Basic Real Analysis
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Lecture 18
Topology of Real Numbers: Compact Sets and Connected Sets Part 3.

For real line a, set is compact if and only if every sequence has got a convergent subsequence converging in a set, or every open cover has a got a finite sub cover or saying that the set is closed and bounded. All three equivalent ways of saying it is a compact set, and what is the big use of saying something is compact is basically in general settings is a last property saying every covering has got a finite sub cover that becomes very important.

Bringing things from arbitrary, any arbitrary covering to a finite, so many of the proof you will find which are true for arbitrary can be brought down to finite by using Heine-Borel property and I think one of the image at use you may find I do not know whether the proof is given in the probability courses or not if you look at the length on the real line, if you look at the length of an interval for interval which is bounded B minus A is the length if and points are A coma, B.

If it is unbounded you can put the length to be equal to plus infinity. So, you get a function from intervals to 0 to infinity and proving that this function is additive is not difficult that if you take a interval which is cut into two pieces in the length of the bigger piece is equal to length of the sum of the smaller pieces, it is something like keep in mind the probability of the set A is equal to probability of B union probability of plus probability of C additive property.

But suppose you cut up a interval into countable infinite number of intervals, can you think of cutting up a interval into countable many sub intervals disjoint and still the union being same, disjoint but union being same. So, let me just give you something where this things are going to be used I do not know.

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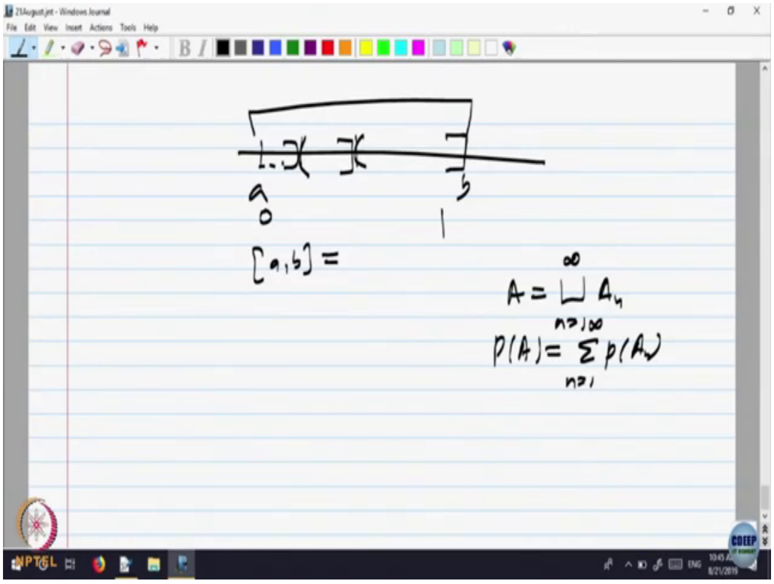

So, for example if I look at this interval A to B what is the interval a b? I want to cut it into countable number of pieces, I can cut it into half this close, this open. So, what I get a smaller piece which is close, I can cut it into that into half. So, I will get this piece and I can go on doing it. So, I will be cutting up the close interval a b into countable many disjoint pieces, disjoint pieces any two pieces are disjoint and the length of the whole thing is equal to summation of length (())(03:39) individual pieces.

So, this is going to be called something this if you think it as 0 to 1 length has the probability of something happening, probability of 0 to 1 is 1 length is 1 and you have got sub set which are events the length of which is the probability of that event. So, it essentially says if an event A is disjoint union of event A n then the probability of A is equal to summation probability of A ns. This is probability going to be used in probability theory called countable additivity.

And the length function is one such on 0 to 1 and proving it is countably additive on uses compactness, so there is the, so anyway. So, when it comes there probably you may get back to it, so we looked at sets which are close which are open and now everything is motivated by sequences now here is a property we want to discuss, which depends more on the geometric aspect of real line that means the real line is a continuity of points, it is the continuity of points, there is no break.

For example, if I take a interval I take at some point then I keep on say moving on the right side then I keep on moving till I come to the end of the interval in some sense, there is no break that it breaks and then again something happens there is a continuity of point. So, intervals characterize by the continuity of points and the we define that way that if x and y are points in the interval and z is a point in between then that must always belong to the set that is how we define the interval. So, such sets are important they are called connected sub sets.

