Basic Real Analysis
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Lecture – 21
Topology of Real Numbers: Connected Sets;
Limits and Continuity – Part lll

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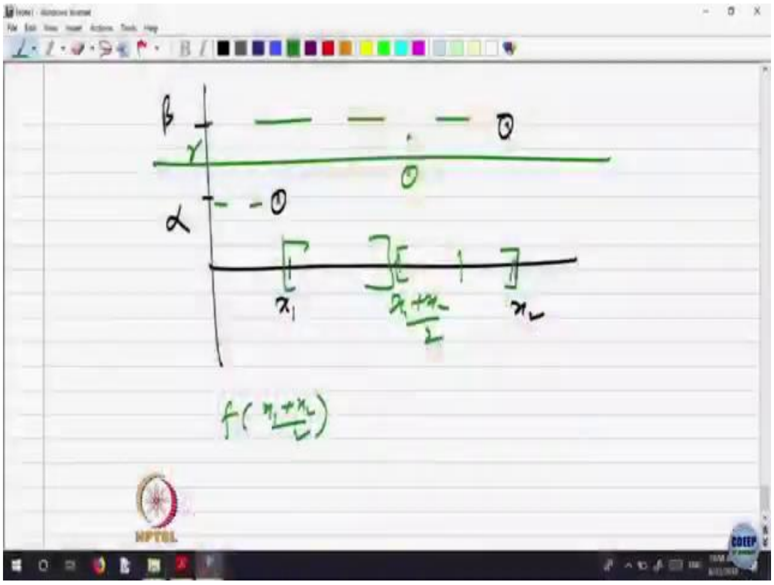

So, this is another way of proving the same thing.

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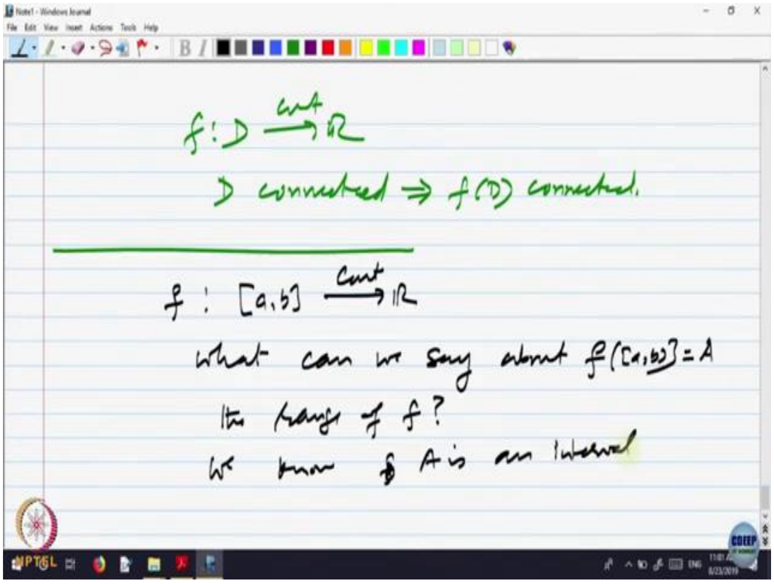

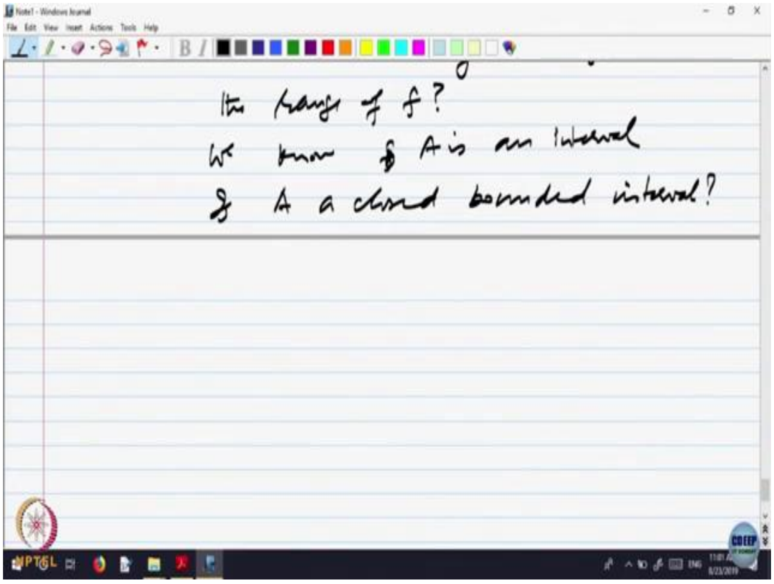

So, we have got as far as connectedness is concerned, we have analyze the image, you can say image of a connected subset of real line is connected. Because connected subsets are only intervals and the image is connected so that is interval. So, another way of saying is continuity preserves connectedness, image of connected sets are, so let us write f D to R continuous D connected, another way of writing implies f of D connected.

