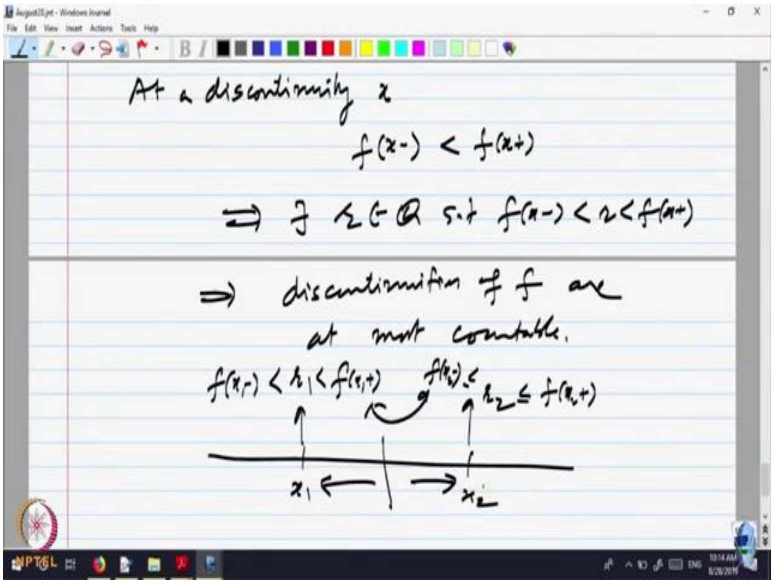

So, f has only jump discontinuities, is it okay? Because at every point left limit and the right limit exists only they may not be equal, so such limits, such a points of discontinuity are called points of jump discontinuity because the graph of the function will have a jump at that point. Coming from the left at some point it jumps to the right limit at that point. We do not know what is the value of the function at that whether it is in between somewhere but that is not of concern because the function has a discontinuity.

We can say something more, see the left limit so at discontinuity at a discontinuity at the point x f of x minus what will happen? This is in general true, if it is a point of discontinuity what should happen? They are not equal that means this is strictly less than f of left limit is strictly less than the right limit.

When it is strictly less than so these are 2 real numbers, one is less than the other, so there must be a rational number in between implies there exist is some R belonging to Q such that f of x minus is less than r is less than f of x plus, why I am doing that? The reason is I am assigning a kind of tag to the discontinuity, a number to a discontinuity, a roll number kind of thing. So, if x is a point of discontinuity then there is a rational number attached to it.

So, how many discontinuity is the function can have? For every discontinuity there is a rational number. So, all the discontinuities form a subset of the number of you can think of that a subset of rational they are at the most countable in many, so implies discontinuities of are at most countable. In fact I can say something more what I want to say is different discontinuities do not get the same roll number, I want to say that also.

Because it is possible for a rational number, there are uncountable number of discontinuities having the same kind of number associated with it, I want to say that is not possible because if there are two discontinuities different ones, the one will be less than the other, then what will happen? There will be distance between them, so can you say the rational associated with each different discontinuities are different.

I want to say for one discontinuity, so here is a point of discontinuity call it x1, here is a point of another discontinuity x2, to this I have associated a rational R1 to this I have associated a rational R2. What is this R1? It is between f of x1 plus and less than f of x1 minus and similarly this one is between f of x2 minus less than or equal to f of x 2 plus. Now, what can you say about these two numbers, the left limit at x1 or the right limit at x1 and the left limit at x2.

Can they be equal? They have to be distinct because for example I can start approaching this start approaching this form some point, say midpoint between the two then everything the right limit at x1 right will be different from the right left limit at x2, are you getting that point or not?