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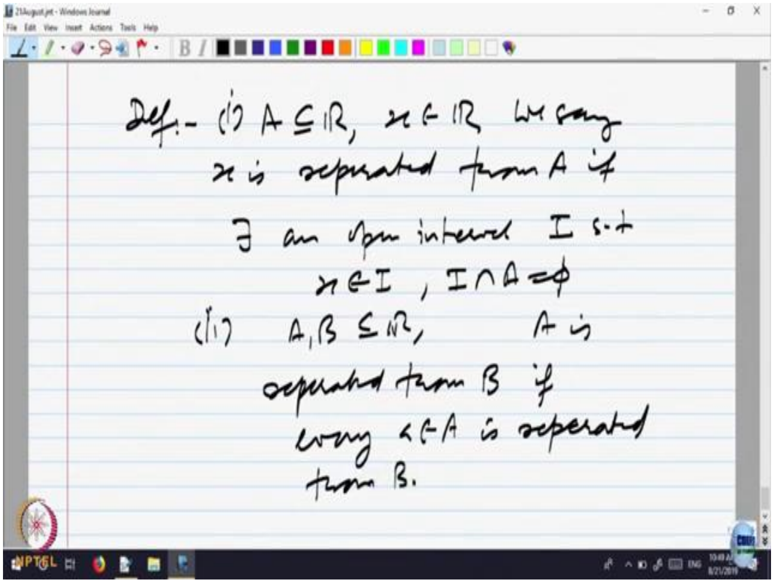

So, what we want to look at it is what are called so definition we want to define, does not make much difference in the real line but latter on it makes a difference. So, let us define, so let us take a A is a subset of R and x belongs to R, so we say x is separated from A, if there exist you can define in R n, but let us keep to the R if there exist an open interval, I such that x belongs to I that means there is a neighbourhood of the interval but this does not intersect A.

There is a neighbourhood which does not intersect A, then you say x is separated from A clear. We want to say a set A is separated from B what should be the definition every point of A is separated from B. So, this is 1, 2, A and B subsets of R we say A is separated from B if every A belonging to A is separated from B.

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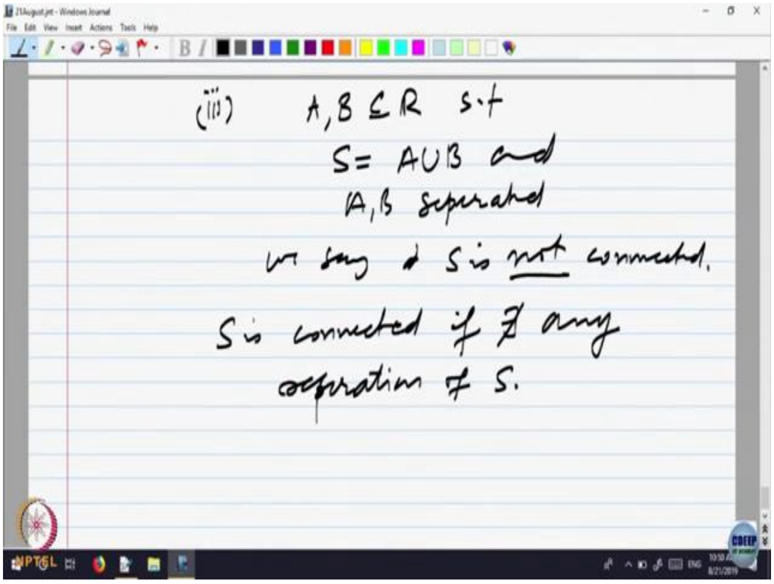

So, let us write say A, B contained in R such that S is equal to A union B and A, B separated if a set S, can be written as, it is cuted it into two pieces where A and B are separated, we say S is not connected. So, what is connected? S, is connected if there does not exist any separation of S. Say, if there is a, if A union B is a separation then they have to be disjoint is that obvious. For every point of A is separated from a point B.

So, if there is a intersection then that point has neighbourhood which intersects both of them because x itself is both of them taken any neighbourhood that will intersect both. So, if A intersection B is non empty then it cannot be separated. So, separately implies it is A and B is there separated implies that they are disjoint already is something more then disjoint separated means every point of A has neighbourhood which does not intersect that and every point B has a neighbourhood which does not.

