Basic Real Analysis
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Lecture No 15
Topology of Real Numbers: Limit Points, Interior Points, Open Sets and Compact Sets
Part 3

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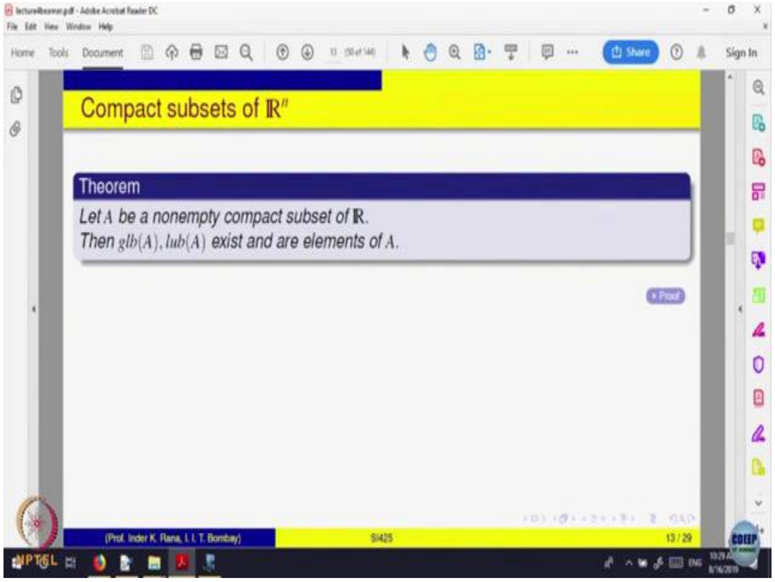

So, let us try to give some more applications of this compactness. It says here is a important thing if A is non empty compact set then we know it is bounded because it is closed and bounded. So, in particular it is bounded so by completeness property of real numbers its greatest lower bound and least upper bound must exist. That is only by boundedness but compact has implies that those points least upper bound and greatest lower bound will be elements of that set if A is compact it is a very important thing. So, let us try to prove that.

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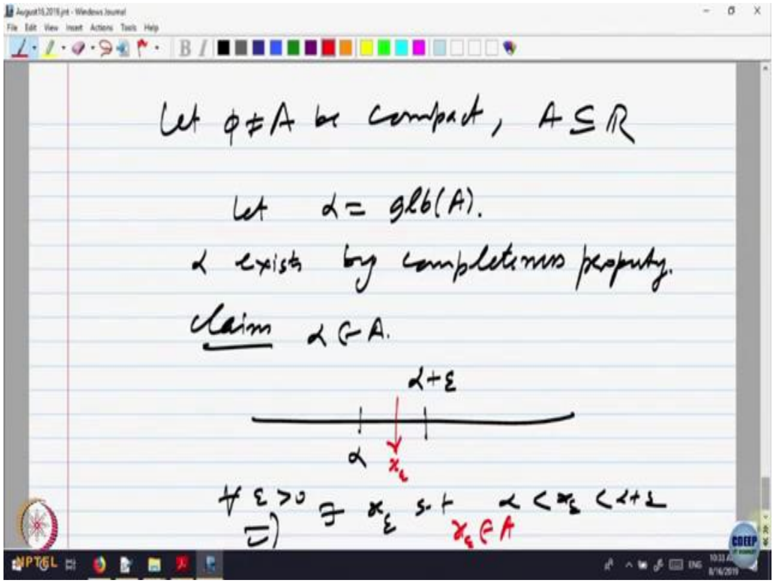

That greatest lower bound and least upper bound both are inside. So, let A be compact when I do not write A, it should be Rn also, A in R are Rn any subset. A is compact, so let us write A is non empty because a ways for empty set everything is okay. Let alpha be equal to greatest lower bound of A. Alpha exist by completeness property. There is a slight logical issue because what is completeness for Rn I have not discussed actually.

Let us keep it on real line for a time being. I can define it, but I think I do not see much point of you knowing that, one can define what is called completeness property for Rn. Let us not go into that, so let us keep this A is compact, A is subset of real line let us do it only for the real line. So, that we do not have to bother about much. So, claim alpha belongs to A. We want to claim that alpha must belongs to A.

Now this is very simple and it is very it just depends on what is the definition of, here is my alpha which is a greatest lower bound. That means what? All the elements of the set A are on the right side of alpha. alpha is the greatest lower bound, it is a lower bound and the greatest of them. So, everything else is on the right side. Now definition of its says that, if it for every epsilon bigger than 0, if I look at alpha plus epsilon that will be point on right side.

