Basically Real Analysis
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Lecture No 02
Real Numbers and Sequences-Part II

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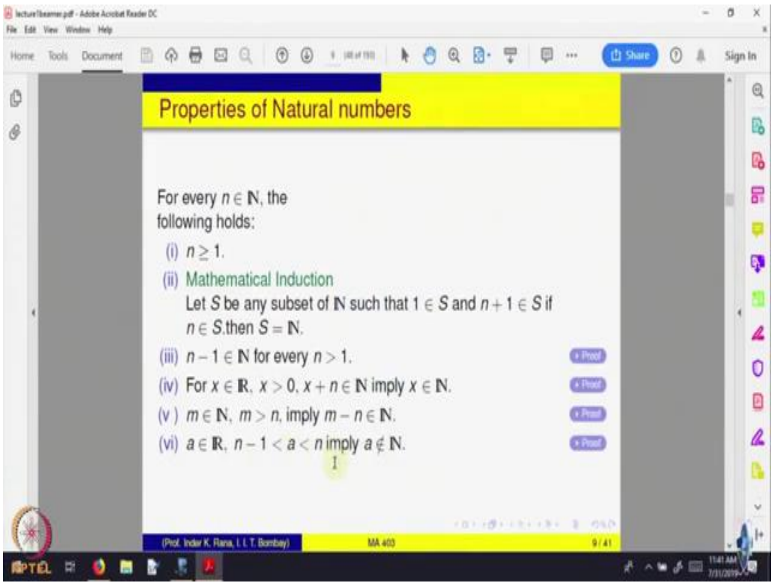

For example you can prove this properties of natural numbers just by assuming what I have said, example there is no natural number but in n minus 1 and n you can prove it actually using those properties of field okay. So, let us assume, so now I want to state that crucial property, for reals, which this, which distinguishes it from the rational.

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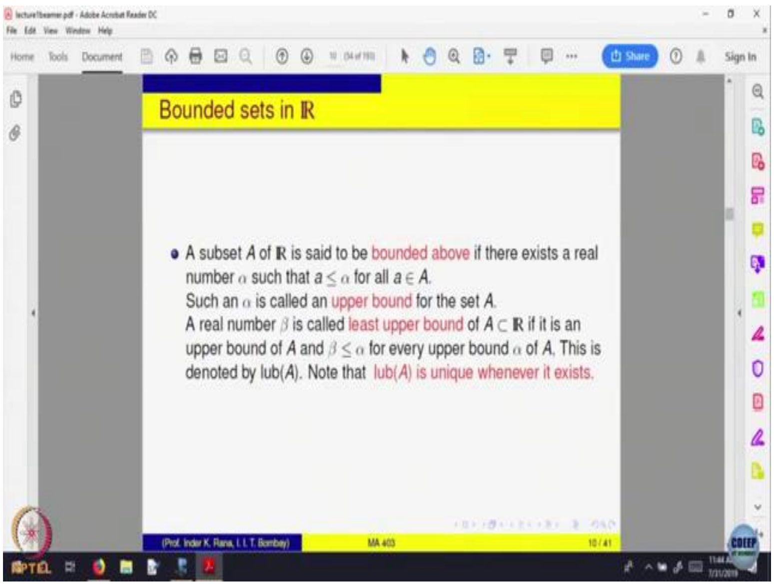

So, and for that we're going to call said to be a bounded A is a subset of real numbers. We say A is bounded above if there is a number, I will call it as alpha, such that alpha is bigger than or equal to every element of A there is order. Given any element A, I can compare it with alpha. So, we want alpha to be bigger than or equal to every element of the set A, when we say alpha is an upper bound for the set A we say is bounded above, if all elements are less than or equal to something, and that something is called an upper bound.

Now, you got a upper bounds there could be more than one. There could be more than one upper bounds. Example, for the number 1, two is an upper bound, three is an upper bound, four is the upper bound. Because we know is all successors are bigger than the number itself. The property, the smallest of these upper bounds if it exists is called least upper bound for that set.

So, let us define what is called the least upper bound for a set it is a number which is an upper bound first of all, we are looking at the smallest of the upper bounds. Among the upper bounds look at the smallest one if it exists, then we say that number is least upper bound of the set A. Now, question is can there be more than one least upper bound? No, because it is an upper bound and it has to be.

So, bigger than less than so, it has to be equal. So, upper bound is always if it exists, it is unique. So, that is why we say the least upper bound normally. So, the lead, the upper bound if it is least it is unique. So, it is unique whenever it exists. Why to say upper only. We can also look at something smaller.

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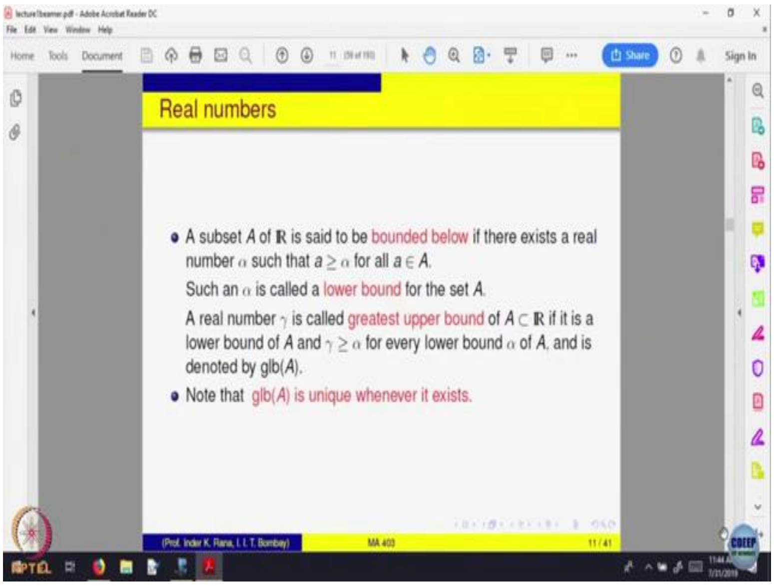