For example look at the rationales as one set, set of irrationals is another set, real line is union of these two, real line is union of rationales and irrationals but can you say rationales are separated from irrationals? No, because every neighbourhood if a rationales has to have a irrational inside it. So, it will intersect it, so rationales and irrational though they are disjoint, they are not separated. So, saying separation is something more than saying they are disjoint.

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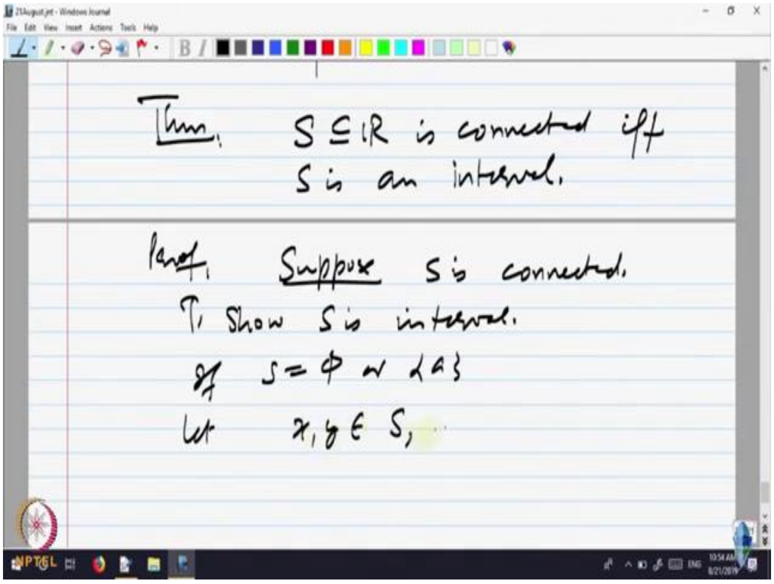

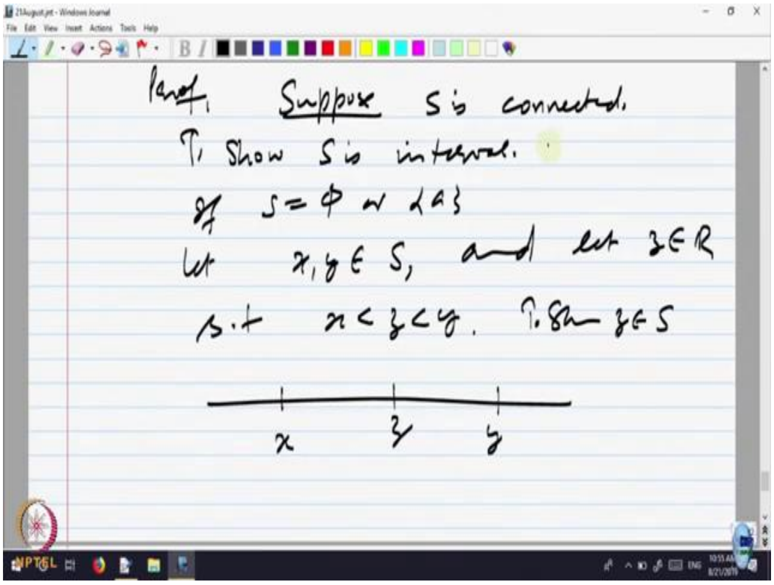

So the theorem that we want to prove time is too short so, theorem we will prove it next time that S contained in R is connected if and only if S is an interval. The only connected subsets of the real line are intervals. So, let us see how much we can proof, so two parts so suppose S is connected to show S is interval, keep in mind we had called empty set also as interval. So, if S is empty set is there anything to proof.

Because the separation does not make sense, separation of empty set into two part does not make sense A union B? Or you can like, you can say empty set is a connected set you can forget about it you can declare empty set as connected, suppose it is a singleton. So, if S is equal to empty or singleton is singleton point that is a subset of real line is it connected that means it should not have any separation.

Only singleton point, so it will be one of the sets A or B only, so there was separation possible. So, is connected, so finally let us take two points let x y belong to S and, so what we want to show? S is connected to show it is an interval we have got two points to show it is an interval what should I show every point in between is also inside and let x less than y, x less than z and let z belonging to R such that x less than z less than y to show z belongs to S.

