Basic Real Analysis
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Lecture 14
Topology of Real Numbers: Limit Points, Interior Points, Open Sets and Compact Sets
Part 2

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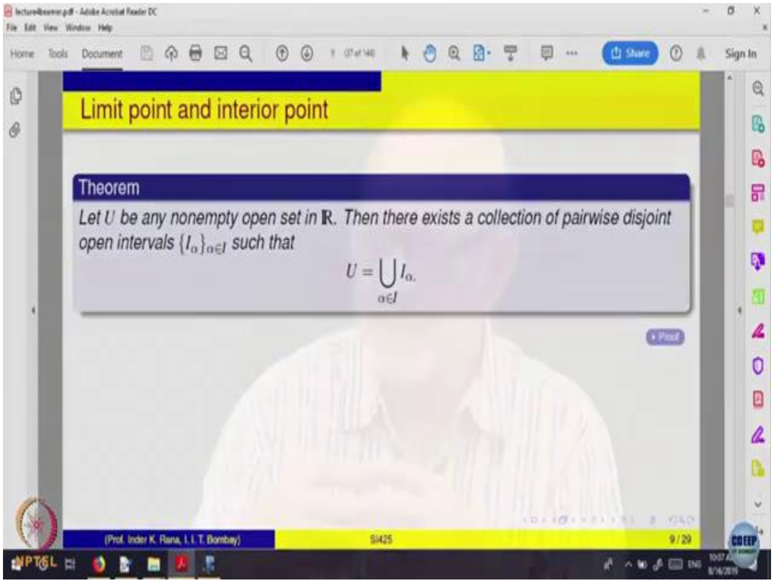

So what we have done, we have looked at basically properties of various kinds of sets. We started looking at a set, the limits of sequences may not be inside. So, we looked at sets, which are called closed sets, what are closed sets? Where the limits of sequences of elements of that set are also inside. So, that we call as a closed set, then we looked at A set may not be closed but you can have something called the closure of a set. So, A is always subset of a closure and closure is the smallest closed set which includes A.

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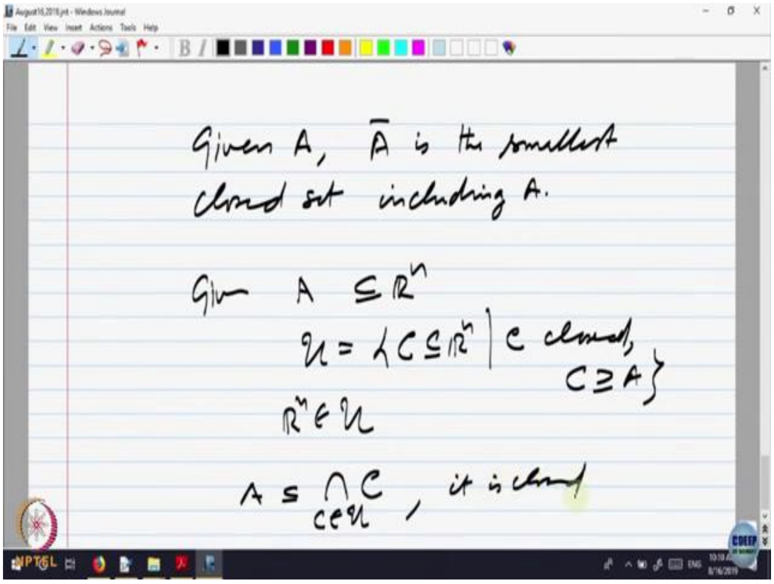

