Basic Real Analysis
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Lecture 13
Topology of Real Numbers: Limit Points, Interior Points, Open Sets and Compact Sets - Part I

Right, so let us start by recalling what we were doing.

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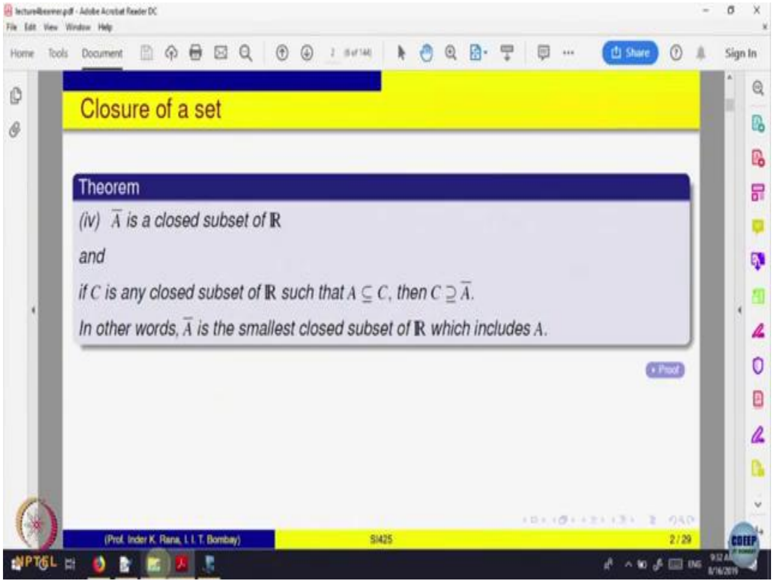

We were looking at the closed subsets of Rn, then we looked at what is called the closure of a set and we proved the result namely, A bar is closed, A bar is the closure of set A, and it is the closed set and in fact we have proved that is the smallest closed subset which includes the given set. Next, we look at some more properties of special points of sets, so let us start looking at what are called.

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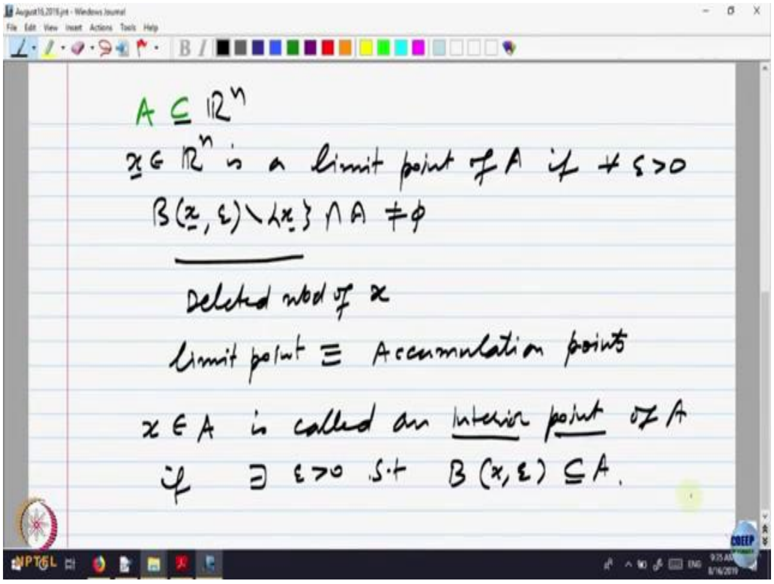

So, we are given a set A contained in Rn, and we say point x belonging to Rn is a limit point of A if, we say it is a limit point of A if for every epsilon bigger than 0, if we look at the ball centered at x of radius epsilon minus the point x that must intersect A, so that is non-empty. So, whatever ball you give at the point x, it should intersect the set A at a point possibly other than x. Because we are not saying x belongs to A or not.

So, in case x belongs to A, then every point will be a limit point, so we are not, we are looking at points which are such that. So, this is what is called, such a thing is called a deleted neighborhood of the point x, from the ball, we have removed the center. So, that is called the deleted neighborhood. So, a point is a limit point if every deleted neighborhood of that intersects the set A, such points are also sometimes limit points is also some people call it accumulation points.

Let us also define a point x which belongs to A is called an interior point of A, if there exist some epsilon bigger than 0 such that the ball centered at x of radius epsilon is inside A.