So, you can look at what are called sets which are bounded below if there is a number alpha, so that alpha is less than or equal to A for every element of the set A. All the elements of A are bigger than that element alpha then alpha is called a lower bound okay. And now what we should be looking at, there are many lower bounds possible the biggest of the lower bounds so, that is called the greatest lower bound for a set if it exists, okay.

So, it is an lower bound and it is the greatest of them all. So, that is called the greatest lower bound okay. Now, question is given a set A, subset of the real line can you say always that the set A may not be bounded? For example, look at natural numbers, is it bounded? It is not because given any if there is a bound something we say that look at a integer and you can go on increasing it. So, it becomes unbounded kind of a thing.

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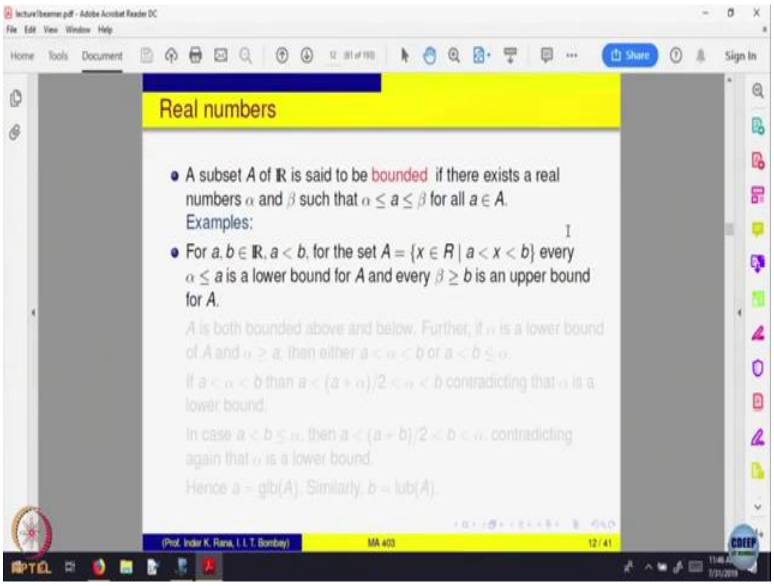

So, here is one example let us look at all x between A and b. A and b are two real numbers fixed and let us look at the set of all real numbers such that A is less than x less than P. Okay. By the construction of that set itself the number A is a lower bound because every element of the set is bigger than A, b is a upper bound. Can you say that there is nothing bigger than A can be a lower bound?

So, try to prove it take it as an exercise, try to prove that if I give you A as the set a less than x less than b a and b are fix then A the set has got greatest lower bound namely A, it has least upper bound namely b. So, try to, no, no we will discuss it in the tutorial session, okay, because I want everybody to think you see the idea is not to get an answer from one person only.

The idea should be that everybody should have time to think about it, analyze and see whether they are able to write down a proof of this fact or not that this has got a lower bound. For example a is a lower bound and actually that is the largest of the lower bounds okay.

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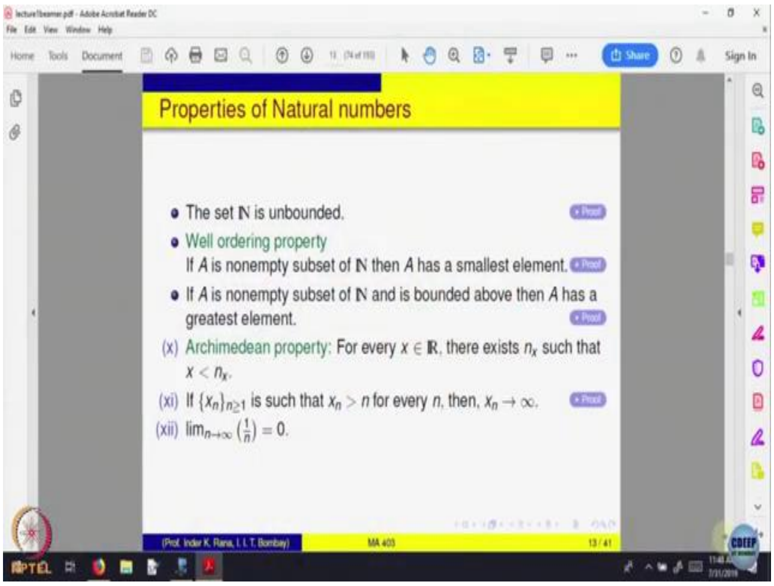

So, let me skip with argument that I wanted to give the saying that this is So, N is unbounded you can prove that these kind of things and you can prove what is called the Archimedean property of real numbers. And that means what? That essentially says that for every x in R there exist a number nx. A integer a natural number nx which depends on x such that nx is bigger than x you will always cross over unbounded is same as saying.

Saying it is bound if x is not a barrier you can go beyond x by some natural number. That is same as saying a is unbounded both are equivalent ways of saying the same thing okay. I have not defined what is the sequence but essentially it says if we look at one over natural numbers intuitively it becomes smaller and smaller and comes as close to 0 as you want it will make this precise slowly okay what does that mean? So, this, this, this thing will make it precise okay. Soon. So, properties of natural numbers okay.