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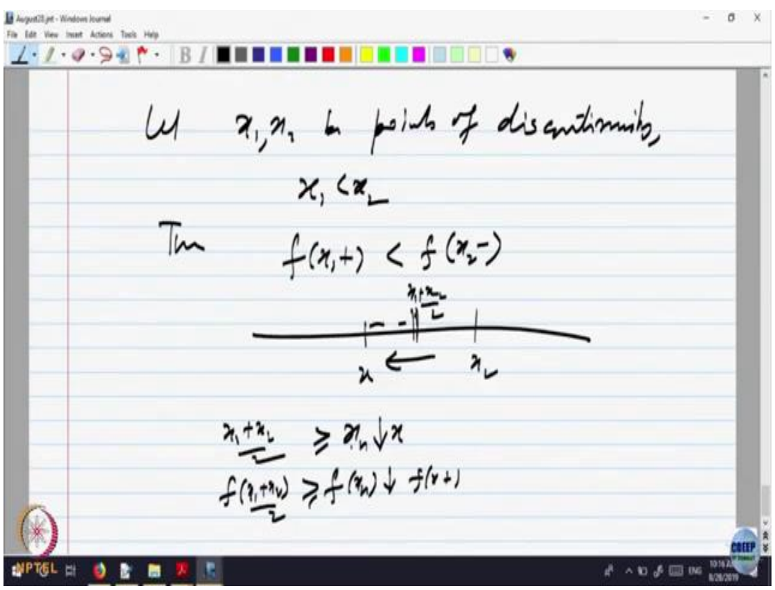

So, let me repeat once again that let x1, x2 be points of discontinuity, x1 less than x2 then my claim is f of x1 plus is strictly less than f of x2 minus. How do you get the left limit at x1? This is x1, this is x2 how do you get the left limit right limit at x1? You are going to approach from this side, and if there is a distance between them x1 and x2 so then all the points in that sequence will be on this side, is that okay? So, what I am saying is when you take a sequence so f of x1 will also be because it is monotonically increasing, so let us take x1 plus x2 by 2.

So, all the image of those points will be less than f of x1 plus x2, so you will be looking in a sequence xn which is going to decreasing to x then what will happen to f of xn? That will decrease to the left limit f of right limit f of x plus but these all xns are less than or equal to x1 plus x2 by 2 function is increasing. So, f of x1 plus x2 by 2 is bigger than or equal to this bigger than or equal to this.

When you are approaching… there is a distance, the right limit at the point x1 has to be strictly less than the left limit at x2, that means what? That clearly says because we are choosing a rational between the left limit and the right limit, so this rationals will be different for, so that association of assigning a tag to every discontinuity we are assigning it to be a rational number for different discontinuities different rational numbers.

So, how many are possible at the most? Countably infinite, so that is what we have said here that only jump discontinuities and they are at the most countable because of this 1-1 association, it is like assigning a roll number to each discontinuity. So, what we have said is every monotone function is having discontinuities only of the first kind they are called the jump discontinuities and they are at the most countable many.

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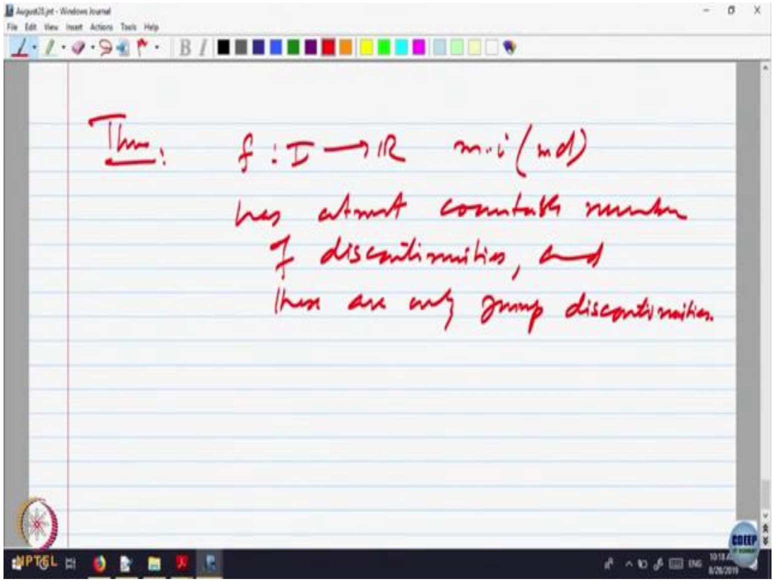