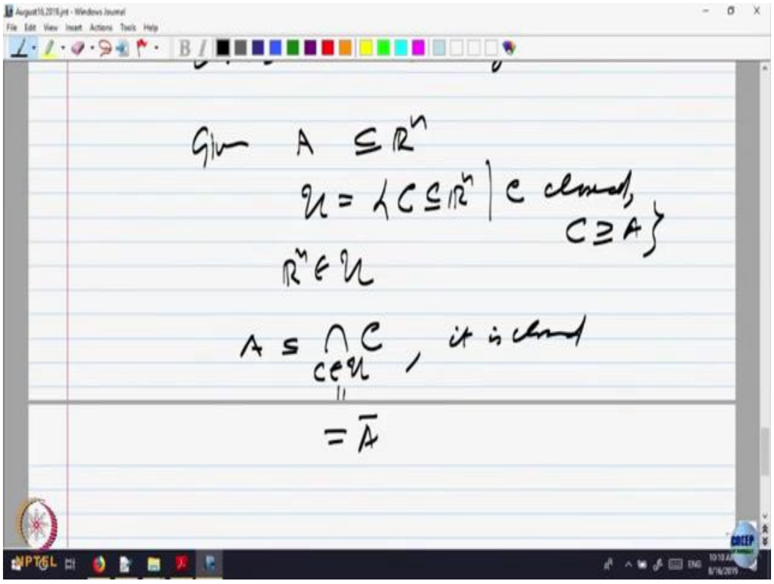

That is interesting, I think I should say that something more about it. We proved a result that given A, A closure is the smallest closed set including A. Remember this we have proved that, so this is basically by proving that A closure is always a closed set, A is inside it and closure of closure is itself. So, that is the smallest. But now, look at this, given any set A, A given is contained in say whole space, say let us write, say Rn.

A may not be closed but it is contained in Rn which is a closed set, the whole space is closed. So, let us collect together all set C in Rn C close and C includes A. So, look at the collection of all closed sets in Rn which includes C, at least we have got one candidate, this set is non empty because Rn is member of, Rn belongs to U. Because Rn is closed and it includes A.

Now if I take the intersection of all set C belonging to U, look at all the sets which are in this collection and take the intersection. What can you say about this set, intersection? That will be a closed set because arbitrary intersection of closed sets is close and each one of them includes A. So, A is subset of this and it is closed.

Like (())(3:13) this is smallest one, because we have taken the intersection of all. So in fact this is nothing but this is equal to the set is equal to a closure. So, this is another way of saying what is a closure. Look at the intersection of all closed sets which included that will be smallest and that is closed, so it has to be a closure. So, that is another way of defining a closure, you will find it somewhere.

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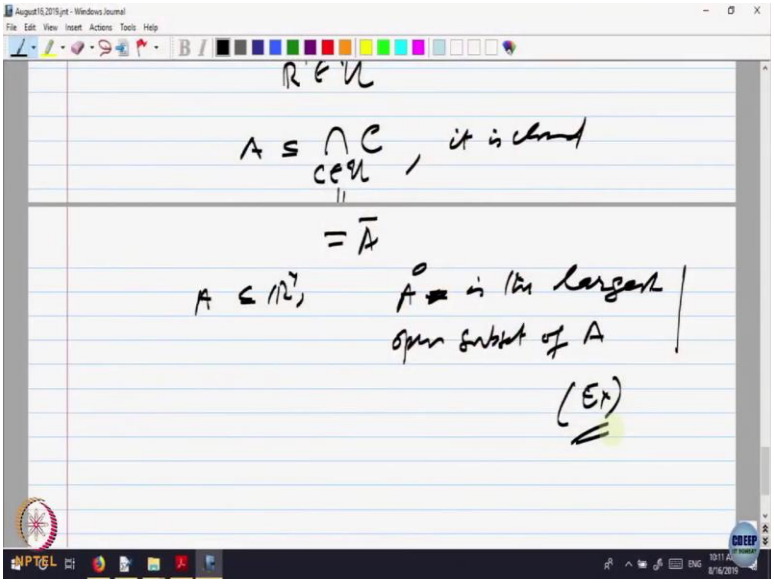

Now can you say what is A interior? Given a set A, can a similar thing be said about A interior? Try to make a guess. Given any set A contained in Rn, what you can say about A interior? It is the largest. Interior is always an open thing inside, but empty set is open. So, I should try to make it bigger.

So, try if you like to proof it, A interior is the largest, you can deduce it from here also if you like is the largest open subset of A. So, what we are saying is look at all open subsets which are inside A, take their union, that must be A interior. So various, there is nice playing around with sets and definitions. So, I am leaving this also as an exercise because we do not really need it but it is good, it will help you to understand interior points and so on and open sets and so on.