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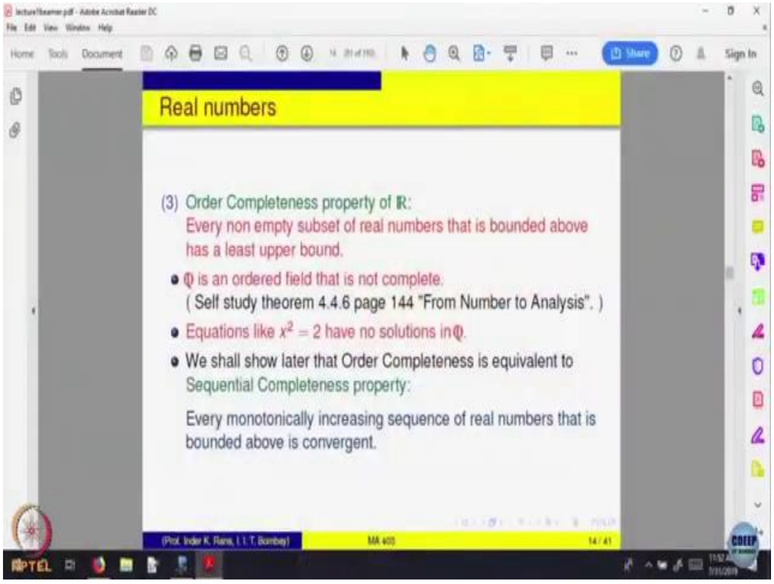

So, here is what is called the order completeness property of real number every non empty subset. Of real numbers that is bounded above has a least upper bound if there is a bound for it, for a set it is bounded then there must be a smallest of the bounds possible. Intuitively it looks okay. It should be so but Greeks discovered that that is not possible always you can look at all rationals less than such that R square is less than 2.

Saying that there is no rationals whose square is equal to 2 is precisely, almost saying that the set of all rational numbers are such that R square is strictly less than 2 is bounded above but does not have a least upper bound. One can prove that if you want to know a proof of that, I will not be doing this proof. You can read this theorem on the book called from numbers to analysis. Okay, we have a copy of that book in the library.

So, look at this just for your knowledge sake, if you want to look at, look at this theorem. It says that you can have subsets of rational numbers which are bounded above but they do not have a greatest lower bound as a rational number. And the completeness property says that if you treat any set as a rationals or not and if it is bounded, then it will have a least upper bound and that will be a real number.

The set maybe of nationals, but the least upper bound will be always there. Sometime it may be a rationals, sometime it may be irrational we do not know but every non empty subset of real numbers which is bounded above has got least upper bound and I would say parallel way of saying that is if a set of real number is bounded below because a set of is bounded above we look at negatives of those numbers that set will be bounded below from up to down you can always go by negatives.

So, saying upper bound negative lower bound, greatest least upper bound, greatest lower bound So, equivalent statement of this is that every non empty subset of real numbers if it is bounded below it has got greatest lower bound both are equivalent statements and real number have got this property either of it and hence both of them. So, this is what is called the completeness property of real numbers and it precisely if you look at that set of rationals R square less than 2 and the least upper bound of that is precisely a real number whose square is equal to 2, one can prove that.

So, that will show the existence of roots of numbers, right in reals which may not be possible in rationals. We will be giving another way of stating this completeness property a bit later, which is more sort of intuitively obvious. And that property which is actually, I will explain the terms of that, which is every monotonically increasing sequence of real numbers I will not define what is the sequence we will do that and we will set then that every monotonically increasing sequence of real numbers that is bounded above is convergent.

So, will, will explain all the terms of this statement and then say what it means later on when I do, what is the notion of a sequence. So, completeness property of real numbers can be described in two different ways. One is least upper bound property or the greatest lower bound property.

That every non empty subset of real numbers which is bounded above must have a above so least upper bound and every non empty subset of real number which is bounded below must have a greatest lower bound equivalently will show it that every monotonically increasing sequence of real numbers if it is bounded above it must converge we will say what are these terms and similarly, every monotonically decreasing sequence of real numbers if it is bounded below it must converge. So, those will prove it and show how is that true okay.

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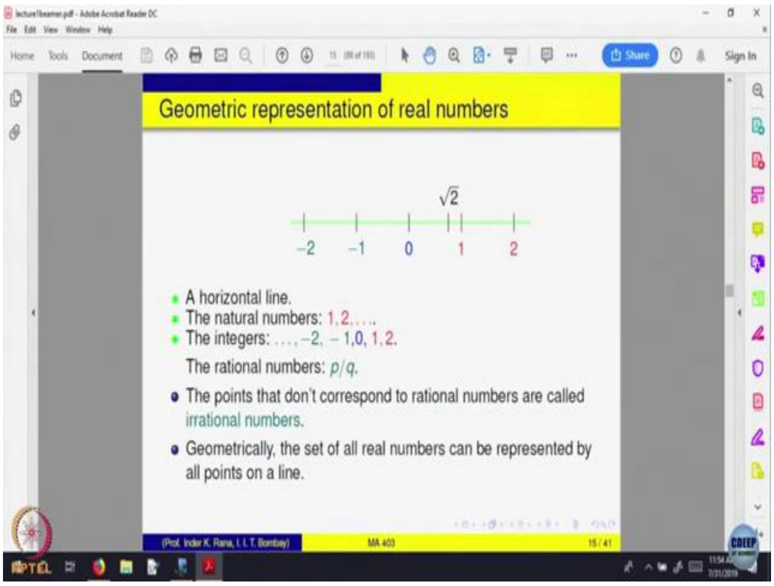

So, here is something which our schools teaches us and we start believing in that, that all real numbers can be put on a line. We said that Greeks discovered that there is a point on the line, which is not occupied by any rational number, namely the diagonal of a right angled triangle with sides one. And later on, so that is a kind of a if you put all rationals on the horizontal line, they go and sit according to their size and positive negative order and everything, but they leave gaps on the line.