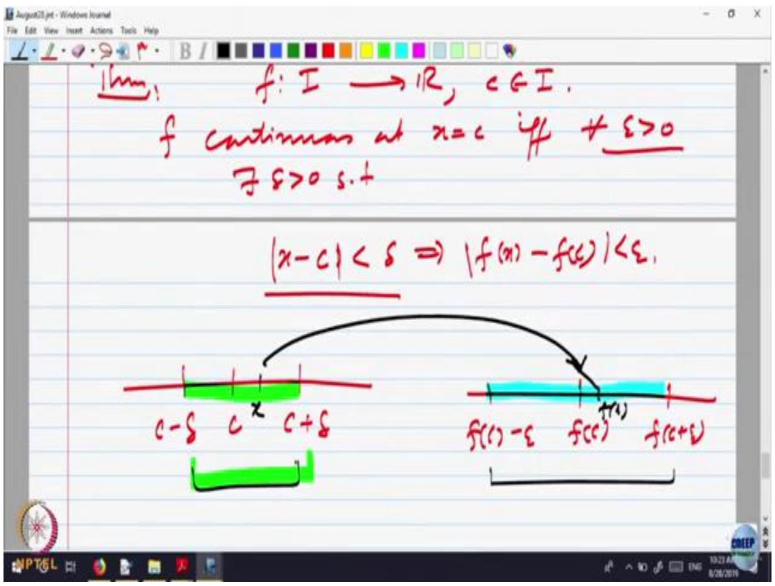

So, let us write as a theorem if you like, so the theorem that we have proved f monotonically increasing or monotonically decreasing as at most countable number of this discontinuities and these are only jump discontinuities. And these are only jump discontinuities. So, is this special this monotone functions have many nice properties, monotone is strictly monotone is 1-1, only discontinuities jump, countably many.

So, we looked at what is called the continuity, here is something I should probably say how did we define the continuity of a function at a point? We looked at whenever a sequence xn converges to x the limit f of xn must exists and that must be equal to the value of the function at that point. There is another way of describing this continuity in terms of neighborhood which is sometimes useful.

So, let me state that and then probably prove it, so the theorem says f is on I say interval to R and we have got a point say let us say C belonging to I, f continuous at x is equal to C if and only if because we are saying it is equivalent and so for every epsilon greater than 0 there is a delta bigger than 0 such that x minus C absolute value less than should imply f of x minus f of C is less than epsilon.

So, let us just look at it what we are saying, when we said f is continuous at C we said if the point C is approached by a sequence we are coming close to C by A sequence corresponding images as f of xn should come closer to f of C.

This closeness is interpreted in other way in this it says see f of x is the value to be predicted of the function at a point nearby, f of C is the actual value so this is a error your making, f of C minus f of x absolute value is the error you are making in looking at the value of the function at the points C.

And we want this error to be small, how small? It says for every I will give you how small I want. So, given this I should be able to find a delta so that whenever any point is close to C by the distance delta then the error that you are making is less than epsilon. So, if you look at this condition this is just saying this is a point C and this is C minus delta 2 C plus delta and this is f of C and this is f of C minus epsilon and f of C plus epsilon.

So, what we are saying is this is what is given to us, so this is what is given neighborhood of the point f of C is given to us we want the value of the function should come in between this neighborhood, for what values? For all values which are closed to C, we should be able to find this neighborhood.

At least one neighborhood so that all this values so everything from here from the neighborhood this goes inside here if this is x here then your f of x is somewhere here. So, I sort of will like you re jumping and approaching the point C that is a sequence, or you are pulling a point near on the line, you are pulling a point near C, both are same basically you are approaching the point.

So, how does one prove this thing so let us give a proof because proof is not difficult but it is interesting because it will so you can interpret it this way for every neighborhood epsilon neighborhood of the point f of C given there is a neighborhood a delta neighborhood of the point C says that all points in the delta neighborhood go inside the epsilon neighborhood.

