Basic Real Analysis
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Lecture 19
Topology of Real Numbers: Connected Sets; Limits and Continuity Part 1.

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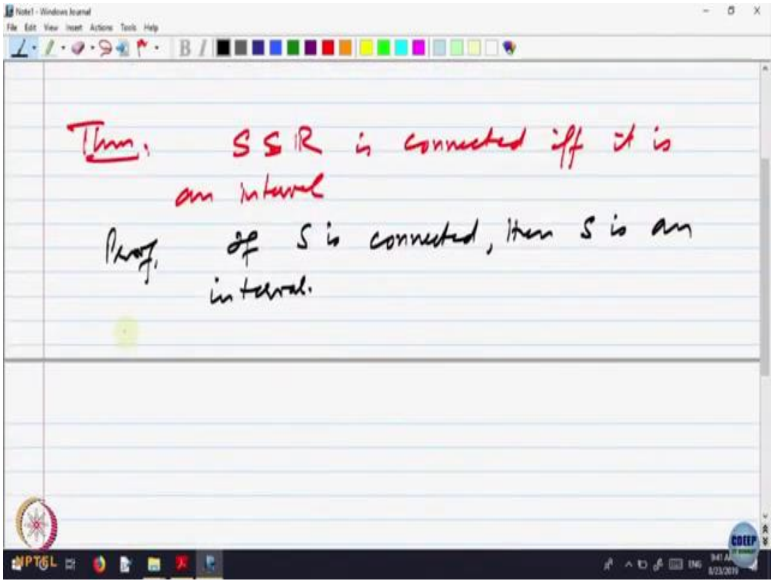