So, the picture here x here is y and here is z, I want to show z belongs to the internal to the set S and given S is connected if z does not belong to S, if z does not belong to S then I should what I should show if z does not belong then I should show S is not connected I should produce a separation of I should produce a separation of S.

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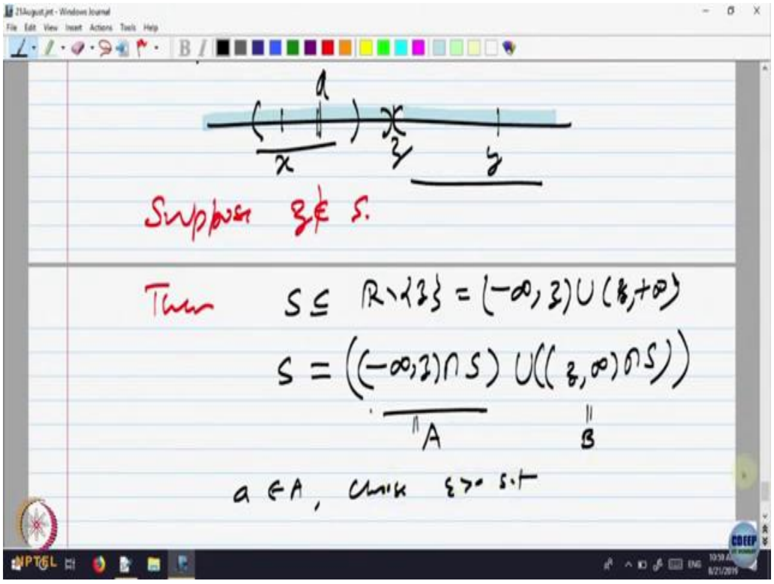