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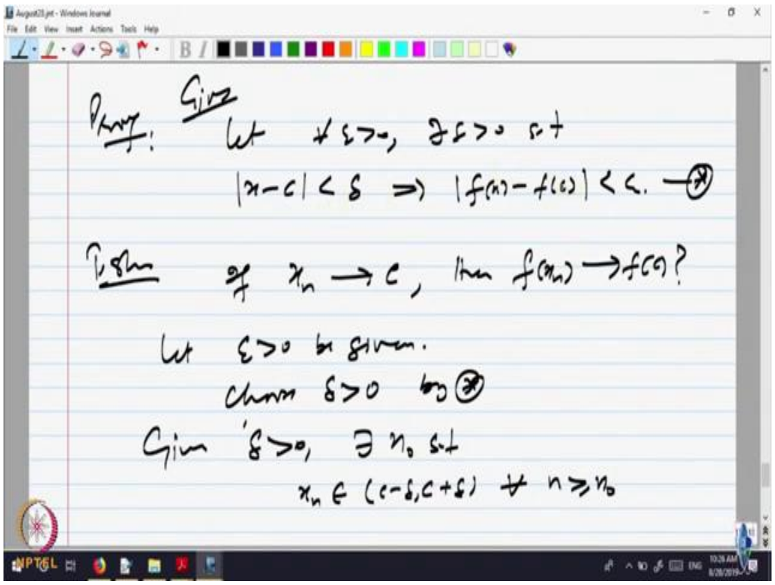

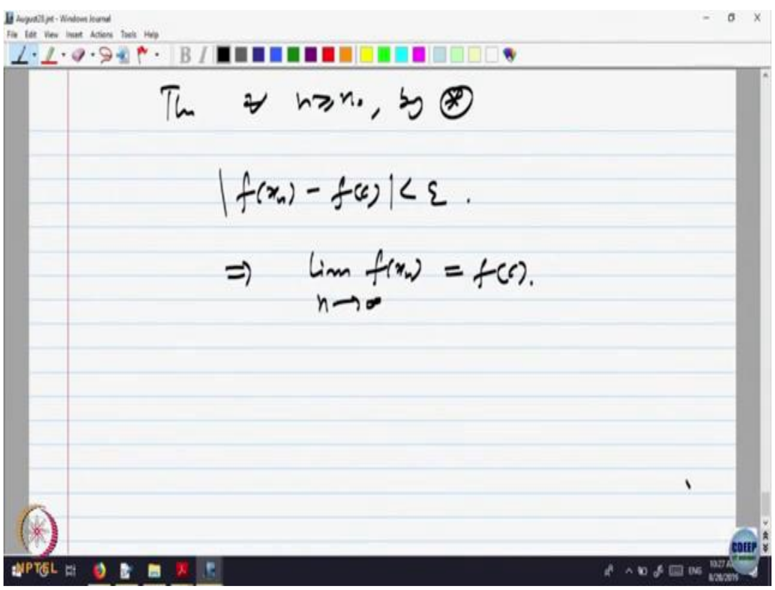

So, let us write a proof of this theorem, so let us say, let for every epsilon be greater than 0 there is a delta bigger than 0 such that x minus C less than f of x minus f of c is less than epsilon. So, this is given. To show continuity in my original definition so to show if xn converges to C then f of xn converges to f of C. So, you can say this is epsilon delta definition or the neighborhood definition implies the sequence definition.

We are saying both are equivalent so one implies the other will show, so I want to show f of x n converges to f of C, so what is to be shown? F of xn must come closer to f of C after some stage, how close you want? So you say, so let epsilon greater than 0 be given, choose delta bigger than 0, so that this is what is given to me.

There is a delta chose delta by star say that this happen, xn is going to C so we have got a neighborhood of C delta neighborhood, what should happen? So, given delta what should happen to the sequence xn? So, given delta bigger than 0 there is a stage n naught such that x n belongs to that neighborhood such that if you want in terms of such xn belongs to C minus delta to C plus delta for every n bigger than N naught.

Is it okay? Because xn is converging to C, so you should come close to C how close we want it, closed by the distance delta because then what will happen? What will happen by star? Then for every n bigger than n naught by star what will happen? F of xn minus f of C will be less than epsilon that was the epsilon delta definition.

Whenever x is closed to C by delta that should happen, now xn has come close to C that is what we are saying, so this is a and that should imply that and that precisely meaning to say so given any epsilon we have got a stage n naught, so implies limit f of xn, n going to infinity is equal to f of C. So, epsilon delta definition implies our sequence definition of continuity, let us prove the other way around.