Next we want to analyze compactness, whether like connectedness whether compactness is preserved or not. Let us look at a very special case, we know that compact subsets of real line are those which are close and bounded, but let us look at very special example when compact set is a close bounded interval, so f is a function on a b to R continuous.

We want to analyze what can we say about f of a b, the range of f. You can also interpret it this way, we took intervals, image of a interval was a interval, now we specialize the interval to be a close bounded interval. And, so the image of this is going to be interval because a b is a interval, the question is, is it a close bonded interval or not.

Earlier theorem at f is continuous, I is interval f of I is a interval. Now we are specializing, if it a close bounded interval can I say f of I is a so we know this call it as sum set a, we know a is an interval, so the question is a a close bounded interval, is it a close bounded interval. It is a interval we know.

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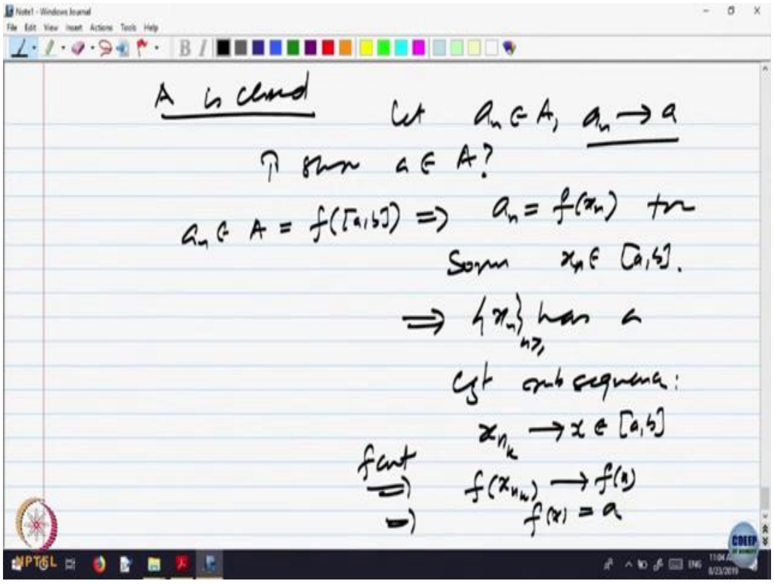

Now, let us a is closed, let us analyze one of them, A is closed, so what I have to show, if A is closed what is to be shown, whenever I take a sequence in a converging somewhere that image must be in a. So, let a n belong to a, a n converge to a, to show a belongs to a. What does a n belong to a imply, which is equal f of a b implies a n is equal to f of x n for some x n belonging to a b. So, implies by compactness x n as the sequence x n as a convergent sub sequence. So, let us write that also x n k converging to a point there is no x used somewhere so x belonging to a b.

So, we are transferring the problem from the domain, range to the domain and doing property there (())(4:52), so we go back implies f continuous f of x and k convergent of x f continuous. But where does f of x and k converge? f of x n is a n. a n is converging to a. So, where does the sub sequence converge? To a same limit. So, implies f of x is equal to a limit is same. Because x n k is subsequence of x n that means a n k corresponding a n k will be sub sequence of a n, a n convergence to a. So, that has to converge to the same limit.

So, that means whenever I take any point in a, if a n converges to a then a most be having preimage that means a belongs to the range. So, implies a belongs to f of a b that is equal to A, so A is closed. A is bounded, I already shown it is closed we have to show A is bounded. That means the range, if I take the function a b in the closed bounded interval a b the range has to be a interval we already know it has to be a closed interval we know, we only want to show it bounded. If not, if the range of a function is not bounded means what? That means in the range there are point where you can go away and away from something.