Can that be an upper bound for A? Can alpha plus epsilon be an upper bound for A? Alpha is the least upper bound, alpha is the greatest lower bound. So, can something bigger than alpha be a lower bound? No, other than definition of greatest lower bound is contradicting. That means what, that means between alpha and alpha plus epsilon there must be a point of the set A because I want alpha plus epsilon should not be a lower bound.

So, there must be a point on the left side of it. So, there must be a point so implies, there exist some point so let us call it as x epsilon such that alpha less than x epsilon less than alpha plus epsilon. So there must be a point here, x epsilon. And that x epsilon should belongs to the set A. There must be a point of that otherwise, it cannot, it will contradict. This is the crucial thing this is the definition of, see at one point I had said that if you want to understand what is true then you should also understand what is false.

So, it is a negation or the definition whichever way you like alpha plus epsilon cannot be greatest lower bound. So, there must come something inside. So, whenever you want to understand what is true, you should also look at what is negation of that statement, what is false to understand that. So, I am going to now, so this is definition and what is given to me A is compact and how is compactness defined? In terms of sequences and convergent subsequences. So, from here I should try to produce some sequence somehow or the other.

And then go to a subsequence and convergent A inside the set using compactness. So, how can we produce a subsequence by using this fact? And I wanted to be convergent, so the obvious thing take this is true for every epsilon. So, specialize epsilon equal to 1 over n, specialize epsilon equal to.

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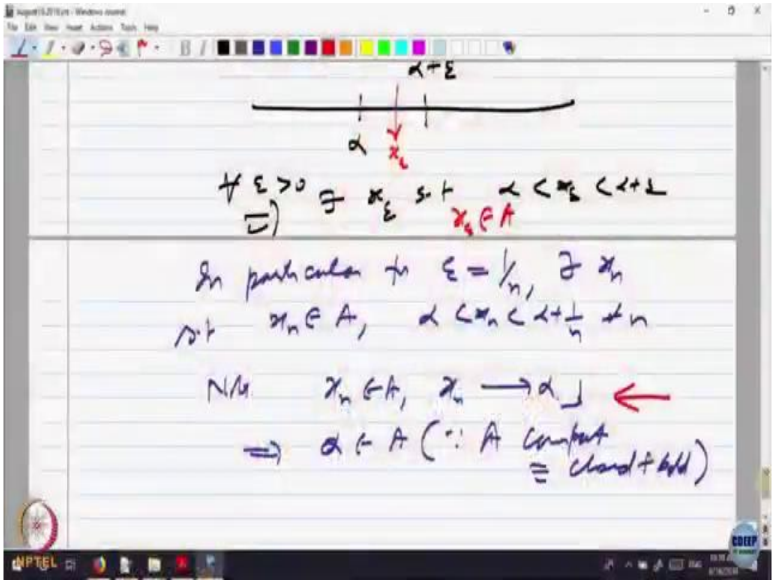