Those gaps are precisely the gaps filled by the numbers which we call now as, irrational numbers. For example, square root 2, square root 3. These are not the only numbers which are not filled for example, the number, there is a number called pi. And if you ask anybody what is pi? Answer given it is the area of the unit circle. Or it is the ratio of the diameter to the, ratio of the circumference to the diameter of the circle.

At least I do not know a school teacher who knows a proof of that theorem. I do not know any one of you know that prove of that theorem or not that if I give you any circle, given any circle, look at the circumference and the ratio of the circumference to the diameter, that ratio always is the same quantity. And that quantity we call it as pi try to discover a proof of that theorem.

Now, you have Internet with you, okay. So, go to internet, try to search for a proof of the fact. No school book gives the proof. Every school kid cramps this statement, every teacher of school level assumes this statement blindly without even bothering whether it is true or false. There is none of them know a mathematical proof. And unfortunately, none of our books give a reference to that you can find a proof in this book at this place. Now, nobody gives. And everybody believes that.

So, that is the way we study mathematics. You should not believe anything that is written in a book blindly, or what Google says blindly. Nowadays, everybody thinks what Google says is correct. Or who is Google? We are Google. We are putting up documents there. We may be making a mistake.

So, do not believe always test whether it is true or false or try to find a proof of that fact that pi what we call as the ratio of the circumference to the diameter is a number on the number line is the real number is the ratio. So, there should be a number is it rational or irrational, you have asked for what is a proof, why it is irrational?

It is nothing of the type of square root of something, it is something totally new. So, it is a irrational number and it is a theorem which is not very easy to prove. It requires some more, it is not as simple as a saying there is no number whose square is equal to two it is a bit more difficult to theorem to prove. Anyway. So, one can represents numbers on the line and no gaps are left that means it says that there is a one to one correspondence between the numbers, real numbers and the points on the.

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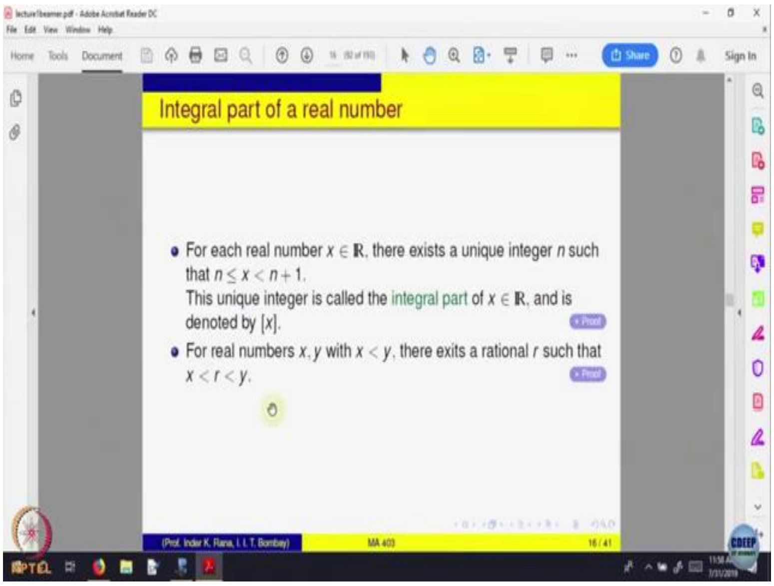

So, that is a geometric representation of the numbers, real numbers. Anyway there are some more properties one can prove namely between n and n plus one okay, there is a unique integer n such that given any number x real number it was live between n and n plus one for some n, less than or equal to n certainly less than and that n, you call as the integral part of that number. That is a beginning of the story of decimal representation. That is where the decimal representation starts.

And here is another important property which we will not prove that between any two real numbers there is a rational number. Rational numbers are everywhere on the line. Given any two, there is a rational number. But still there are many gaps left out. So, it is an interesting question to ask, how many rationales are there? How many points are there on the line? How many irrationals are there? You can think of this question.

Can I sort of compare the size of rationals with the size of irrationals with the size of the, their union is the whole anyway. So, what is the comparison? Again, we will not be doing this, but it's very interesting to know that there are as many rationales as there are points on the natural numbers. What do you mean by as many? Both are infinite things. So, how do you compare infinities?

Same ways I will compute how do you say there are 34 students in the class pick up one, number one, role number one, role number two, role number, one to one correspondence. So, given any two sets, you say they have same number of elements, you do not say number now, say the same cardinality. If there is a one to one on to map between the two, you can associate this with this, this with this and one to one on two, so there is a one to one on two map between natural numbers and rational numbers and one can prove there is no one to one on to map between the rationales and reals.

Rationals are inside the reals, they are dense, they are everywhere, as far as order is concerned. But if you try to count if you find out the size, the size of rationals is much much smaller that means there is a copy of rationals sitting inside reals of course, but what is left out is much much bigger.