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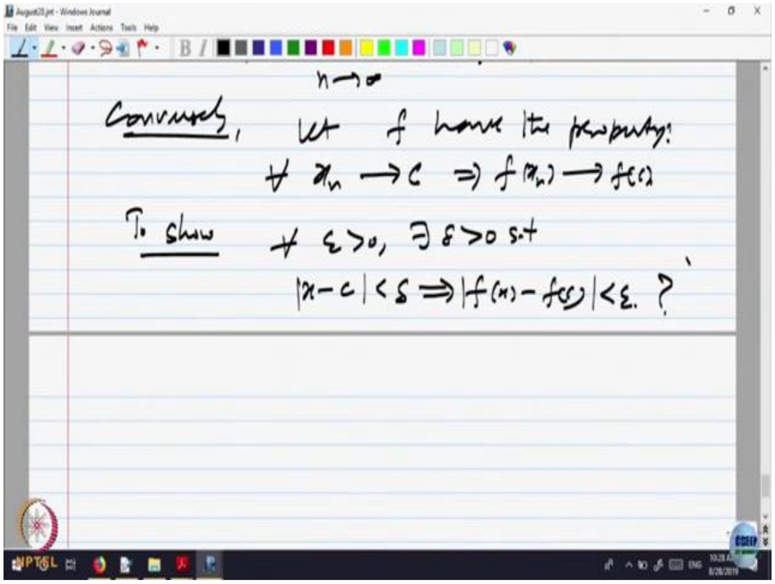

Conversely let f have the property, what is a property, for every sequence xn converging to C should imply f of xn converges to f of C, so that is given to me. To show for every epsilon bigger than 0 there is a delta such that x minus C less than delta implies f of x minus f of C is less than epsilon, so that is what is to be shown. The sequence definition implies epsilon delta definition.

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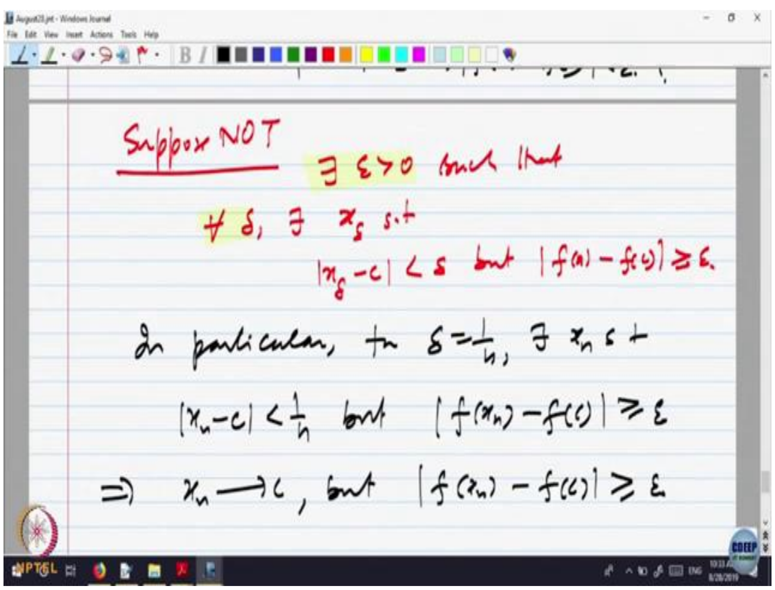