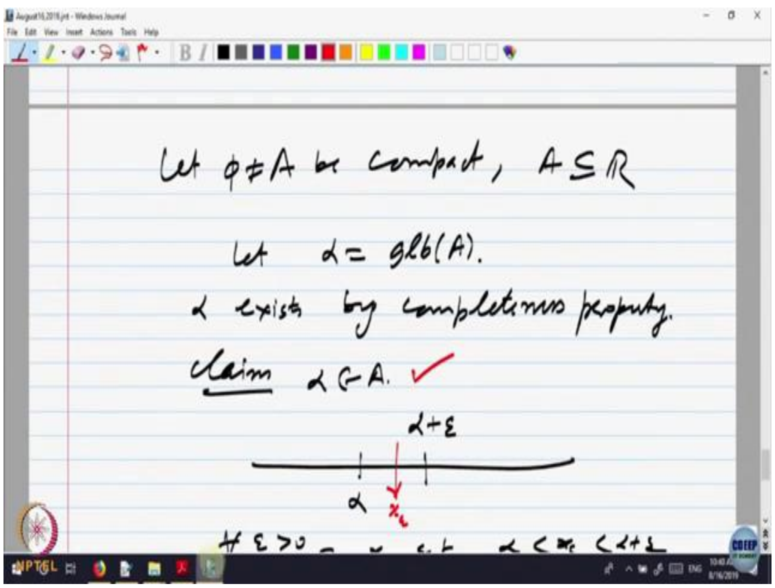

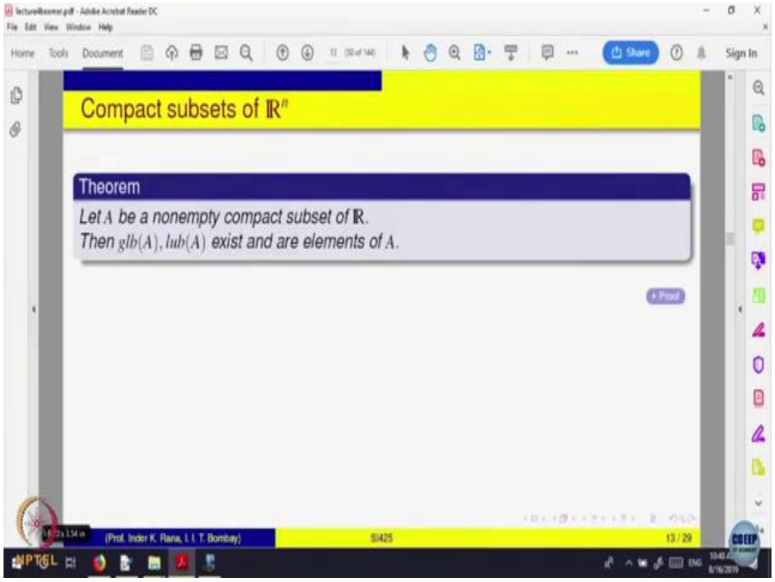

So in particular, so we write, in particular where epsilon equal to 1 by n, there exist some xn such that xn belongs to A and alpha less than xn less than alpha plus 1 by n for every n. Because something is happening for every epsilon so I have specialize it to epsilon equal to 1 over n. What are the sequence now xn and that sequence xn is between alpha and alpha plus 1 over n.

So, can I say sequence is convergent? Yes, convergent where? To alpha. So, note xn belongs to A. xn converges to alpha. What does compact, till now we have not used any compactness or anything, we have only used definition of alpha. Now compactness says that this must have a convergent subsequence, converging in the set. But the limit is alpha and the sequence itself is convergent that implies alpha must be inside A. Because sequence has a subsequence which is convergent but that subsequence will converge only to the same limit has a sequence converging and that is alpha. So, by compactness alpha must belong to A.

So, by compactness or if you like xn is a sequence converging to alpha A is closed, so limit must be inside, you can shortcut using the fact that compactness means closed and bounded, implies alpha belongs to A. Because A compact equivalent to closed plus bounded, if you like either way you can write. So, what we have done. Is it till now till this point what I have done is if alpha is a greatest lower bound of a set then there is a sequence converging to it.

Till this point we have not used anything, only definition. So, if a number alpha is greatest lower bound there must be a sequence converging to of the set elements of that set which you are taking the greatest lower bound converging to alpha. In fact, something more I can do I can make the sequence xn a monotonically decreasing sequence. I can specialize it, because I can always choose something on the other side if it does not work go to the next number. And go to the next one.

Because if once I will chose an x1 between x1 and alpha there must be an element of the set because x1 cannot be, so there may be x2 bigger than x1, similarly x3 and so on. And you can go on doing such kind of things. So, you can improve this but this is very nice and important thing that if in term of sequences that if alpha is greatest lower bound then there is sequence of elements of set converging to that alpha and compactness says this must belongs to the set.

So, alpha belongs to A, so this proves that if alpha is least upper bound, if alpha is greatest lower bound then it belongs to a set. Same property true for least upper bound. Why? We will be on the other side only you will be instead of waving your, there on the right side everything now you will be seeing everything on the left side. So, if beta is least upper bound then beta minus epsilon cannot be least upper bound, so there must be something in say. So, take epsilon equal to 1 over n again, so alpha minus 1 over n you will be doing and same analysis you will do.

So, this proves the theorem that if it a compact set then greatest lower bound and least upper bound are elements of that set. This will be useful later on when you want to maximize and minimize functions. In many situations your functions will have domain which is compact and the functions will be continuous have not really defined but all of you know continuity and so on. Then you will get that the range is compact and hence upper bound and function must have a, the range must have a least upper bound and get us lower bound, and that must belong.

So, that says that will give you a theorem for continuous functions. Every continuous function must attain it is bounded and must attain his maxima and minima. So, that will come in calculus later on. So, compact sets are defined as sets, in term of sequences we are defining, every sequence in that set has a subsequence which is convergent in the set that is compactness. Using that we said every compact set must be close and bounded that is one.