So, one says rationals are countably in finite and reals are unaccountably infinite I, we will all be going much into these things we just to make you understand the relation between rationals and reals. Rationals are dense everywhere as far as order is concerned, their size is much much smaller as far as reals are concerned. Okay. So, that is the relation between the two, okay.

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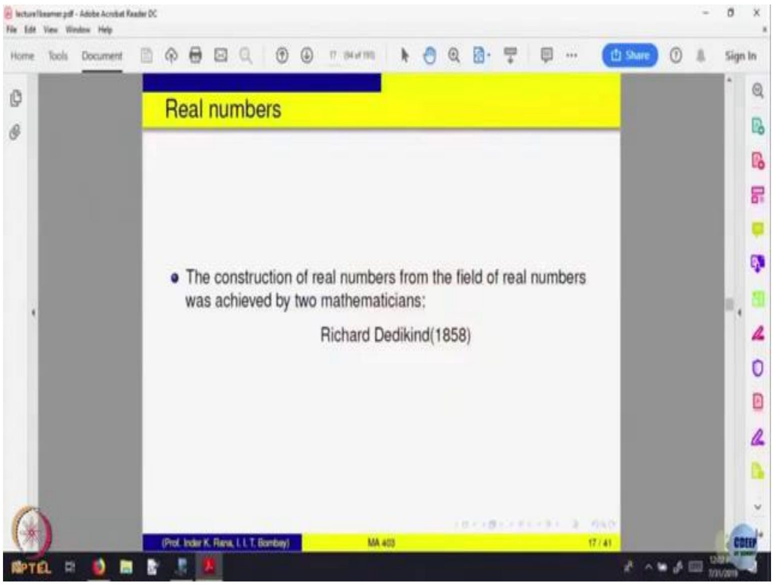

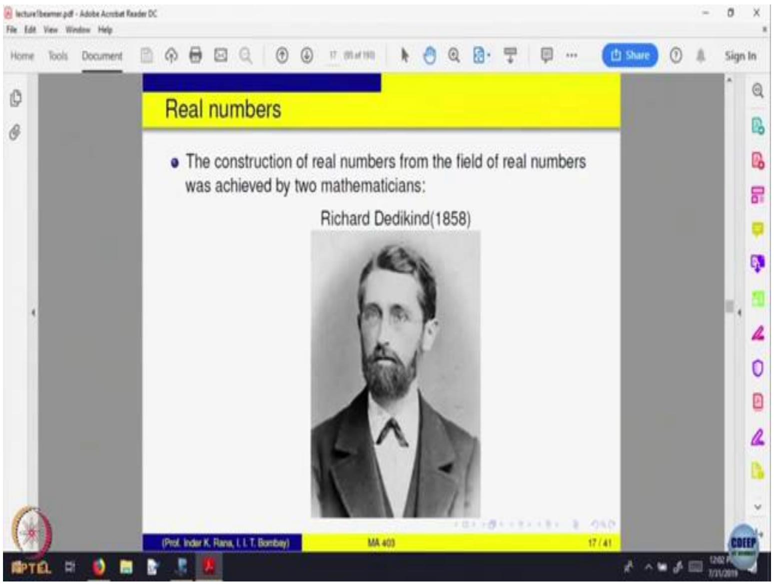

So, this construction as a set of real numbers was obtained by Richard dedicated 1858 which is not very far away. 1858 is only 150 years back. So, you see after the Greeks, it took almost 2000 years to define what is the real number, okay.

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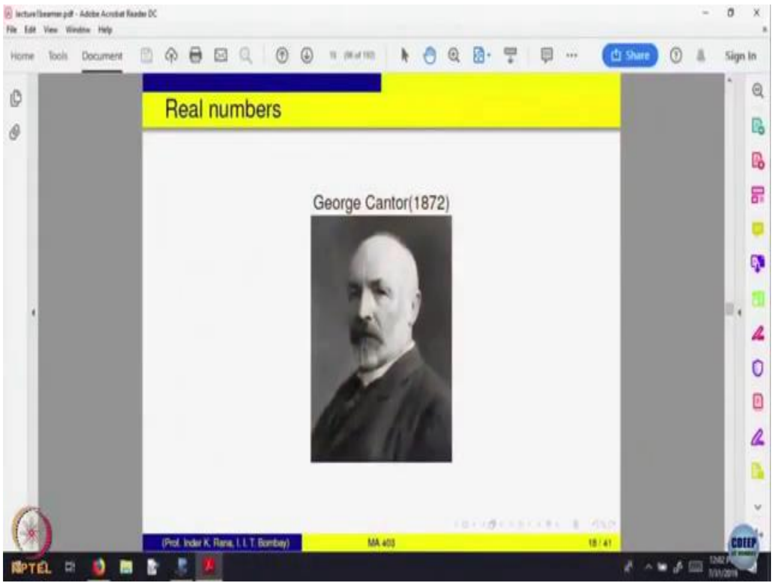

And independently it was done by a mathematician called George Cantor in 1872. Both the approaches of construction are different. Both start with a copy of rational numbers and both construct a new object which is a complete ordered field which has a copy of rational sitting inside it as a dense set, okay. So, not, nothing more than that.