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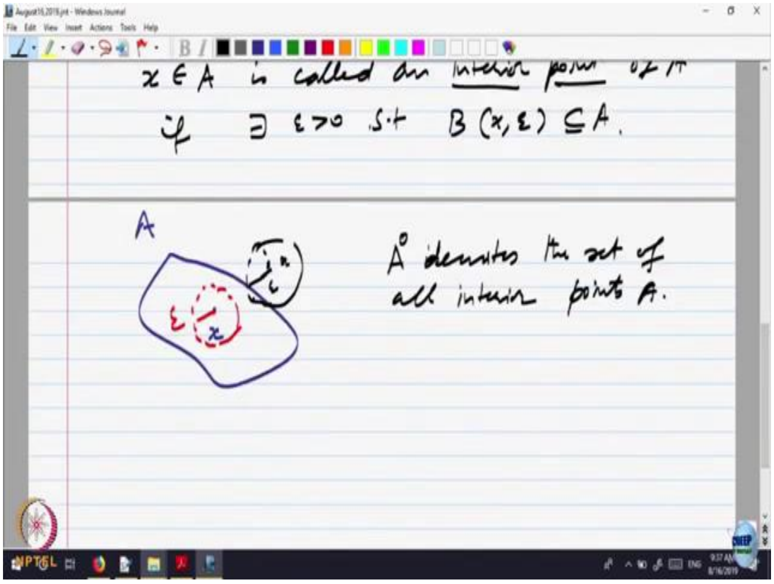

So, if you are looking a kind of picture, it says that, so if this is a set A, then a point is a interior point, if there is some radius epsilon, such that the ball, open ball around it of radius epsilon is completely inside, and we say x is a limit point, so this is x is a limit point if I take any ball of radius epsilon, then it should intersect the set A at some point, so that is called the limit point.

So, let us have a notation for, A0 upper 0 is or denotes the set of all limit points, interior points of, not limit points, interior points of the set A.

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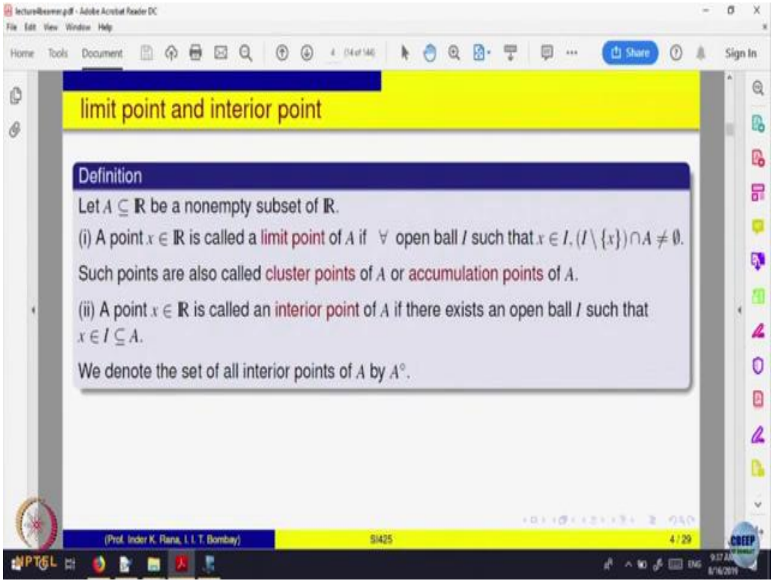

So, here are some obvious properties, so let us look at those properties, so interior points and limit points, okay. So, a point is a limit point if the open ball I have called it “I” here, minus the center intersects A for every ball every open ball, whatever the radius we have and interior point if there is some ball which is inside centered at x which is inside A, so that is a interior. Clear?

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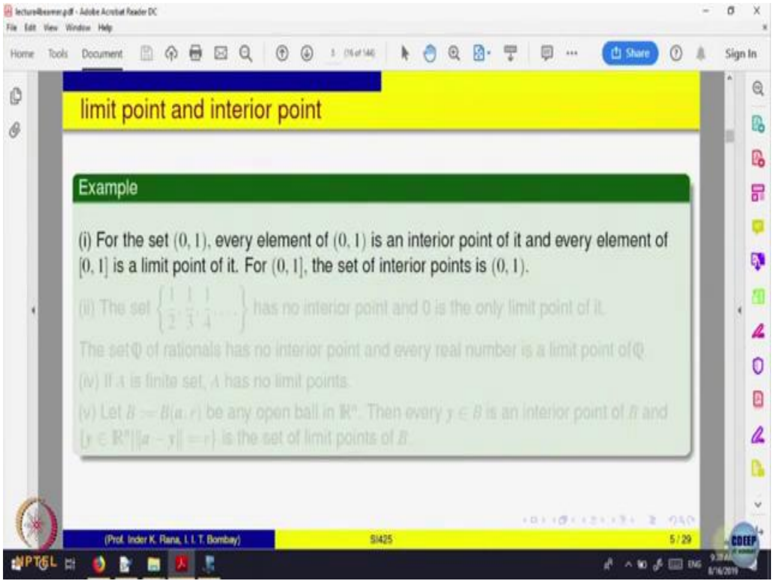

The relation between these things, okay, some properties of some sets, for example if you take the open set interval 0, 1. Then every point is a interior point. Because we take a point x it will have some distance from 0, some distance from 1 look at the smaller of whichever it is, so that open interval will be inside the open interval 0 to 1, okay?