Second if a set is compact then very interesting and application of that is that it is bounded so greatest lower bound and least upper bound is exist as real line so they must be attained. That means they are in set itself right for like an open interval 01 the greatest lower bound is 0 but that is not part of the set. Similarly, least upper bound of open interval 0 1 is 1 but it is not inside the set and that is because open interval 01 is not compact, we know that right here. So, let us look at some more properties of this.

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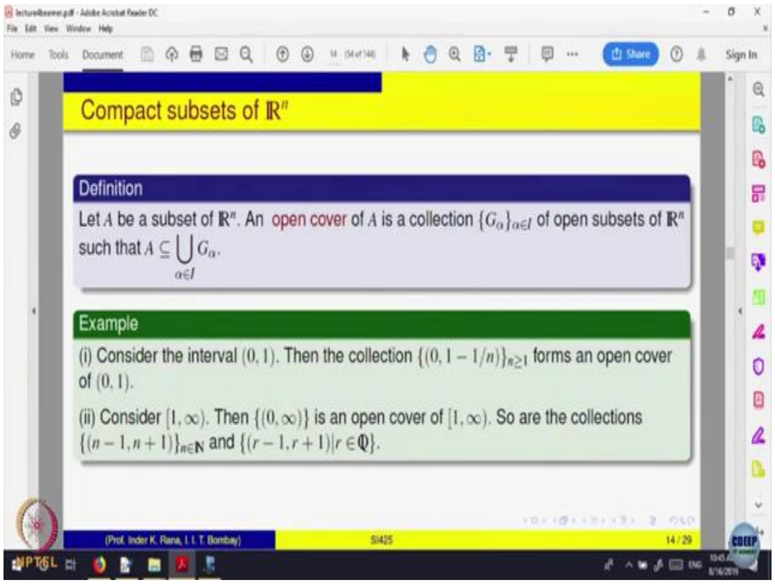

So, here is another way of defining compactness which, finds why we should be doing this at all? Say there is something you should be doing that because it is a part of syllabus. But there are some things which, why it is a part of syllabus because there is some other reason for that the reason for looking at something called like cover of a set is that in this way will using, will define what is called a open cover of a set and define some properties of compact sets in terms of this and then that will make it independent of sequences and such things.

And there are spaces where you can define compactness where there are no notion of a sequence but compactness becomes important. So, I will probably indicate this later on. So, let us just look at for the time being that it is mandatory for us to know about what is a covering, and how compactness is defined in term of, it is important. So, what does a English word cover mean? It should cover what else it could be. So, we want to say given a set there is something that covers.

So, if A is a subset of B, then we will say B covers A but in general one set may not cover another one, may not be subset but there may be a collection of sets their union covers that given set, then we say this collection is called a cover for the given set. So, set theory purely, a collection of say here it is G alpha of subsets of Rn is called a cover for A, if A is inside the union it covers. If each G alpha is open then we will say it is an open cover for A, we have specializing, cover by sets, that is a cover and if I can say each G alpha is open then we say it is a open cover for A.

Let us look at some examples of this you can manufacture many, for example if I take the open interval 01 and look at 0, 1 minus 1 over n, 0 to half, minus 0 minus 1 by 3 so that is increasing. I am making that point slowly increasing, increasing their union will be the open interval 0, 1. Union of open interval 0 to 0, 1 minus 1 over n forms a cover of the interval 0, 1. Because it is starting at 0 and ending at 1 minus 1 over n. Slowly, it becoming going to the right and right. So, every point of 0, 1 sometime or the other will be inside on the left side of 1 minus 1 over n.

Is it clear, because 1 over n goes to 0, so 1 minus 1 over n goes to 1. So, it is increasing to the right side. So, this will be a cover and each is a open set, is an open interval. So, this is an open cover of 0, 1. Some another examples are given, so look at 1 to infinity then you can look at 0 to infinity is an open set which covers it, is an open interval, so open cover of it and so on. There are many examples you can construct yourself many more. I am just giving you some to get an idea of open.