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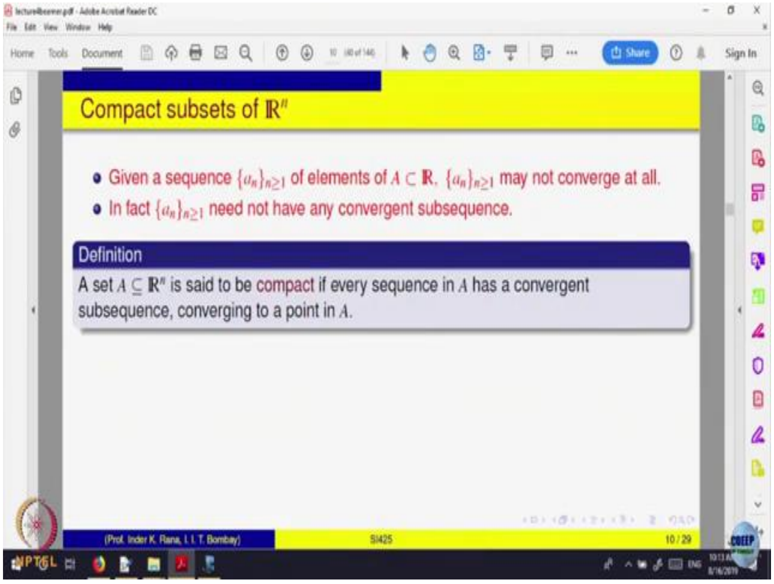

So let us see, now let us come to a very important concept. We started looking at sets whose limits may not be inside, limit of sequences but there can be sets, which have no conversions of sequences at all. There could be sets, which does not have any conversions of, conversion sequence at all. But, possibly there is a conversion subsequence of every sequence. There are no conversion sequences but every sequence has a conversion subsequence.

There could be sets with that property. So, they turn out to be very important collection. So, we say this is a such a set is called compact set. A set is called compact if every sequence, the sequence may not converge at all but it should have a subsequence converging, where, to a point inside the set. That is a condition are you understanding what I am saying, yes, subsequence converges and the limit is inside that set.

Student: Why the sequence is called convergent?

Professor: Well look at the sequence minus 1 to the power n, that sequence is not convergent. Every sequence need not converge. It may not have any convergent subsequence at all also, look at the sequence 2n, neither the sequence converges nor any subsequence converges. So, convergence of a sequence is independent of anything but some sets have that property.

There are sequences, some of the sequences may converge, some may not converge. Those who converge collect the limit points and put them together in a box we called it a closure and so on. But now, we are looking at those sets which have property given any sequence, it should positively have a convergent subsequence, at least one.

So for example, let us look at examples, look at, but convergent to a point inside the set, that is important, keep that in mind. Look at the closed interval say 01. Look at the close interval 01. Take any sequence in that close interval, take any sequence in that close interval. We proved a theorem that given any sequence there must be a monotone increasing or monotone decreasing subsequence.

So given any sequence in the close boundary interval 01, there is a subsequence of it of the sequence, which is monotonically increasing or decreasing and it remains between 0 and 1. So it is bounded, so the close interval has the, close boundary interval 01 has a property that every sequence has a subsequence which is monotone and bounded, so it must converge by completeness property. So, the close bounded intervals have that property that we are looking at namely every sequence has got a convergent subsequence converging to a point inside that set. So, any close bounded interval AB is a compact sub set of the real line by this definition.

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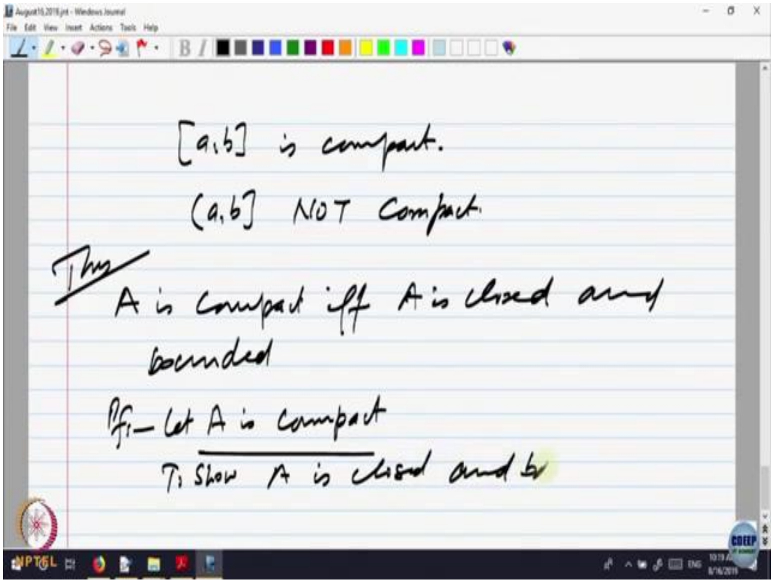

So, I am giving examples now. So example 01 is compact, It is a compact set because, given any sequence by our earlier theorems on sequences, it has a monotone sequence and it remains between a and b, so it is bounded, so it must converge. So, every sequence has a convergent subsequence.