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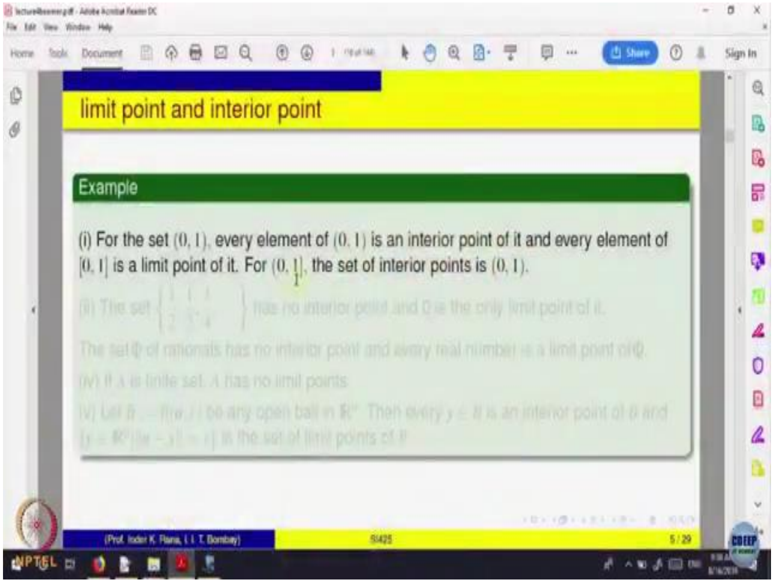

If you look at open 0 and close at 1, then 1 is not a interior point. Because at 1 if I take any ball that will go outside, okay? So, there is no ball centered at 1 which is completely inside, so 1 is not a interior point, but 1 is a limit point. Because whatever ball I take. At the point 1, it will intersect it will have some point of 0, 1, okay. Look at the set, for example 1 by 2, 1 by 3, 1 by 4 and so on.

So, what do you think is a limit point of this? 0 is a limit point. Because if I take a any ball at center at 0, okay then 0 does not belong to A. So, as I removing from that ball, if I take a open ball centered at 0, any ball because this sequence whenever n is going to converge to 0, so after some stage all the points will be inside that ball, so whatever the radius of that ball maybe, that is going to intersect A minus the center of course, so that is a limit point.

Can you say what are the interior points of this set? Interior point has to be a point of the set first of all, so can 1 by 2 be a interior point? No, obviously not because I can find a. Whichever open ball I take around 1 by 2, it will go outside the set 1 by 2, 1 by 3, and so on, so there is no interior point, interior is empty.

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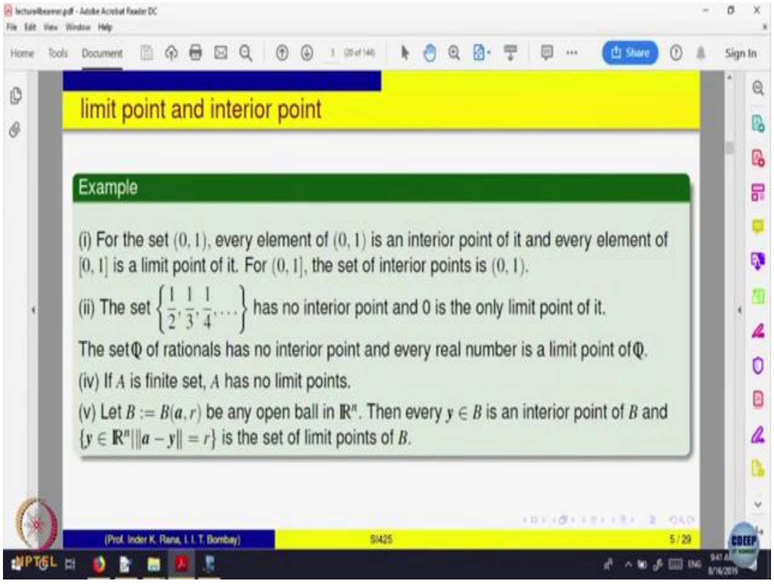

So, similarly, you can show that set of rationals. There is no interior point for set of rationals, what about limit points? Limit point is a point in real line such that whichever neighborhood I take of that point, it must intersect rationals, and we know rationals are dense. So, that means whichever ball I take, it will positively contain a rational, so all real numbers are limit points for the set Q, okay?