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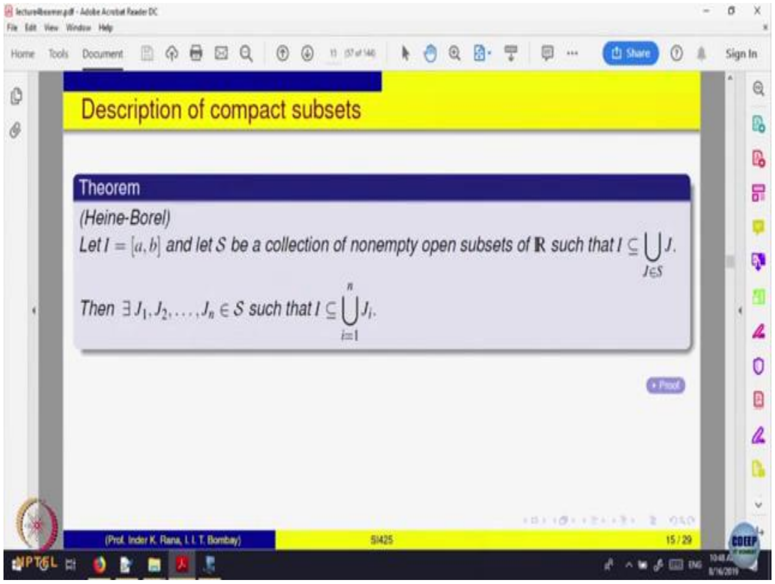

So here is a many important property is called Heine-Borel theorem. It is for the real line. We are looking at it says in a closed bounded interval a,b is given. And you are given an open cover of this, by open sets JI. I is an interval that is a,b and there is a covering of I by open sets J. J belonging to some collection, some collection of in open sets covering. Claim that we do not need a full cover, there is a finite number of them. So that means there exist J1, J2, Jn such that I is contained in this finite union only.

So, from a arbitrary collection you are come down a finite sub collection which covers it is a very important thing. You understand what we are saying, given any arbitrary open cover of the closed boundary interval a,b there is a finite subcover of it. There is finite sub collection of the given collection which covers it. And this goes by the name of Heine-Borel theorem. So, let us try to prove this theorem. The idea of prove is rather simple, so let me let me try to give you the idea and then prove it.

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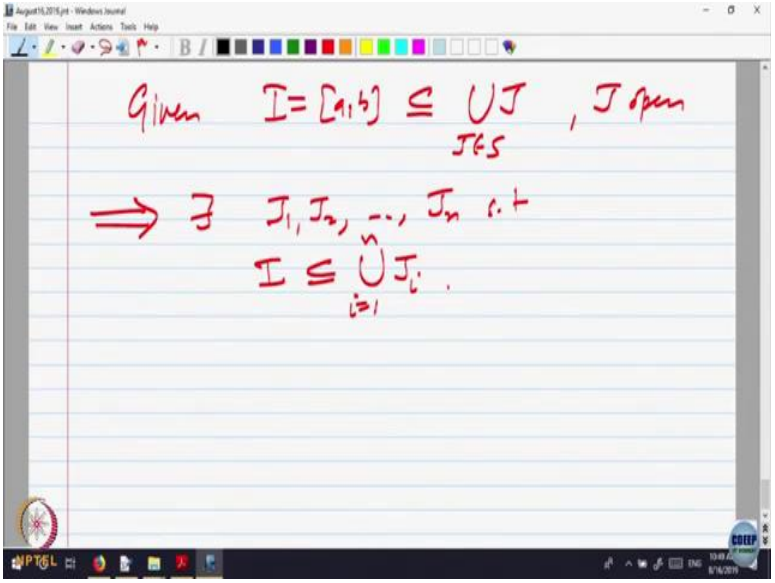

So, what we are given, given I equal to a,b is inside union of J, J belonging to some collection S. J open, each J is open. Implies there exist J1, J2, Jn such that I is contained in union of Ji, I equal to 1 to n. Given any open cover, cover of a close bounded interval by open sets, there is a finite sub cover, there is a finite sub collection which is enough to cover it. You do not have to do all.

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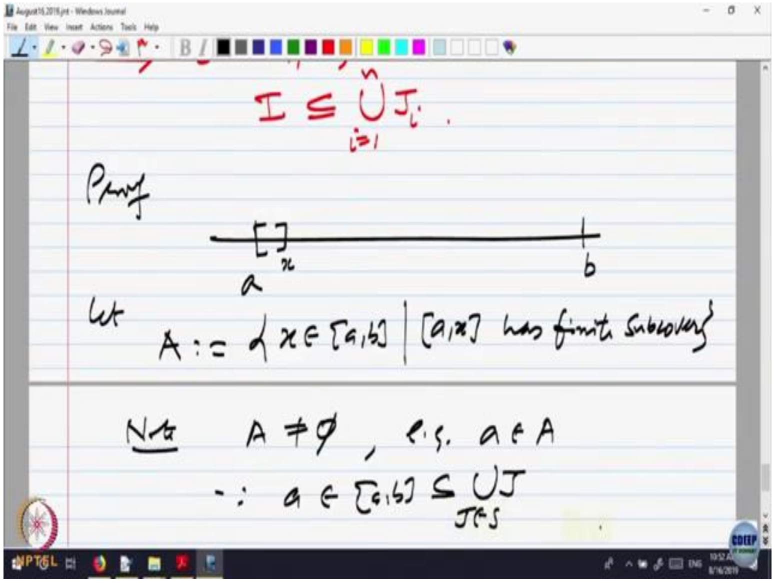