Let us try to look at, say a b. Let us take any sequence inside, let us try to copy the earlier proof. Where it goes wrong if at all or it works for this also. Take any sequence in the open a close b. Then it has a monotone sequence, subsequence. It is bounded by a and b, so it must converge, that is up to completeness but the limit may or may not be inside open a and closed b.

Because earlier for the subsequence every term was between a and b. So, the limit could be equal to a or could be equal to b but that remains inside that interval. So, limit is inside here, the limit can become for example a, if you look at the sequence a plus 1 over n, then that is going to converge to a, any subsequence that also will converge to a because the sequence itself is convergent, but a is not inside the set. So, this is not a compact set because we have produced not compact.

So, it looks like this, this close bounded interval seems to be a prototype of compact sets. So, let us try to prove a result, we guess a result now that a set in Rn is compact if and only if it is close and bounded. So, we want to prove a theorem. So, A is compact if and only if A is closed and bounded. So we are, let us try to prove, if we are able to prove then it is true, if not then we have a contradiction. So, let us try to prove, so let us say A is compact. So, this is given, so let, this is given, to show A is close and bonded.

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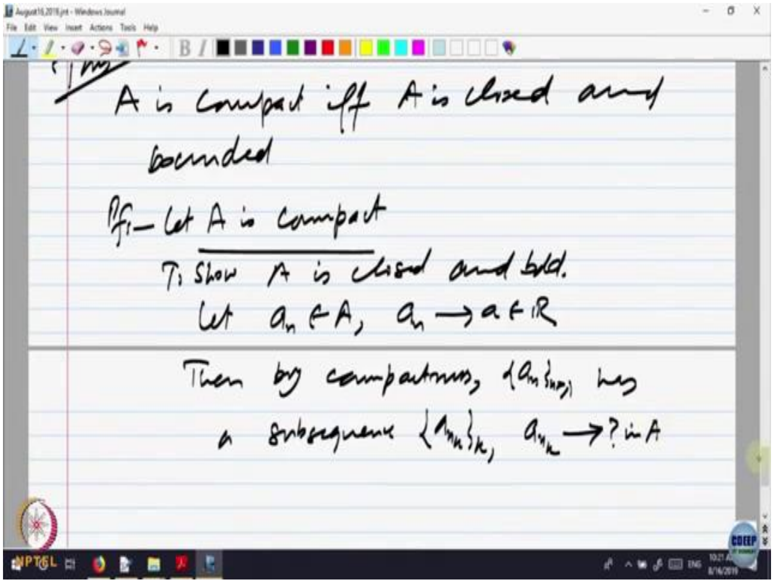

So, how do I prove A is closed, definition, what is saying A is closed, if a sequence of elements of A converges somewhere, then that must be inside the set A. So, let a n belongs to A, a n converge to a belonging to R. Then we want to show a belongs to, small a, element a belongs to the set A. What is given to us? A is compact and the compactness is every sequence has a convergent subsequence converging in the set.

So, a n is a sequence which is itself converging. By the compactness property it must have a subsequence which is converging but to a limit which is inside the set, but the sequence itself converges, so every subsequence has to converge to the same limit and that is in A because it is closed. So small a belongs to, so let us write then by code compactness a n has a subsequence, so let us write a nk, a nk converging to something in A.