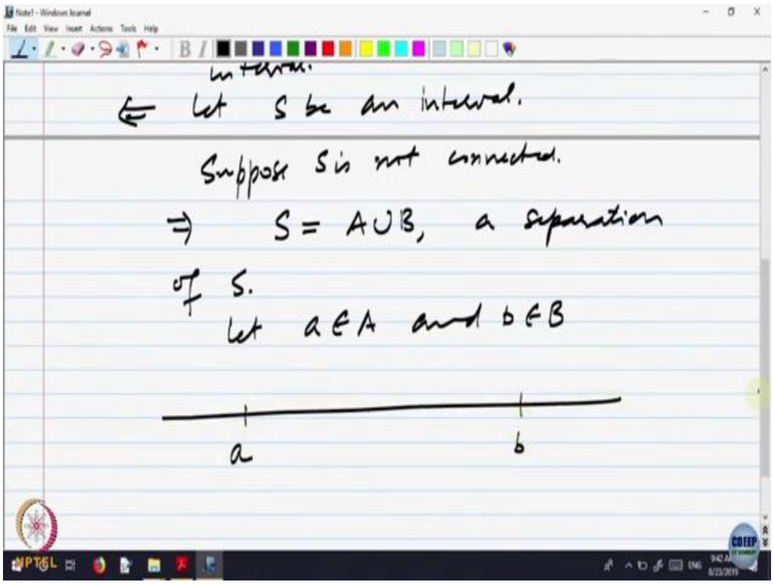

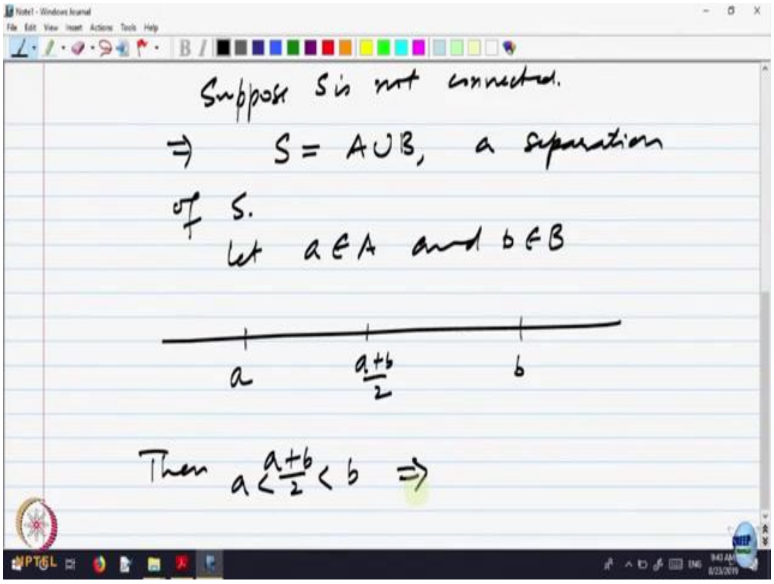

So, last time we were trying to prove a theorem that a subset S of real line is connected if and only if it is an interval. So, we have already proved one way. So, namely if S is connected, then S is an interval.

So let us proof the converse part of it. So, conversely so let S be an interval and suppose S is not connected. So, that implies if S is not connected it must be having a separation. So, let us say S is equal to A union B, a separation of S. That means S is written as union of two sets where A and B both are separated. So, let us choose an element a belonging to A and an element b belonging to B. So, here is a and here is b. Because it is a separation, so S is equal to A union B when A and B are of course non-empty sets and A is separated from B and B is separated from A.

So, let us choose any element a in a and b in b and look at the element which is the midpoint of the two. So, then the midpoint should belong to, it is between a and b. So, let us write that is a point we know that this is in between a less than b implies, we have got S as a interval. So, anything in between must be a part of an element in the interval. So, imply is a plus b by 2 belong to S. Now, if it belongs to S, S is A union B. So, it will either belong to A or to B. So, two possibilities.

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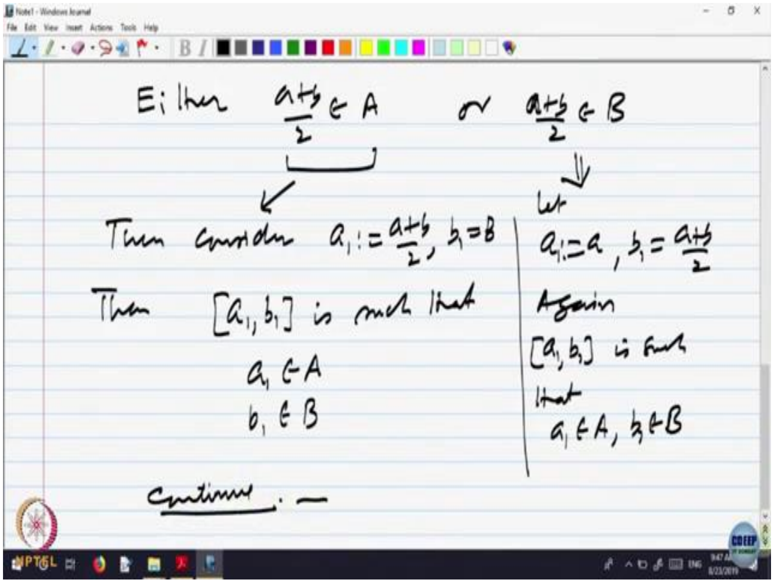

So, either a plus b by 2 belongs to A or a plus b by 2 belongs to B. So, if this is the case, is this is the case. So, then consider a1 let us call this point as a1 a plus b by 2 and b1 equal to b. So, if it belongs to a then I call it as a1 and the right point b that point b we call it as b1. So, what does it give me then a1, b1 is such that a1 belongs to A and b1 belongs to B. So, what I have done is taken two points a and b look at the midpoint, midpoint must be a part of the interval.

So, it must belong to either a or b if it belongs to A call it a1 and call b as b1. So, what is second possibility if this is the case then a1 then this belongs to b. So, define a1 equal to A, so let and b1 equal to a plus b by 2 again a1, b1 is such that, so what is happening is such that again a1 belongs to A and b1 belongs to B.