So, let us write the proof. So, here is my a and here is my b. So, idea of proof is I start with the point a and go to a nearby point say x and look at this interval ax. Look at a part of the interval a,b starting from a. close interval a to x, possibility is a to x is covered by finitely many members of that given collection.

This interval which is a subinterval part of ab has the property that a to x has a finite sub cover or it may not have. It may or may not. So, the idea is try to collect such points and show that they have a largest element that is b. So, a to b will have a finite sub cover. So, idea is let us construct, let A be the set of all x belonging to a,b. Such that a to x has finite sub cover.

What is our aim to show? b is an element of this set. Aim is to show b belongs to it. Because if b belongs a to b will have a finite sub cover will be through. So, we are reformulating the problem in some way so that we can get a hold of it. This set A, so note this set A is non empty set. Why it is non empty? That means what I have to show there is at least one element of x between a and b which belongs to this set A.

I can take x equal to a itself. For example, a belongs to A. a, a x is equal to X, so it is a closed interval a, a. It is part of a,b and it is only single point. So, I can pick up any element in that cover, open cover. One is, one element is enough to say that a,a is covered by a open set J, is only one is enough, why finite, only one is enough for the single turn Because a, the point a belongs to the interval a,b because a belongs to a,b is contained in union of J, J belonging to S.

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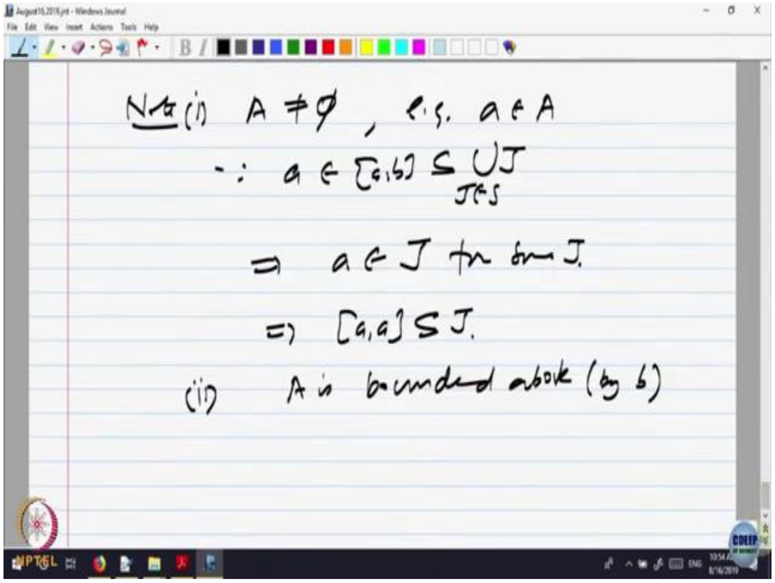

So, what does it imply? a belongs to J for some J because it belongs to this, this is inside the union, so a has to belong to one of them, pickup any one, you like. So, implies a,a is inside J. So, interval a,a has a finite subcover. So, a belongs to the set A. So, it is not empty. So, this is the first observation. Second A is bounded above, it is bounded above. Why it is bounded above? Where are the elements x. You are picking up elements of the interval a,b. So, x cannot be bigger than b. They are all inside a,b. So, they are bounded above by b. Is bounded above by b.