So, let us look at probably one of the last ones, if you take any open ball centered at in Rn, I take a open ball, then every point is a interior point. You have pointed out yesterday also, given any point in the open ball, you can have a small radius, I will have that as an exercise so do it in the problem sessions that given any point in an open ball, okay.

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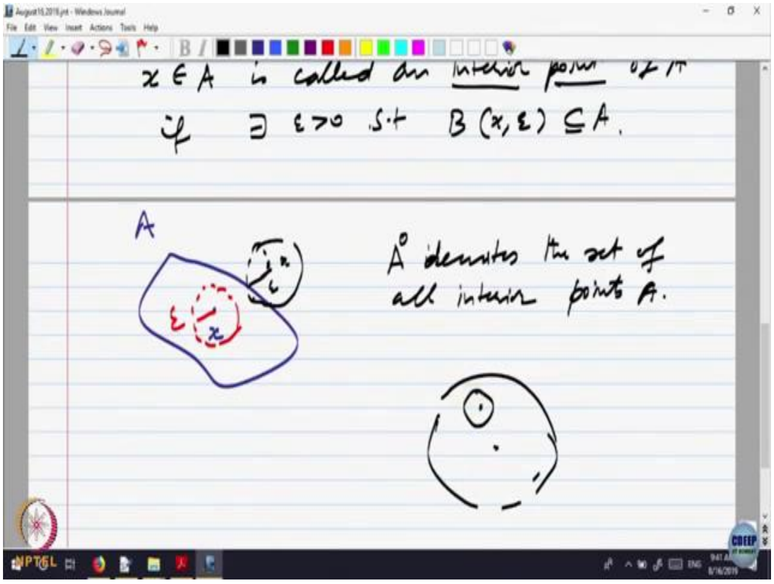

For example, if I take a open ball like this, this is centered at some point if I take any other point, then I can find a small radius, what that radius should be, you have to figure it out. That is not very difficult to figure out what could be that radius. So, every point is a interior point for the ball.

What are the limit points? For the open ball? Every point of the ball is a limit point anyway. Because whatever ball I take, at any point it will have points other than the center also inside A. Whatever the points on the geometric boundary the distance equal to if the ball is of radius r, so radius r all points at a distance exactly equal to r, they are also points of they are limit points, because if I take any ball at the boundary that will come inside also. Entire points inside.

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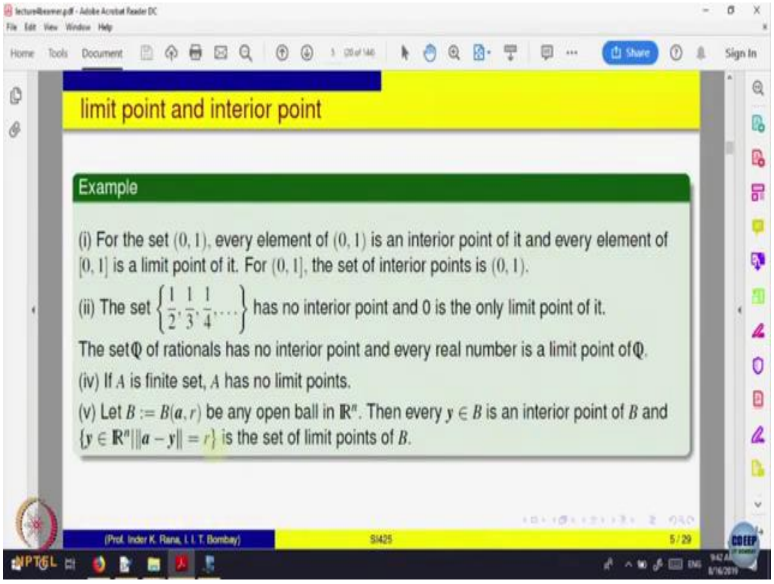

So, limit points are interior point and these are possibly limit points, and of course the ball inside also are limit points.