So, either case I have gotten is interval a1, b1 such that the left hand point belongs to A and the right hand belongs to B, whichever the case may be and other property that a1 b1 is half of a b. The interval a b, we are taking, one of them is an midpoint. So, either it is a to a plus b by 2 or a plus b by 2 divide to b1. So, it is only half of it. So, length of this interval a1 b1 is half of the length of the original one.

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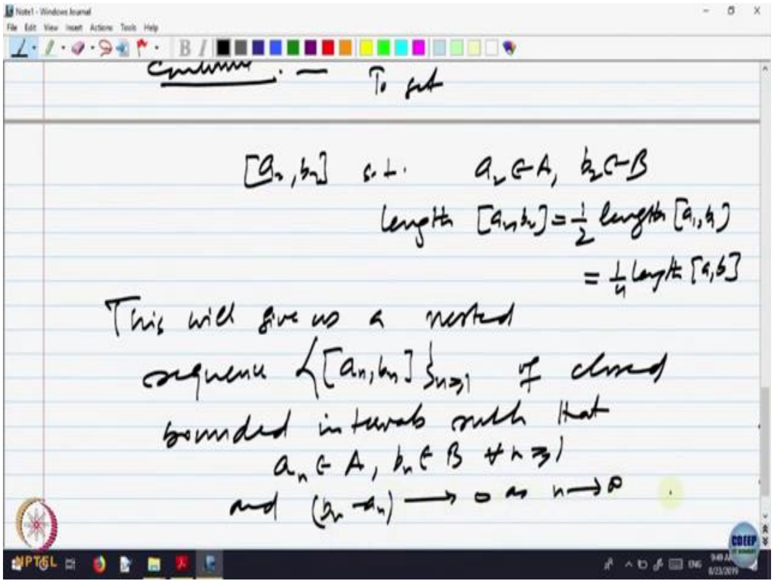