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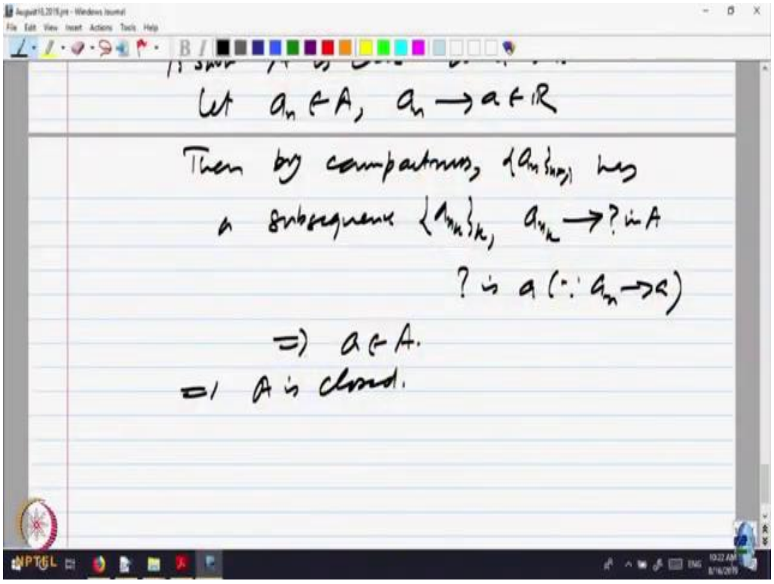

It only says, it has a convergent subsequence converging to something which is in A, but that something is a because a n converges to a and it is a subsequence, because it is a subsequence implying that a belongs to A. So we are using the fact, compactness says there is a subsequence which is convergent, but the important thing is compactness says that the limit must be inside the set. On the other hand, the sequence itself is convergent, so every subsequence must converge to the same limit and the limit being a, a belongs to capital A, so A is closed, so A is closed.

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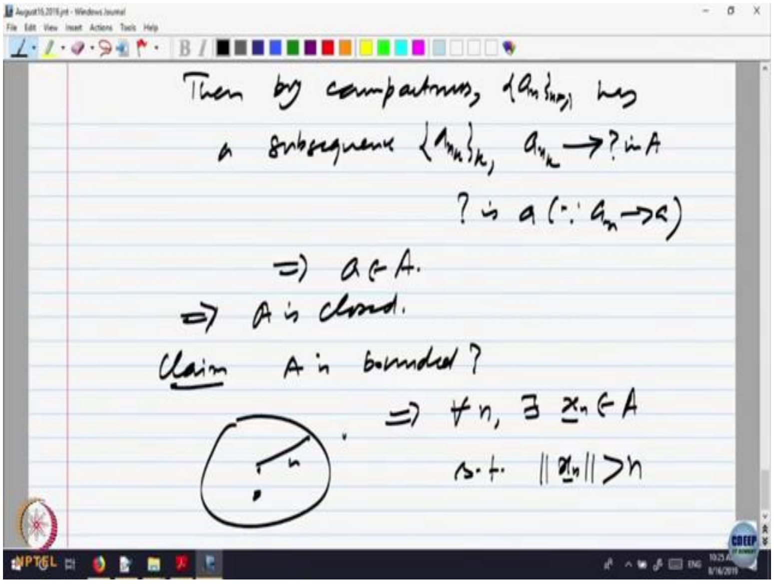

Next I should prove that A is bounded. If A is compact it must be bounded. If not, if it is not bounded what will happen? If it is not bounded, it is a subset of Rn, subset of Rn. What is, when do you say subset of Rn is bounded? We are not defined it as such, for real line we defined bounded, that it is between alpha and beta. When would you say a subset of Rn is bounded?

Because you can go in any direction, so there is a ball which includes A. Is that a good enough definition, we will say A set is bounded, if there is ball of some radius, you can take it centre at origin or it does not matter which is inside. So, if you want very precise, you can say, we will say set A is bounded if there is some radius r, say that the ball centred at origin of radius r includes the whole set A.

Good enough, so let us take that as a definition of bounded. Now I want to proof if a set is compact it must be bounded. So our definition of bounded is there is a ball which includes A, so if it is not bounded what will happen? Whole of A is not inside any ball, whatever ball I take there is something which is outside. So,, what will be the distance of, so if I take a ball of radius n, n natural number then there is a point outside.

So that point outside how much away it will be from the origin? Distance at least bigger than n, because it is outside, it is a ball of radius. So, there is a ball of radius n at origin. So, this is n and if this is a point outside what will be the distance of that point? It will be greater than n. So, saying that A is not bounded implies, so I am saying this implies for every n there exist some x n such that norm of x n is bigger than n. Just what I was discussing, we have just written it out.