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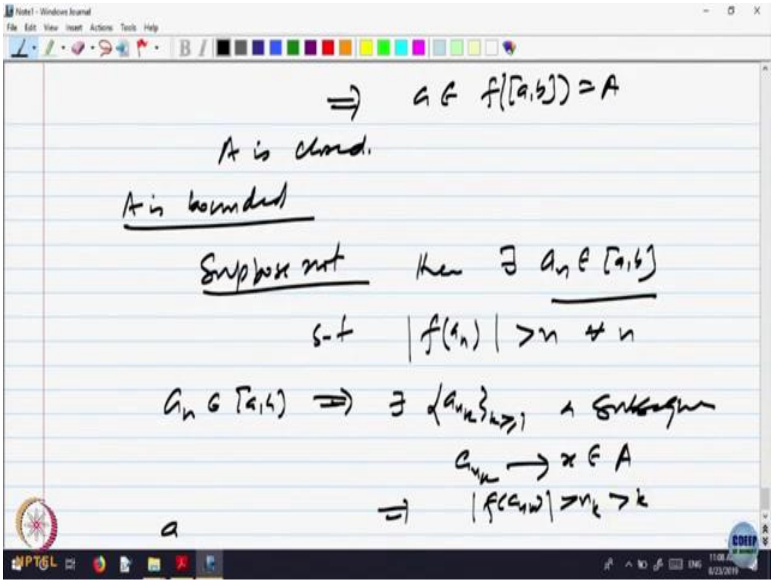

So, let us right is a bond suppose not than there exist, if not there exist points a n belonging to a b say that f of a n, let us right mod f of a n goes to infinity, is that ok? If the range is not bounded there must be a sequence in the domain. Which is either going to plus infinity or minus infinity, so let us say mod of that goes to plus infinity, is that ok? Now, a n belongs to a b, so again compactness, a n must have convergent subsequence, converging in a b. So, a n belonging to a b implies the (())(8:02) a n k a subsequence a n k converging to a belonging to A, oh sorry, I should not write a because a is end point.

So, something else x belonging to A. But I have got continuity so f of x n must, f of a n k must converge, but can f of a n converge, the original sequence is, or if you like divergent let me write to be more precise instead of this let us write is bigger than n for every n, let us write that way. Let me make this very clear, if it is not bounded I can find points where value is bigger than otherwise it will be less than or equal to bounded. So, at a n the value is bigger than mod n, so what is value at f of a n k, it will bigger than n k which is bigger than k anyway. Is it ok?

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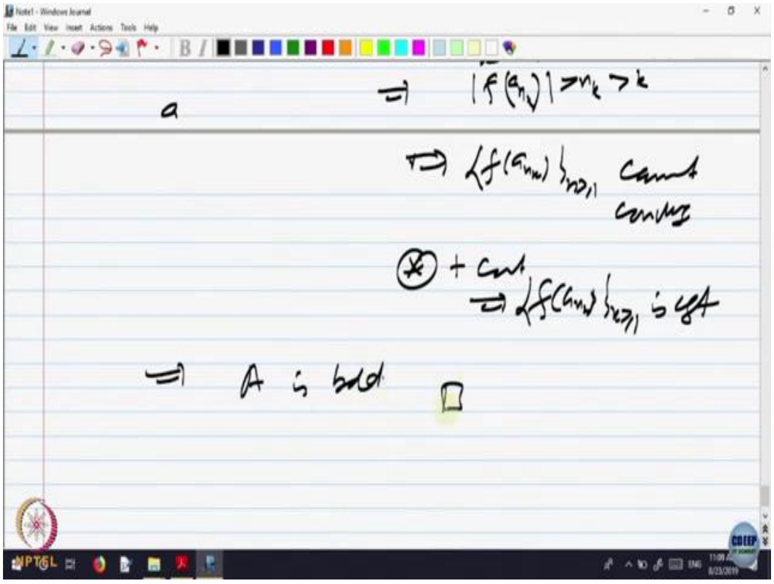

So, for every k absolute value of a n k, so it cannot converge but others, other has said it should converge to implies f of a n k cannot converge. But this star, star plus continuity implies f a n k is convergent so that is a contradiction, because a n k converges to x, f is continues so f of that must converge, but here it is bigger than n k, bigger than k so it can not converge, so that is a contradiction. So we assume if not that means if not bonded then there is a problem, suppose not so that must be must not true implies A is bounded.

So, what we have said that if the domain of the function is a closed bounded interval then the range also is a closed bounded interval. Can I say something more about close bounded interval what that should be anybody can guess?