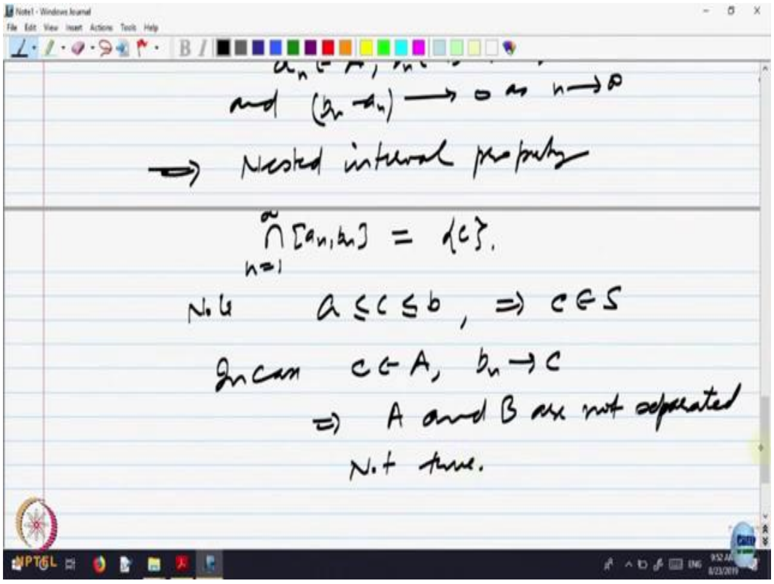

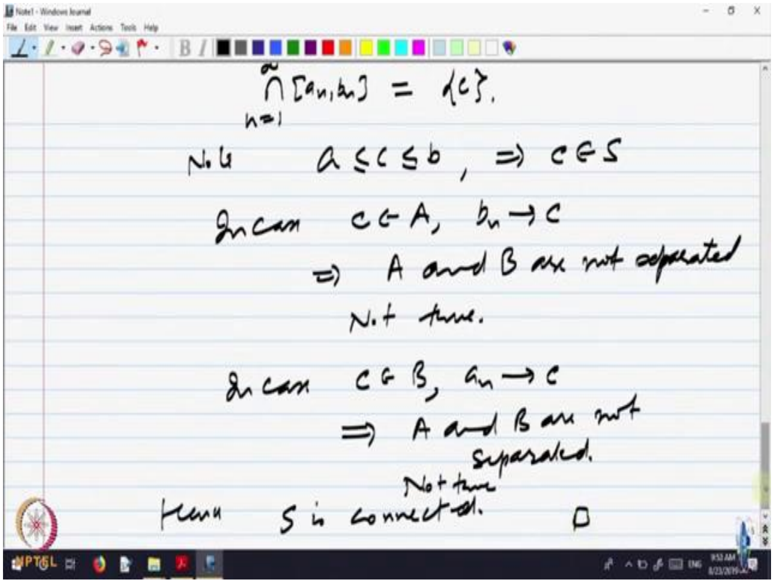

So, let us continue this process, so what does it mean continue I got a1 b1 and now I will take the midpoint of a1 b1, both a1 belongs to S, ab1 also belongs to S so midpoint must belong to S again. So, whatever we were doing for a, b I am doing the same thing for a1 b1. So, continue to get a2 b2 such that a2 belongs to A, b2 to belongs to B and the length of a2 b2 is equal to half of length of a1, b1 which was equal to 1 by fourth length of a b.

So, I have got a way of generating smaller interval from ab a was in A, b was in B I am able to generate a interval of half the length with a same property left hand point is in A right hand point is in b whatever the case may be. So, continuing this process what I will get I will get a sequence of intervals An, Bn which is nested, each one is the previous one, they are closed bounded intervals and the length is going to, what is happening to the length of them?

It is becoming smaller and smaller it is going to 0. So, let us write, so this will give us a nested sequence an bn, of close bounded intervals such that an belongs to A bn belongs to B for every n and bn minus an goes to 0 as n goes to infinity. So, I manufacture a nested sequence of intervals. Now we approach nested interval property of the real line that says there must be a point in the intersection of all of them.

So, imply by nested interval property, intersection of an bn, n equal to 1 to infinity must be equal to some point let us call it c. So, obviously a less than or equal to c so less than or equal b. Now a and b are both in S so c must belongs to implying c belongs to S. Because a and b both belong S is a interval, so it must belong. But S is equal to union of is a separation A union B.

So, either it will belong to A or it will belong B. So, in case c belongs to A then what happens what is c basically. It is the limit of bn, so close to c there will be points of B. So I cannot find a neighbourhood of c which will be disjoint from B and this is the point in A, but A and B are separated. So that is not possible. In case bn, converges to c implies A and B are not separated is that clear?

Because every neighbourhood of c must have some bn after some stage onwards actually all, because bn is going to converge. Nested interval property bn are right hand points must be converging, the nested sequence, the decreasing sequence, that means there is no neighbourhood of the point c, c is in A which is disjoint from B because 1bn at least will come in. So, A and B are not separated not possible, not true.

So, another possibility is c belongs to B same thing, now you go to the left hand points an converges to c implies A and B are not separated, so not true. So, in other case you get a contradiction, so our assumption so what was our assumption? Our assumption was that S can be written as A union Ba separation of S is possible we have shown it is not possible that means S is a connected set.

So, implies so let us write hence S is connected. So, we have shown that we have characterize all connected subset of real line a subset of real line is connected if and only if it is an interval. A similar question one can ask for Rn, one can proof some theorems will do it later you need more techniques more properties of some other concepts. So, we will later on show for example that in Rn what could be a generalisation of interval ruby ball if you take interval then the relation in Rn is ball. So, you would expect that every ball is connected at least that much should be true.

So, that can be show on actually, there is a concept called path wise connectedness some geometry comes in, if any two points can be join by a curve in a subset of Rn, than that subset is called path connected and one shows every path connected set is also connected, but every connected need not be path connected. So, this kind of things happens. So, slightly more complex slightly more involved to study connected subsets of Rn or even in the plane.

So, we will do some examples and some theorems when we come to continuity and such properties of Rn, functions. So, for the time being we are doing it only on the real line, so we have. So, till now what we have done is we have looked at subsets of the real line and special properties of those subsets. We started with the real line as a complete ordered field and then after having done that we looked at what are called intervals, they are some basic open sets called intervals and then we looked at sets, we looked at sequences whether they converge or not when do they converge.