Student: (())(17:40)

Professor: That, so can this sequence x n have a convergent subsequence? Because it is a sequence in A and A is given to be compact. So, by that compactness, you should imply this should have convergent subsequence. But can this sequence xn have a convergent subsequence? It is going away and away, the points are going away and away from 0. It is, or you can say the sequence x n is unbounded. Norm of x n is bigger than n, so as n becomes larger, actually this is becoming larger and larger.

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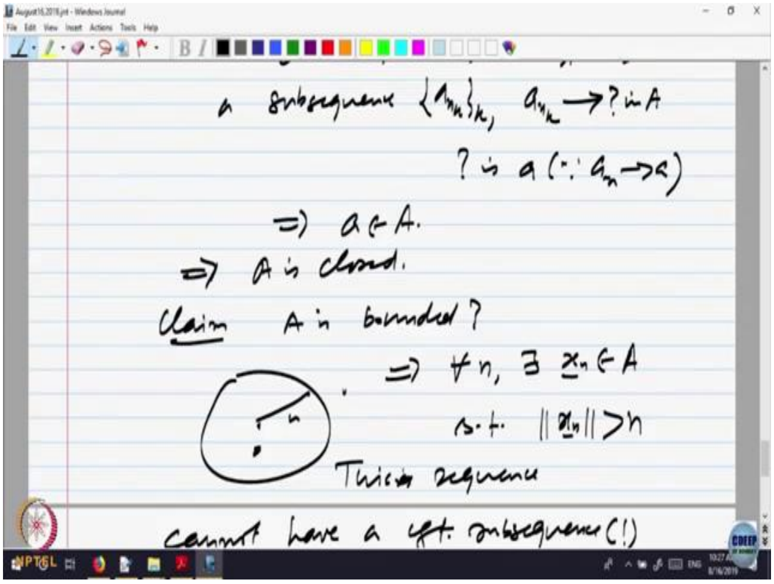

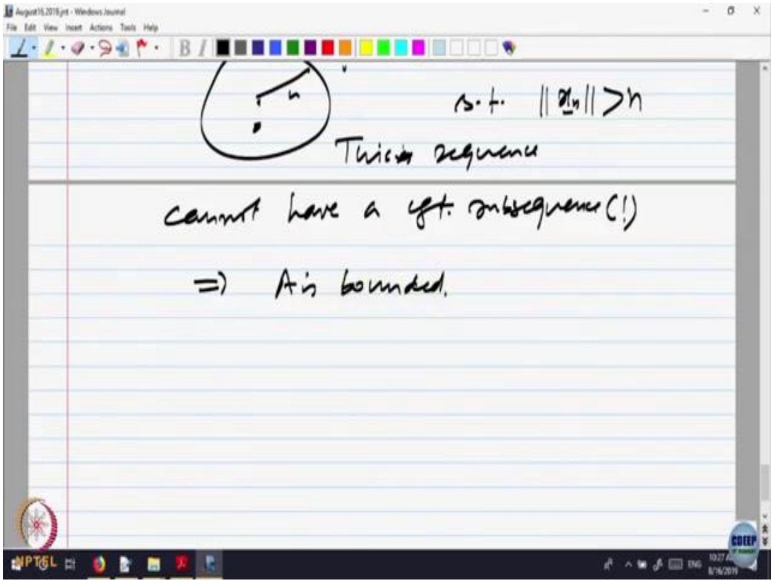

So, this does not have, this sequence cannot have a convergent subsequence. If you find visualizing in Rn to be difficult, you can in mind your real line. You got points on the line which are going away and away from 0. So, sequence is unbounded and convergent implies, that is a necessary condition, every convergent sequence should be bounded. So, this is unbounded, so cannot converge by that also if you like.

So, saying the sequence cannot converge because it is unbounded. So that means our that assumption, there is something outside every ball must be wrong, that means the whole of A must be inside one particular ball. So implies A is bounded.