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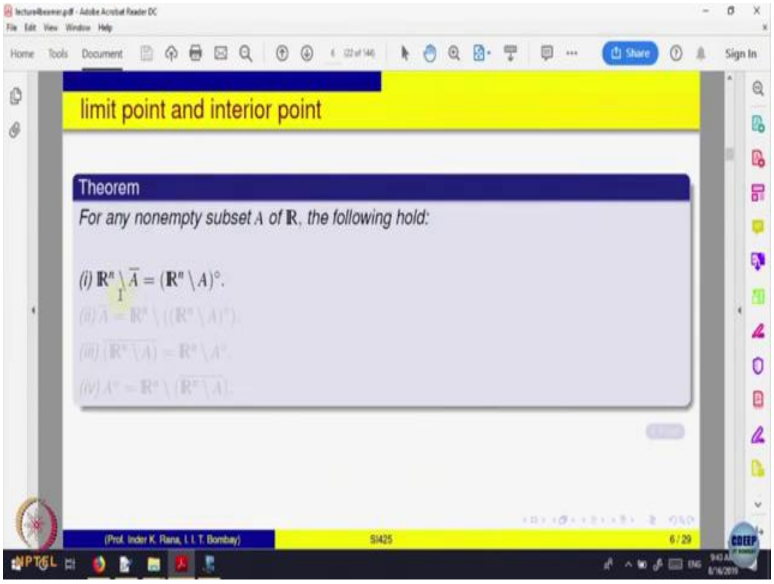

So, look at the complement of the closure, given any set I am looking at, Rn minus A what is that? That is the complement of the closure, claim is it is related to the interior of something, it is interior of the complement, okay?

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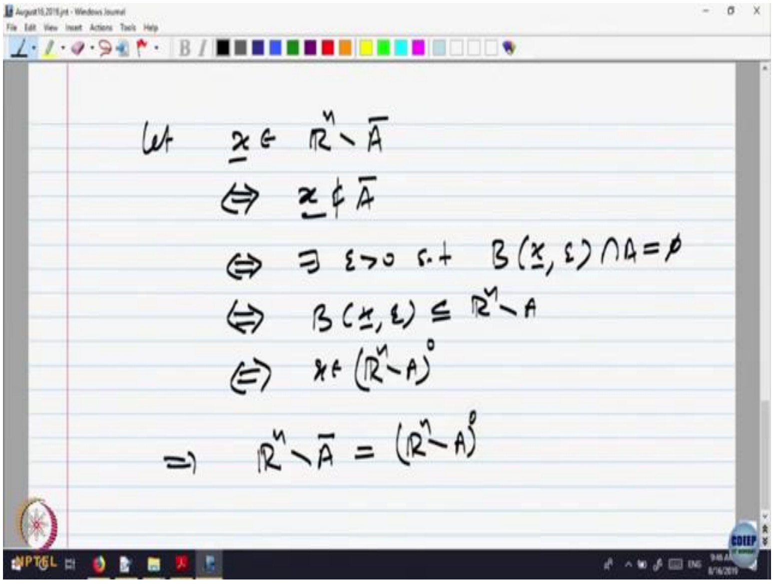

So, let us see why it is so, we are looking at the complement of, so let us look at so let x belongs to Rn minus A closure. So, what is the meaning of this? It is in the complement, that means x does not belong to A closure. It is in the complement of A, A closure. So, what does that meaning say it does not belong to A closure? What was A closure? It was a set of all points, such that if I take a ball, it should intersect A, every ball should intersect A, that is a closure.

Closure of a set over the points such that every ball centered at that point should intersect A. Okay? So, if it does not belong to closure means what, negation of that statement? That means there exist some epsilon bigger than 0, such that a ball centered at x of radius epsilon intersection A bar is a intersection A is equal to empty set, is that okay? That is by definition of a closure, but what does that mean, this intersection is empty so where is that ball? So, where is that ball? That is in the complement of A.

So, what we have shown, if x is a point here this is if and only if x is a point here, and what is the meaning of this? That x, the ball at any, at the point x there is a ball. There exist some epsilon such that the ball is inside that side, that means it is a interior point for that set. So, it says x belongs to Rn minus A interior, radius set theory nothing more than that.

Definition of the closure, so definition of the closure says, x belongs to the complement of the closure, that means x does not belongs to the closure, it does not belong to the closure means there is at least one ball centered at x which does not intersect A.

So, that is what we have written, intersect that A. That means it must be in the closure and that means this is a interior point. So, this says so implies Rn minus A bar is equal to the complement and take the interior, is it okay? Not doing much, just writing the definition and using the complement property.