And then we looked at sets which have the property that whenever a sequence of elements converge to some point whether that point is inside the set or not. If all the limits are inside that was called a close set then we studied some property of close sets even the sets is not closed you can make put it inside a set with his smallest close subset of real line called the closure and then we define open sets as those sets whose compliments are closed.

The set is open if and only if it is compliment is closed, so concepts property and so on and then we looked at properties of sets which are called compact sets. So, a compact set is something more than a close set namely a sequence of elements of sets, may not converge but at least there is a subsequence which converges inside the set. So, these are called compact sets, and then we characterize compact sets namely we said every compact set.

A set is compact if and only if it is closed and bounded and then we looked at a characterization of compact sets called Heine-Borel property, that every open cover has got a finite sub cover and that we proved it all if in the real line true for Rn and probably you will see if time is permits us we will do it later on. All this are, these are concepts, which are also valid in general space is called matrix spaces.

So, we will probably introduce, so that also a bit later. For time being let us concentrate on real line and properties of real line. So, we have looked at real line as such sets in real line and special subsets of course we looked at lastly the connected the subsets.

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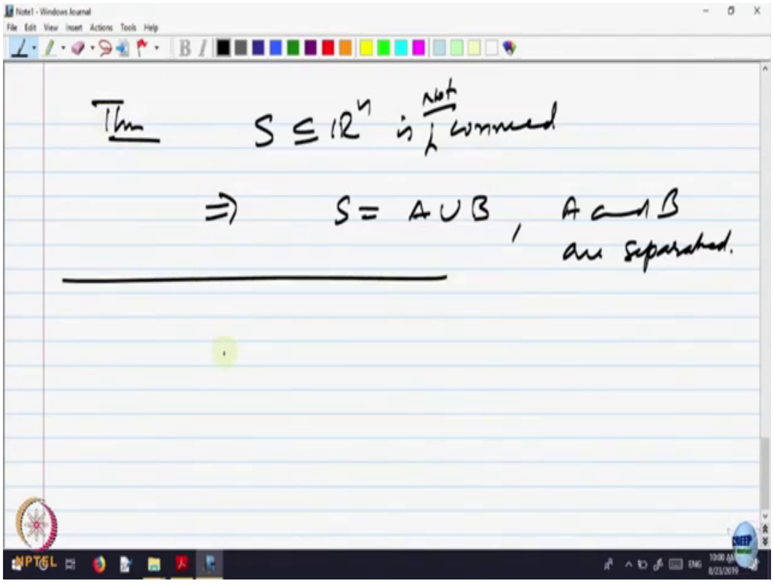

Now here is something very obvious probably I should state that which is good. So, it is true for Rn also, S contained in Rn is connected implies. Let us write not connected means what? S is equal to A union B, where A and B are separated. Now that separation means what every point of A has at least a neighbourhood which is completely inside A, it does not intersect B because it is separated. So, every point of A must be open, every point of A must be open, it was an interior point and hence A must be an open set.

Every point of A it is separated from B, so there must be a open neighbourhood which does not intersect with B, there is a neighbourhood of which is completely inside A it does not intersect B. It does not intersect B, that does not say it should be inside A it does not say it is inside so let me revise this statement, so what I am saying is not correct, I have to go to subset actually that is not good idea what I want to say is something need sub spaced topology but let me say a milder version of it.