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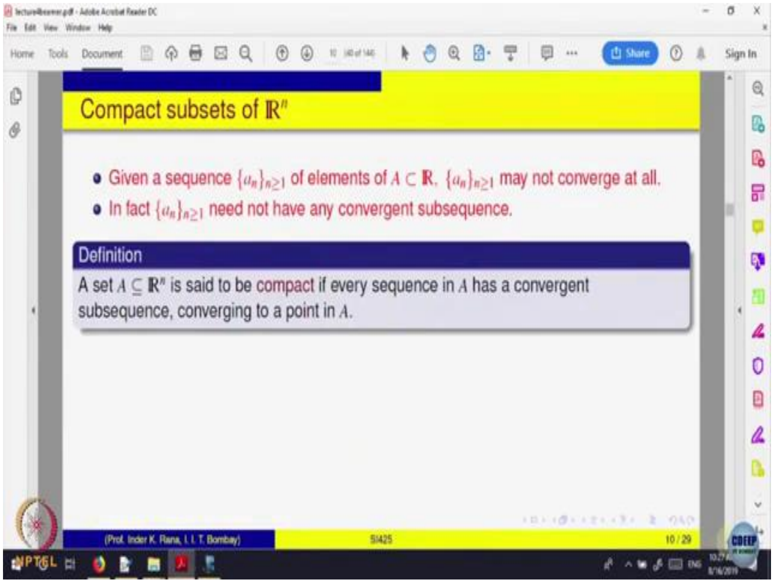

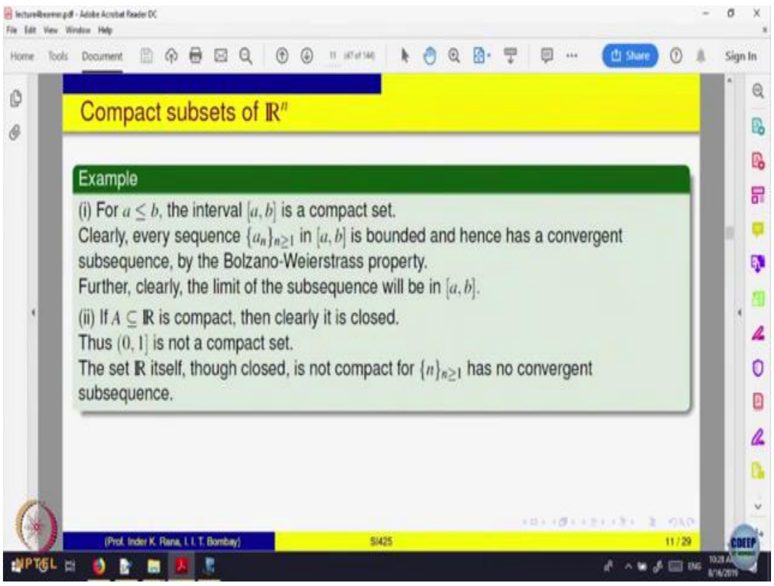

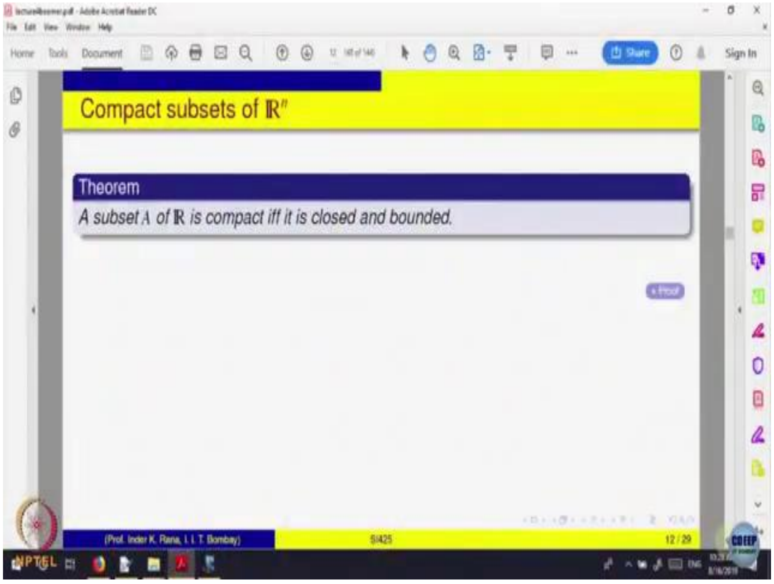

So what we have done, we have proved the theorem, that a subset of Rn is compact if and only if it is bounded and closed. That a less than b that is closed interval, that is compact, then it is closed. So the whole space is not, R or Rn is not compact, real line is not compact because it is closed but it is not bounded if you like, well same reason Rn is not compact. What way it is useful, so I will let us come to, it is closed and bounded that we have proved.