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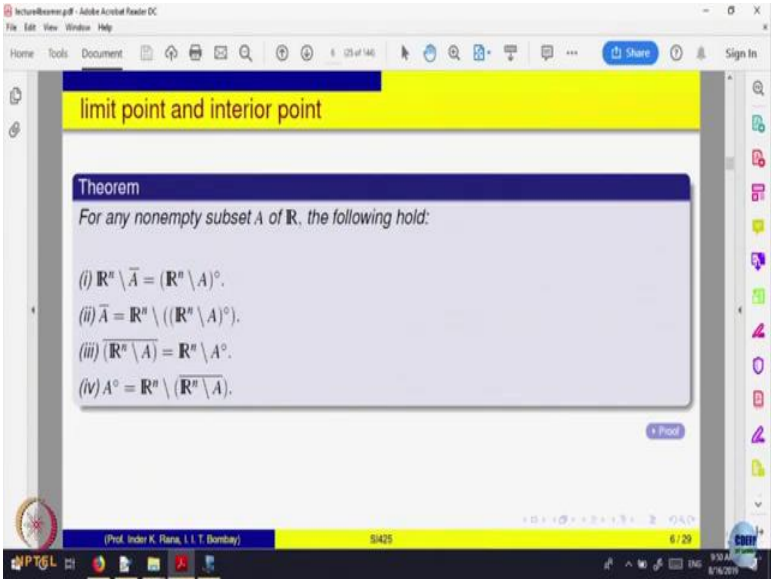

So, this is what we were saying here, the first one the complement of the closure is the interior of the complement, okay? If I take complement of this, what will I get? That means A closure is equal to complement of the right hand side, so that is the second thing, just by complements I get the second statement.

So, a similar statement, if you take here complement interior, here complement closure, okay that is equal to Rn complement of the interiors. Basically, same thing, definition. It belongs to closure that means there is a ball which should intersect A this set. And hence, go to the other way around.

So, just note it down and try yourself I have given you the idea of the proof of the first one, all remaining are same actually, there is nothing much involved in it, just definition of the closure and definition of the interior. Closure means every ball at that point must intersect that set, that is a closure. Interior means there is at least one ball at that point which is inside the set, okay. So, that is the interior.

So, using that, all these are easy to verify, first one we have already verified, other one you can do complements and for example, from here to here third to forth you take complement and you get that thing either by complements or by definition you can prove. So, now let us define, we will call a set to be an open set, till now we have not defined what is called an open set, a set is called open if it is equal to its interior points, given a set, set of interior points is a subset of it. With some property, okay.

So, what we are saying, a set is open the definition is, if all its points are interior points, A is equal to A interior. So, if A is equal to A interior, look at this thing, what does it give you? What does third give you? If I replace A by A interior, then right hand side is complement of A. So, what is complement of A? It is equal to closure of that set, if A is equal to A interior, what does the third equation tell you?

Third tells me that, the complement of the interior is equal to its closure, that means the complement is a closed set because its closure is itself, is it okay? In equation 3, if I say that A is equal to A interior, what does right hand side mean?

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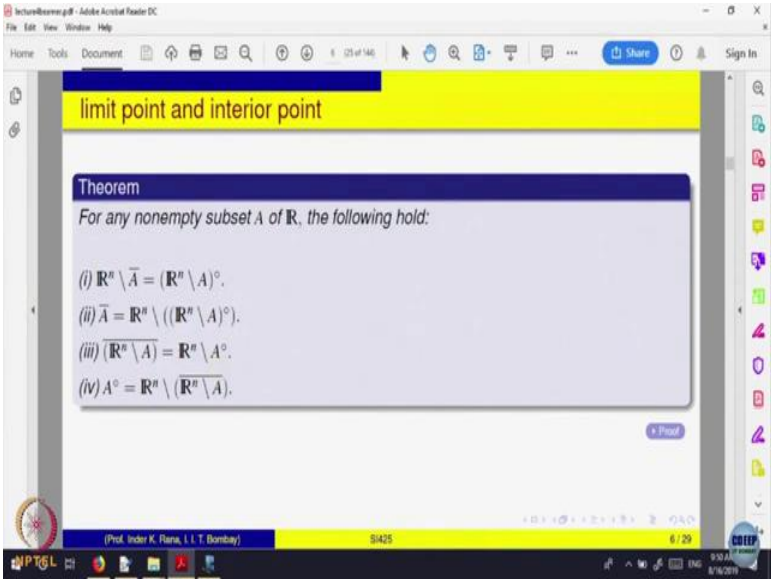

Right hand side becomes complement of A, complement of A is equal to the closure of the compliment. Same set, complement of A closure is equal to complement of A, what does that mean? Whenever for a set A is equal to A closure, we prove that is a closed set a set is closed if and only if A is equal to A closure.

So, a set is open if and only if its complement is closed, you get that property, from third you get that property, that is set is open if and only if its complement is a closed set, okay? Normally sometimes it is taken, this has taken as a definition, one defines an open set and takes closed set to be the complement of A. Closed set is defined as the sets whose complements are open, we are doing other way around, we have defined what is a closed set first, how is a closed set defined?