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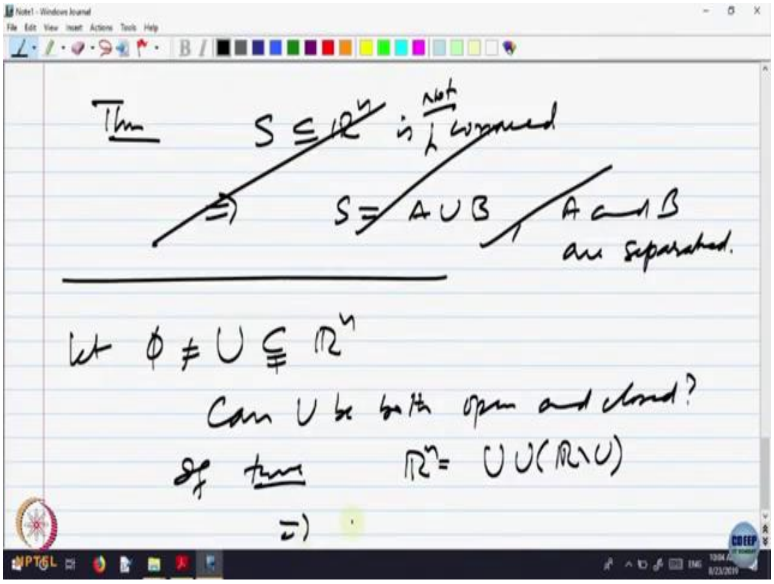

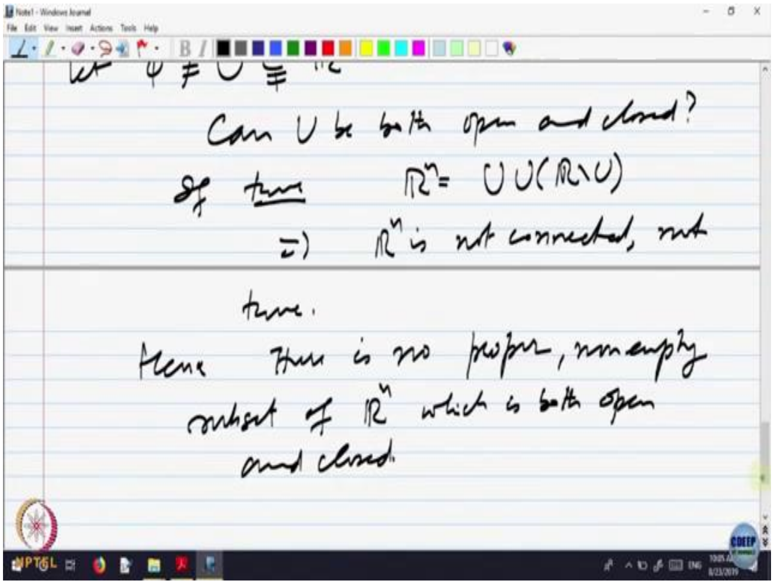

Let us take a set U contained in so forget this one. So, U is a subset which is non empty and contained in Rn let us take. So, let U be an nonempty subset of Rn can U be an proper subset it is a subset of Rn it is not empty so it has at least one element it is not whole of Rn then can you be both open and closed? Suppose it is suppose U is both open and close than what is real line equal to?

It is U union U compliment. If true than Rn is equal to U, union R minus U, my claim is this is the separation of Rn, why it is a separation of Rn because U is open take any point in U then there must be a ball inside U, because U is open, every point of U will be inside a ball which is completely inside U. So, U is separated from U compliment and same if it is U is both open and closed then U compliment is open. So, every point of U compliment is also separated from U is that clear to everybody.

Definition of openness says every point is a interior point. So, if I give you a point of U it must be inside a ball which is completely inside U. So, I will not intersect U compliment that means U is separated from not only it is disjoint it is actually separated from U compliment, and now let us go to the other way round take a point in U compliment, U compliment is also open, because U is both open and closed.

So, U is closed so U compliment is open. So, every point of U compliment will be inside a ball which is completely inside U compliment. So, U compliment also separated from U for the same reason. That means real Rn is not connected because you got a separation. So, conclusion is in Rn there is no nonempty proper subset which is both open and closed as a consequence of connectedness.

So, let us write not possible implies Rn is nor connected not true because it is a interval. So, hence there is no proper nonempty subset which is both open and closed. So, Rn connected implies this kind of property you cannot have subsets which are both open and closed of course non-empty and proper.