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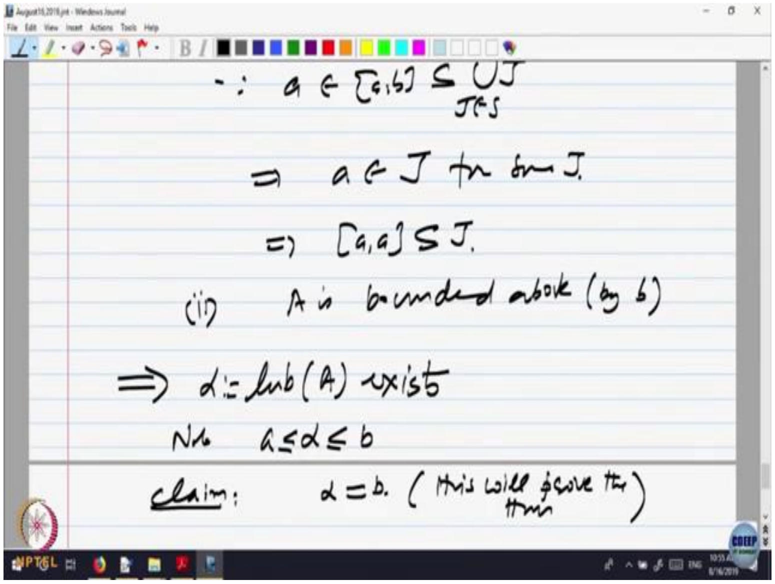

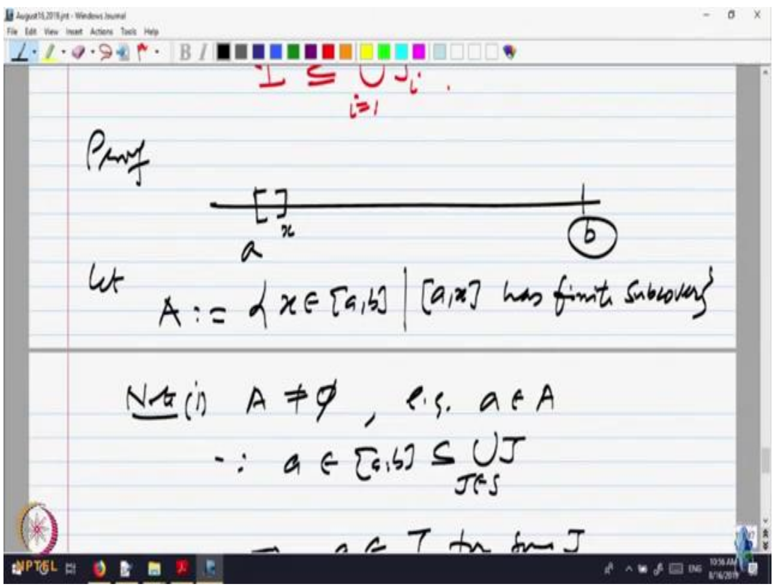

So, we have got non empty set which is bounded above implies lub of A call it alpha exist. lub exist, by the completeness property of real number it is a non-empty set which is bounded above so lub must exist. Note alpha is between a and b. Because what is, b is an upper bound, b is an upper, b an bound so least upper bound cannot be bigger than b. It has to be less than or equal to b and all element are bigger than a. So, this is between a and b, alpha is between a and b.

Claim that alpha is equal to b, that will prove the theorem. This will prove, will that prove also? Because if a is equal to b then a,b will have finite sub cover. So, I have to only show that this least upper bound, so look at the picture that is what I said that a to x has got finites cover, finite sub cover. Pickup all this x's and try to take the largest possible x with that property and we try to show the largest possible is has to be b it cannot be something smaller. So, a to b will have a finite sub cover. So, that is the only thing to show that it has.

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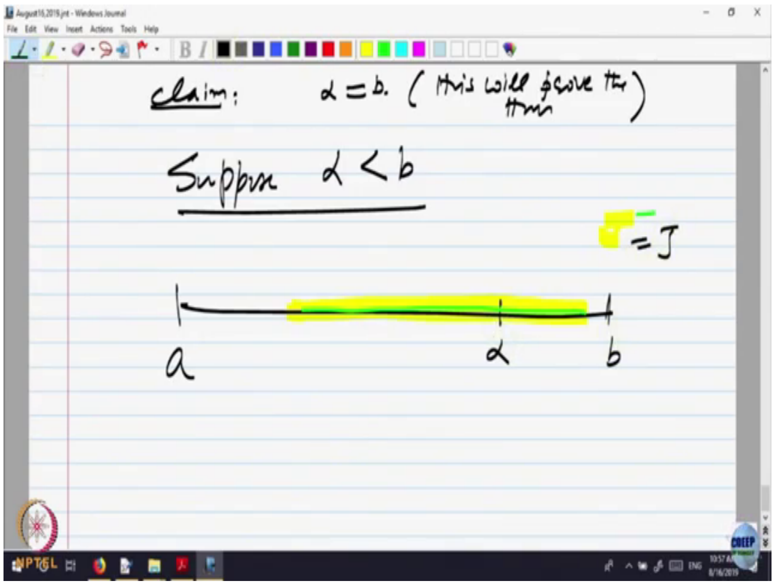