We defined what is A closure. The points which can be approximated by approached by limits of points in A, that was closure, then we said A is closed if A is equal to A closure and we also proved it is the smallest closed set which includes A, so that is the closed set, so we defined closed sets.

And now we are saying we define a set to be open if its interior is equal to itself, for example, open interval, what is the interior of the open interval? So, open interval that is why it is called open, that interval round bracket A, comma B is called open interval because it turns out to be an open set.

Every point is a interior point, and similarly if you look at the closed interval, what we call as a close interval A, B. Its closure is equal to itself that is closed. And what is the complement of that closed interval? That is minus infinity to A open union, can we say that is an open set? I know, each part is open so we are saying then we need to analyze something like if A and B are open, can I say A union B is open or not.

So, we have defined open sets, now let us look at the properties of open sets like we looked at the properties of closed sets. So, let us look at properties of closed sets.

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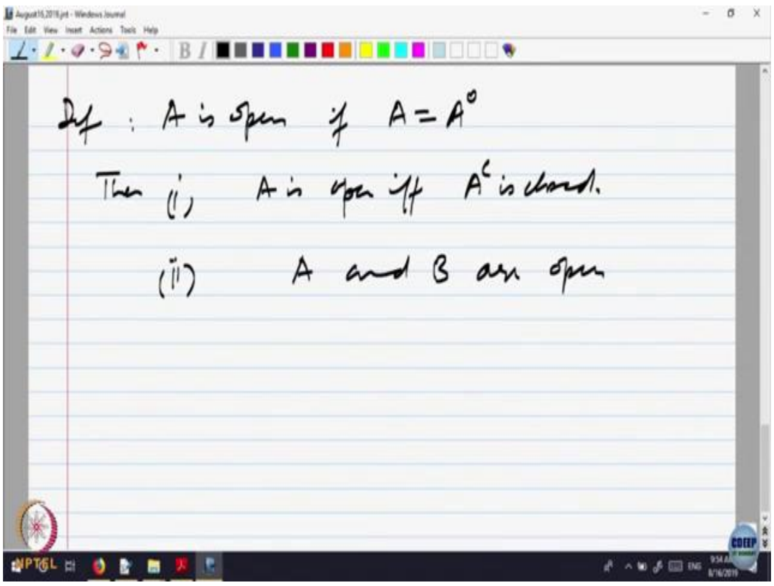

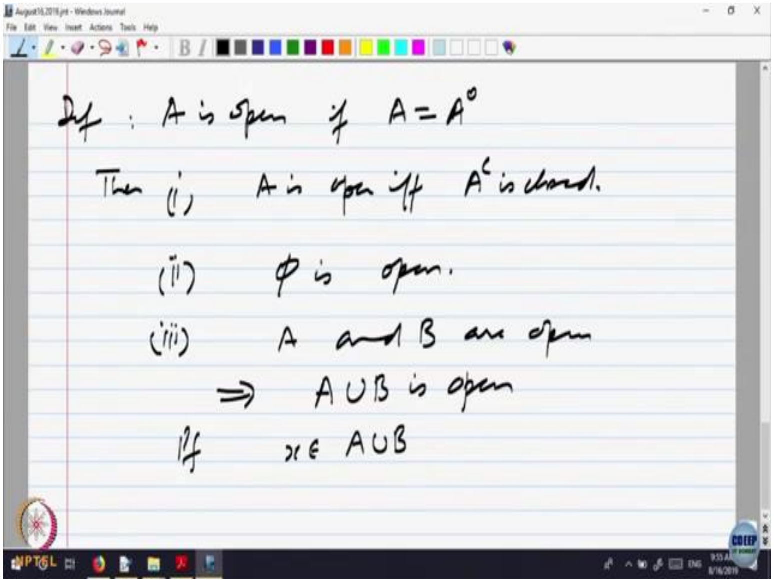

So, we defined A is open that was a definition we have said, A is open if A is equal to A interior and consequence is, so if this is a definition then A is open if and only if A complement is closed, okay? So, that way, okay. So, let us look at some properties, let us look at second, if A and B are open, what about, can we say that empty set is open set? Is there any problem if we say, so probably somewhere, okay.

So, before that, let us just write empty set is open, why is it open? You can look at complement I closed. If you want to say what is the interior of empty set? It is itself. There is nothing inside, so interior is equal to itself whatever way you want the say it is open, so let us look at the third that if A and B are open, then that implies A union B is open.