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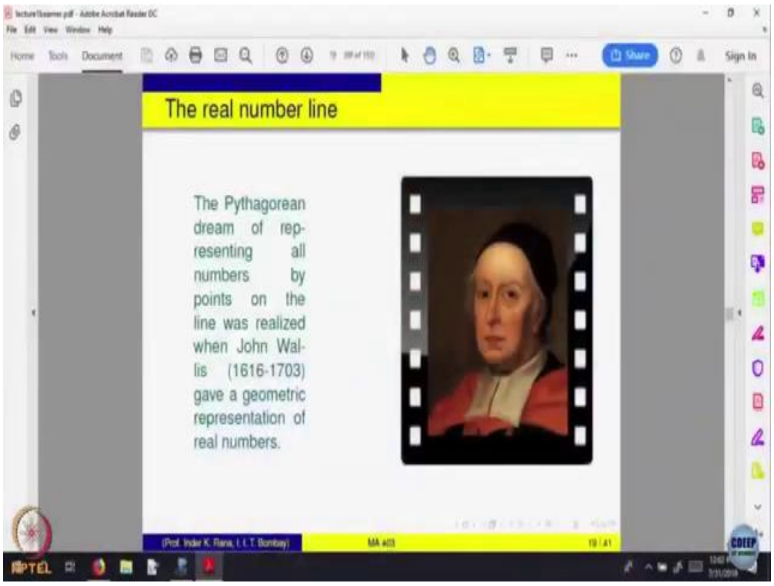

And the geometric realization that Greeks believed every number must be a point on the line that was realized by Wallis. Mathematician John Wallis, who gave there is a one to one correspondence between set of real numbers and the geometric object of points. So, dream of Greeks was realized by the work of Wallis.

So, whenever we want to give examples on the real line, which help us to understand we take real line as, real number as a line, real number as a set, real numbers as a line that is only for visualization purposes, to visualize some things. That helps okay. So, that is the real numbers.

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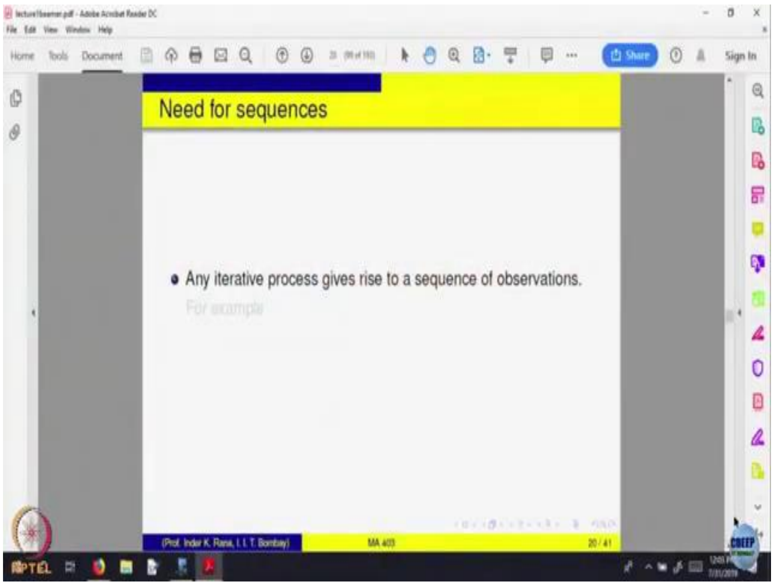

Now, let us come to what is called sequences. Why do you think one should be studying sequences of anything? Are the sequences around us? They are everywhere. You go and buy one to buy a ticket of something. You stand in a queue, what you are doing? In English you will say if you are forming a queue, but I will say you are forming a sequence and then that whenever somebody tries to come ahead of you, you say queue is behind. There is a order in this queue, I am the first, he is the second, he is the third and you are the last, you go in the end.

So, a ordered collection of objects and order, a collection of all objects which is ordered meaning there is order to every object, the first object, second or third and so on is called a sequence of that objects it could be of anything, sequence of human beings, sequence of numbers or anything.

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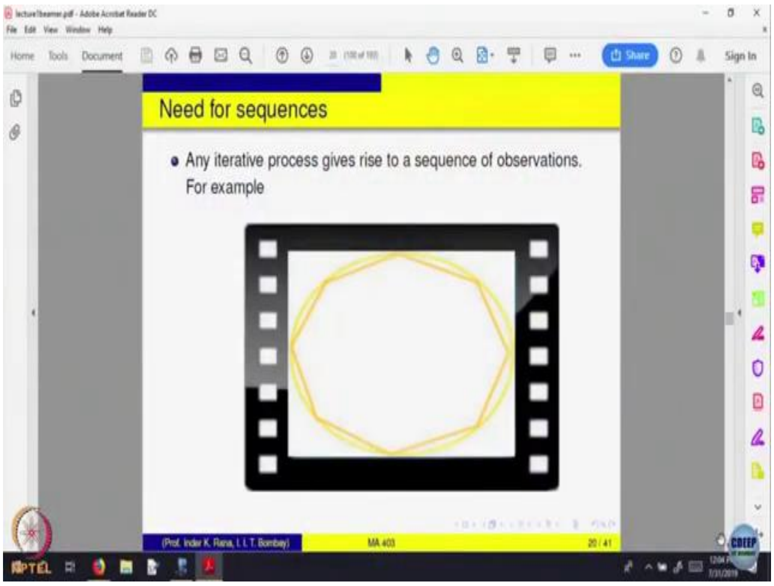

So, here are some examples, which are very, which goes back to what we were discussing. Let us see if I can show you just.