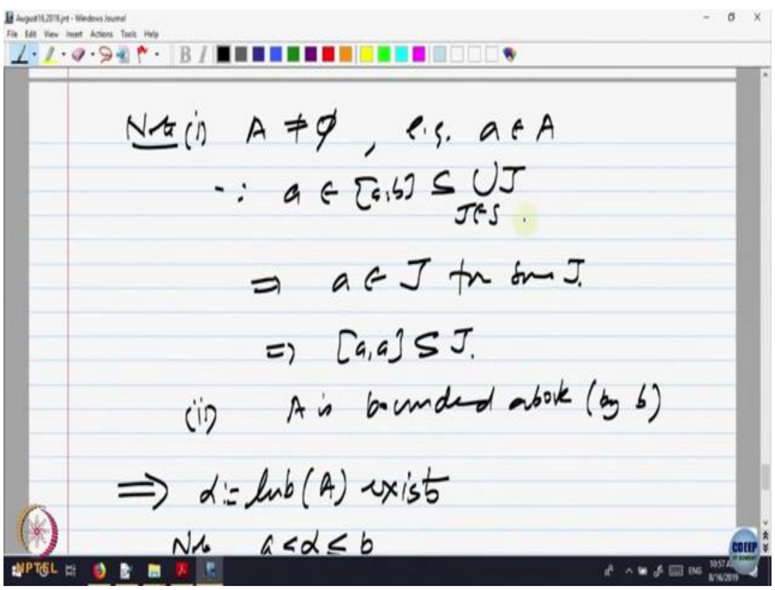

So, let us prove that and most often than not such proofs goes by contradiction. So, suppose alpha is strictly less than b. So, here is the picture a here is b and here is my alpha, here is my alpha. Claim that alpha is less than b that is a assumption, we will try to prove a contradiction that it cannot be strictly less, it has to be equal to b. Now if it is strictly less than b it is in the interval a,b then it is in the interval a,b and a,b is covered by those open sets J. So, alpha will belongs to one of the J's. So, let me just draw a picture. So, here is my something, this is my J this yellow one is.

Looks very bad, so this yellow one is J. That is a open set because a,b is covered by J's open sets J's and alpha is inside a,b so alpha will belong to one of them. If it belongs to one of them, it is an open set so there will be open interval which will include alpha. So, call it as something c to d. So, what I am saying is there is an open interval c to d which includes alpha. Now if this open interval includes it then there must be some point beta on the right side of alpha.

There are many but there is at least one and on the left side c to alpha, alpha is the least upper bound of all those x's. So, there must be an x belonging to a, there must be x belonging to a which is on the left side of alpha. Because alpha is the least upper bound of those x's. Now look at the interval a to x, x belongs to a. So, finite number of J's will cover a to x, and this beta is covered by c to d which inside J this c to d, c to d is inside J, J was an open set.

So, one more J if I include that covers beta also. So, that means A to beta is covered by finite number of J's. Because a to x is covered because x is in a. So, by the definition of a, a to x is covered, and beta belongs to c to d which is inside a J. So, if I take that J also earlier finite plus one more, they cover alpha to, a to beta. But beta is bigger than alpha, beta is bigger than alpha and we said alpha is the least upper bound of those which are covered by finitely many. And we have found another something bigger than alpha which is beta, which has the same property that means beta must belong to a.

That is not possible because alpha is least upper bound of a that is a contradiction. Simply nothing more. So, probably repeat it next time it is very easy to understand and alpha is the least upper bound. So, assume alpha is less than b, so this alpha must belong to one of the Js, and alpha is an open set. So, you must have a open interval including that point inside. So, c to d is the open interval including alpha. On the right side pickup any point beta on the left side there has to be a point a because alpha is the least upper bound.

So, a to x is covered, x to beta is covered by this. So, this finitely many cover a to beta that is a contradiction to the fact that alpha is least upper bound of this, and that proves the theorem. So, probably I will repeat it next time again the proof, very simple nice proof. So, what we are saying is that closed bounded interval a to b has the property every open cover has got a finite sub cover and next lecture we will prove the converse is also true. In the sense that for a compact set this is one of that equivalent ways of saying, every open cover has a finite sub cover.