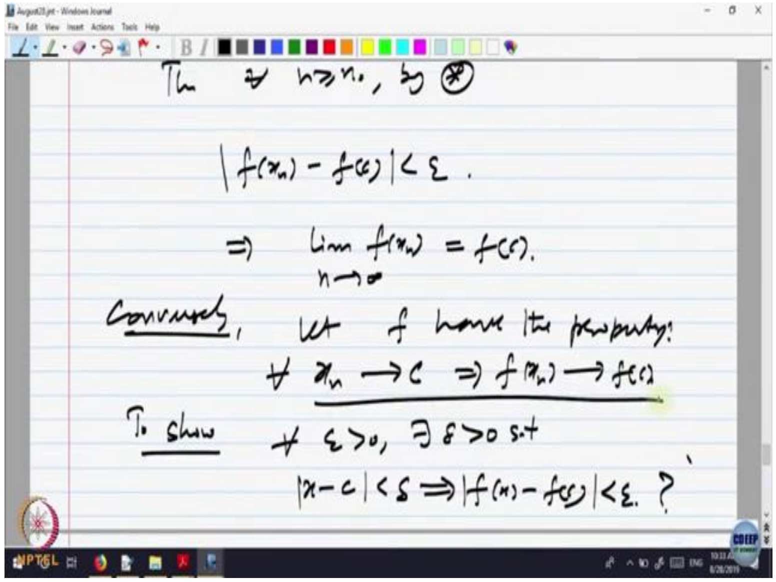

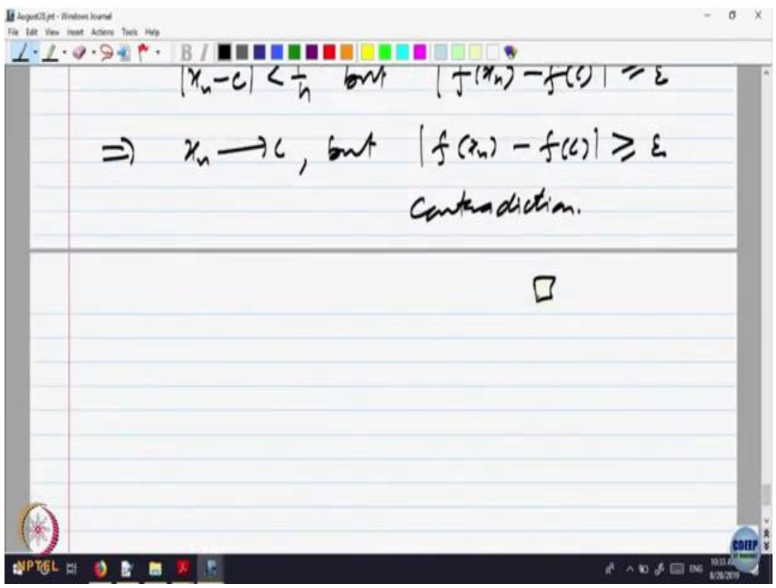

So, here is suppose NOT, so here is some time I had made a remark if you want to understand what is true we should also understand what is false, so what is the meaning of saying that this statement is not true? The statement is true for every epsilon something is happening, so not true means at least there is one counter example, so there is at least one epsilon.

So, it means there exists epsilon bigger than 0 such that, such that what should happen? The remaining statement there is a delta and all that should not be true, and what should so for every point x less than this delta something is happening that means there are points in the neighborhood delta where this thing will not be true, where the required inequality will not be true, so such that so there exists so for every delta there exists a point x delta such that x delta minus C is less than delta.

But f of x minus f of C is bigger than or equal to epsilon, is that okay? For every epsilon something was happening so that goes to saying that so that h becomes there exists an epsilon or something goes wrong what goes wrong? This f x minus f of C less than epsilon should go wrong, for what? For at least one point in that neighborhood and for what delta, for every delta there exists should become to, so this there exists becomes for every.

For every delta it should be violated, so this is what it means in the negation of that statement, so this is what is if we assume epsilon delta definition does not hold then this is what is given to me. So, I am able to select points x delta closed to x for every delta and what is I am trying to hit, I am trying to construct a sequence we should converge but the image should not converge, that is the contradiction I will have.

So, how do I get a sequence by specializing delta? Something, is happening for every delta, so the obvious thing is delta equal to I want a sequence to converge to C, so delta should become smaller and smaller also, one obvious choice is take delta equal to 1 over n, so in particular so one writes in particular everything is motivated what we want for delta equal to 1 over n there exists a point xn such that xn minus C is less than 1 over n.

I am specializing that equation now, x delta I am calling it as xn but f of xn minus f of C is bigger than epsilon. So, what does the left hand side say, x n minus C is less than 1 over n for every n, the sequence xn has the property that the distance between xn and C is less than 1 over n, so that means what?

That means that the sequence xn is converging to C by Sandwich theorem if you like it is less than 1 over n, so implies x n converging to C but mod of xn minus f of C is bigger than or equal to epsilon. So, we said that there exists an epsilon so what we have done there exists an epsilon, there exists a sequence, so in the sequence converges to C but the distance between f of xn and f of C always remains bigger than that number epsilon.

So, f of xn is not going to converge to f of C. So, contradiction to the given thing, so what is given should happen for every sequence, every sequence that should happen. But we have produce the sequence which is converging to C (())(31:39) so that proves that the two are equivalent.