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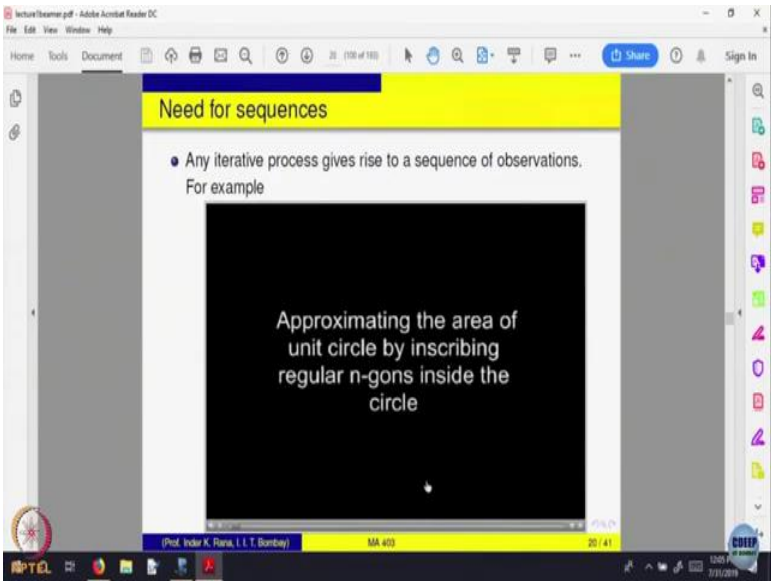

Just look at this clip. Blue is the circle area I want to find out. And I am approximating it by putting in a rectilinear figure inside it. Because for rectilinear figures, we know what are the areas I can compute. And if I increase the number of sites, I am filling up more and more part of the circle, this is what Euclid had done. That he tried to find out what is the area of the circle by this method by putting the regular n gons inside the circle and finding their areas.

So, what is this giving me? When I put a triangle inside I get a one, area of the triangle, when they put a square inside it, you get another observation that is my second approximation a two. Third, a five, five sided and pentagon, so a three, a four. So, I am getting a sequence of I am getting a sequence of observations and what I want to know about the sequence of observations this eventually these n gons should fill up the circle.

Or n large enough these approximations, give me a good estimate of what I can call as the area of the circle. They do not help me to define yet I have not obtained what is area of the circle, unit circle, I have only got approximations. But I know that geometrically it is coming closer and closer to the object I want. So, what is, how do I formalize that notion that these things are coming closer and closer.

So, another way of doing that would be the following. So, let me do, try to do that. Slightly better what Euclid done, not better than what Euclid done in slightly different way. So, let us look at the unit circle.

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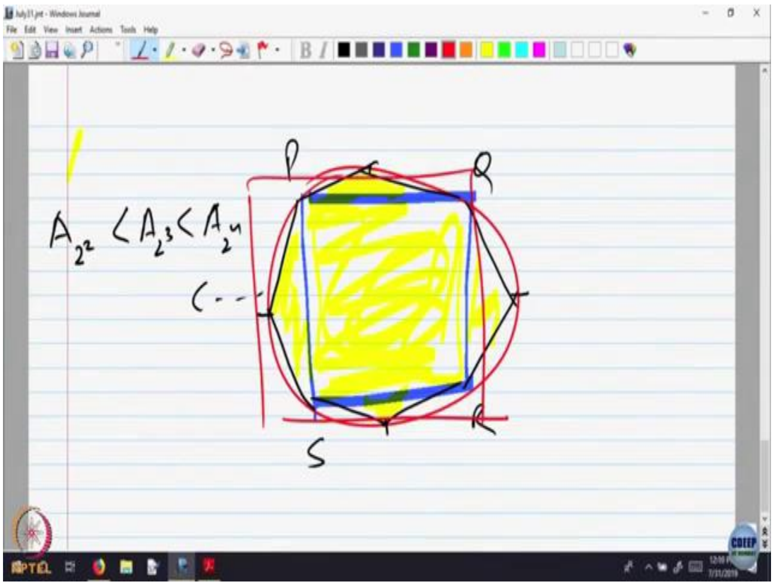

So, let us, I start with the regular four sided thing inside it, that is a square. So, that will give me. So, that will cover this part of the area. So, what is left out, so, let me call this as something P, Q, R and S, so what is left out is those sectors, they are not covered. I want to cover them. So, what I do is, instead of a Pentagon I do the following I take the midpoints of this and join them.

Now, look at the area of this how many sites are there? Eight sides of the octagon and this octagon not only covers the square actually it covers something more these, these are also covered these triangles are also covered now. So, first iteration area of the square, second iteration a two area of the octagon clearly, it is bigger than the area of the circle. Geometrically it is clear and what should be my next step? From octagon I should go to 16 gons.

So, what I will do is, I will write down this is my, for the sphere less than, less than. And geometrically, it is quite clear these are the numbers. Each one is strictly bigger than the previous one. And all these areas are inside a square with a circumscribing the circle. So, all these numbers, a four, a eight, a sixteen all are increasing. Each one is bigger than the previous one, but they do not go beyond something.

So, what do you think should happen to them? They should get cramped and cramped eventually, and they all should come closer and closer to something and that is precisely the notion of convergence. So, that is the completeness property of real numbers it says a monotonically increasing sequence of real numbers, which is bounded above must come closer to a value. So, let us make this as a bit more precise. So, that we can go ahead.