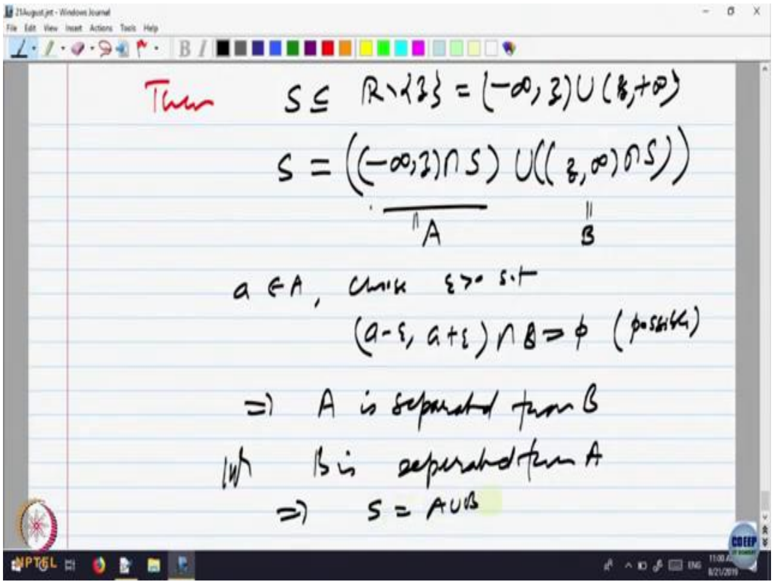

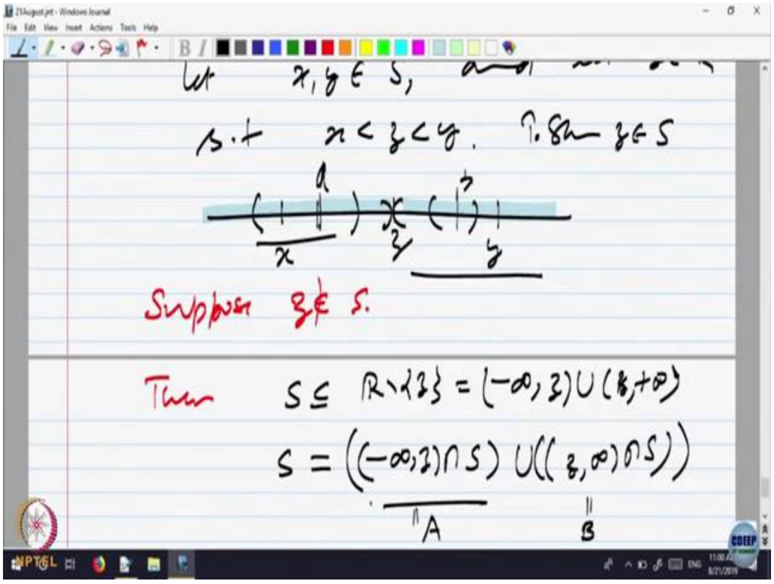

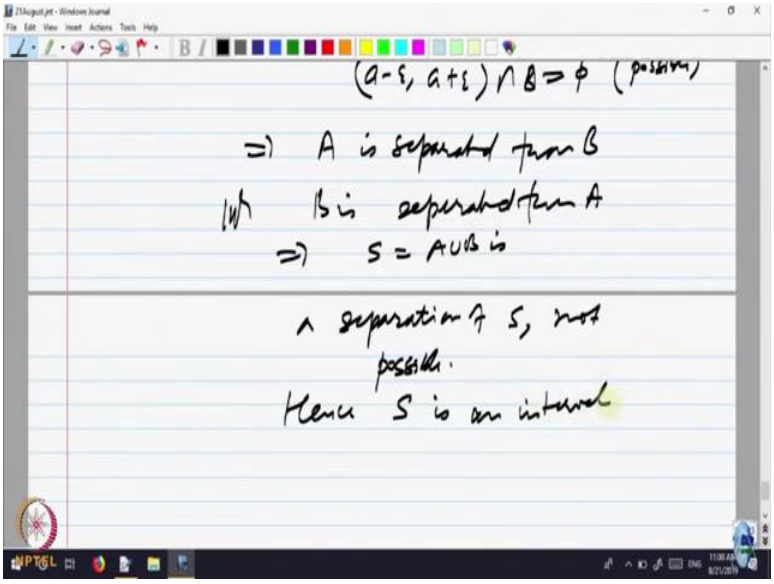

So, let us write, so suppose z does not belong to S then z does not belong to S that means where is S? S is in z compliment singleton. So, that means than S is a part of R minus z, that does not belongs, so it is the part of the compliment. So, what does a compliment look like it look like two parts one parts is here and another part is on the other side. So, 1 is here the other is here. So, let us look at which is equal to minus infinity to z union z to plus infinity.

So, what is S equal to intersection of both sides. So, minus infinity to z intersection A union minus infinity to z intersection B. Because S is not containing z so some part will be on the left side, some part will be on the right side. So, this is the part on the left side and this is the part on the, intersection A where is A there is no A there is S. So, let me write properly some intersection S union z to infinity intersection S.

So, intersection of S on the left side of z, intersection of z on the right side, so that will be everything. Now so, here is a set call the set as A and called the set as B. So, S is written as a union of two sets A and B, they are disjoint is it a separation of A is it a separation of A, a separation of S. Let us see, so let us take a point on, in A so if A belongs to A where it will be given somewhere here.

So, I can take a interval because if this it is not an it is not equal to z. So, it is bigger than z so this will be some interval I can take around A which does not go beyond z. So, for every point here I have got a interval which does not intersect the right hand side. So, every point A in A will be having a neighbourhood disjoint from p is the picture clear. Choose epsilon bigger than 0 such that a minus epsilon a plus epsilon intersection A intersection B is empty.

I am saying possible why it is possible? Because if you take any point A it is going to be less than z. So, you can always take an interval on that side, so that it does not go beyond z. So, that means what every point A that means A is separated from, every point of A as a neighbourhood which does not intersect.

So, A is separated from B, similarly B is separated from A, similarly other way around it will take away the right side I can similarly do it, is it clear to everybody similarly means what if I take a point A here B then there is a neighbourhood which does not intersect it does not go on the left side.

So what I have got if z does not belong I have got a separation of S but S is connected. So, contradiction, so implies S is equal to A union B is a separation of S not possible hence S is an interval. So, we will prove other way around tomorrow. Namely what we have shown today is if S is connected it as an interval, we will show that if A take a interval than it is connected.

That means no interval can be cut up into two parts, such that they are separated the two parts are separated from each other. So, that will prove only, that is why, this is the reason and the real line connectedness though important is not very interesting, only intervals are points, we will see what it means in Rn tomorrow.