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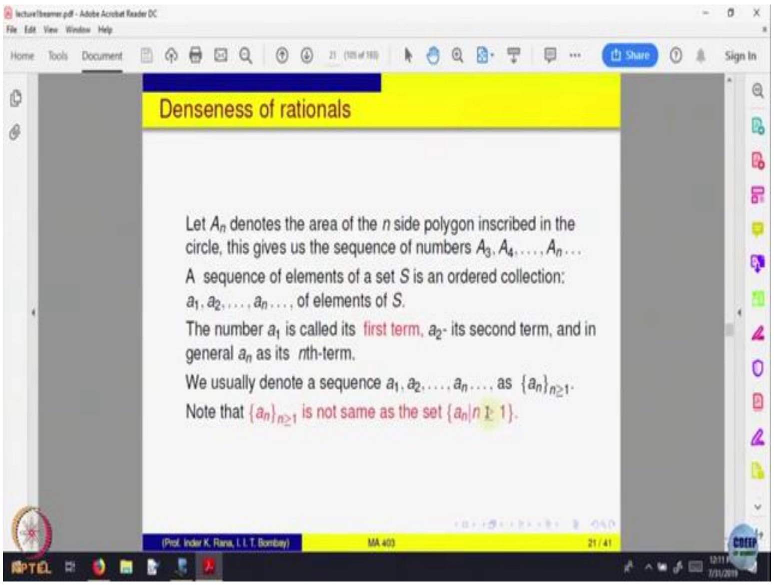

So, the sequence of, so everybody understand what is the sequence of numbers. The sequence number is the ordered collection of numbers that is a sequence, first term a one, second term, a two, third term a three and so on. Okay. And this is the way we write it, a sequence is written as an, an is nth term with these curly brackets and bigger than or equal to 1, this is not the same as the set an by the way.

This rotation is much different from, this is the notation for saying you are looking at the an as elements of a set this is looking at the sequence an. So, let me make it slightly more clear these two are different things.

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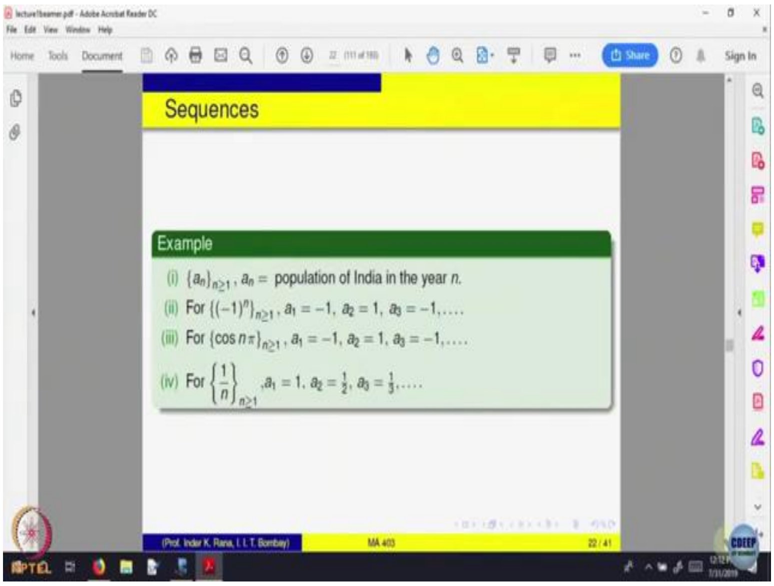

For example, I can look at an as the population of India in some year and nth year. Let us look at minus 1 to the power n the sequence what is the first term n equal to 1 that is equal to minus 1, second term plus 1, third term minus 1. So, as is set it is only two elements set plus minus 1, but as a sequence it is minus 1 plus 1, minus 1 plus 1, minus 1 plus 1. So, that is a difference between a sequence and sequence as I said, Okay.

You can have for example, this cos n pi. You can find out the values again, is this the same as the previous sequence? Second and third both are same. No? Think about it, n equal to 1 what is cos pi? So, what is the first term there in the second one? What is the second term? I will not say anything more for example this is another sequence okay.

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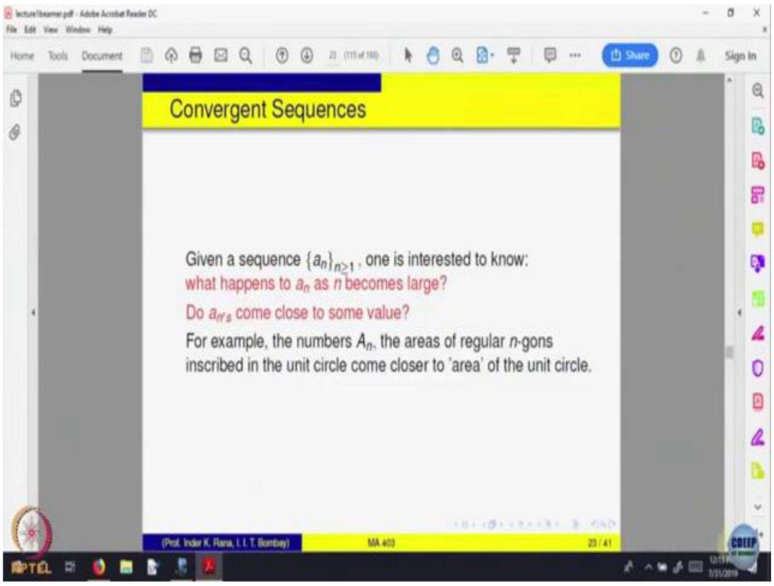

So, let us write down what is the meaning that what happens to n when an, n becomes large and large in the question, in the part of that circle thing we hope that the N gons, n equal to, right two n gons they increase and bounded.