So, what will be an argument for that? If x belongs to A union B, I want to show every point is a interior point. Or if you can use your A and B and go to complements and then complements back and so on, but let us try definitions straight, if x belongs to A union B then either x belongs to A or x belongs to B, A is open, so there will be an open ball at x included either in A or in B depending on whether A is open or B is open. So that open ball will be inside A union B also. So, that no problem at all.

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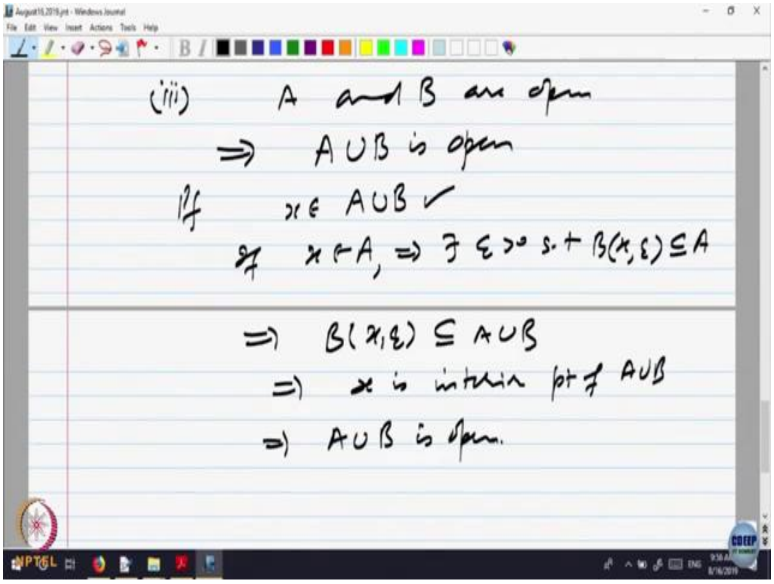

So, if x belongs to A, implies there is epsilon bigger than 0 such that the balls centered at x of radius epsilon is inside A implies the ball centered at x of radius epsilon is also in the A union B because it is in A, so implies x belongs to A union B. So, implies, x is interior A union B. So, every point is a interior, every point x is a interior point, so implies A union B is open. No problem?

What about if I add a third set C also? If something belongs to A union, it will belong to one of them that is important. Either A or B we are saying, so you can take arbitrary union also, it does not affect the proof at all, take arbitrary collection of open sets A alpha then their union is also a open set because if x belong to union A alpha, it belongs to one of them. And that being open there is a ball inside that particular one and hence in the union also. Obvious.

Keep in mind, properties of closed sets, we said arbitrary intersection of closed sets was closed, here arbitrary union because they go complements set is open if and only if it is closed, so keep that in mind.

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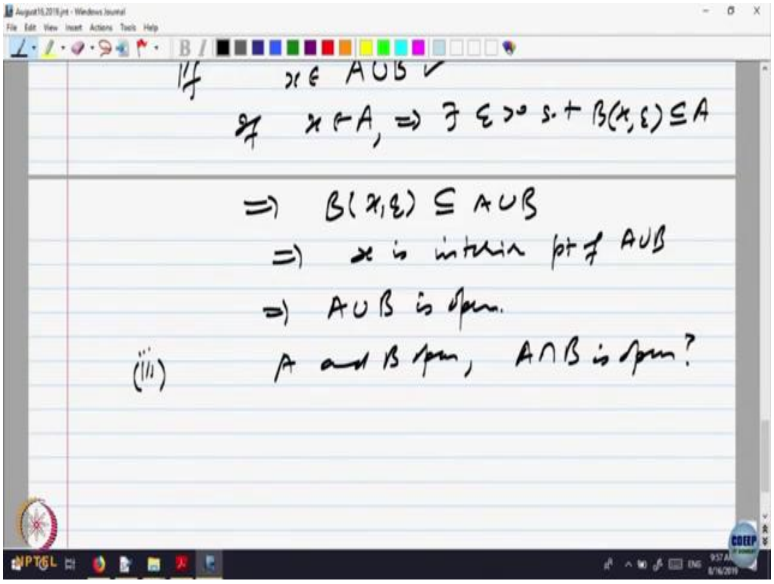

So, probably we should have another property about intersections, if A and B are open, can you say A intersection B is open? Yes, no problem, we can go by definition if x belongs to A intersection B, then it belongs to both of them. That means there is a open ball at x of some radius epsilon 1 contained in A but it also belongs to B which is open, so there is some open ball of some other radius epsilon 2 centered at x which is inside B, at same point there are two radii, one bigger one smaller, I can take the smaller one, that ball with a smaller radius will be in both A as well as in B, so it will be inside A intersection B.

So, every point in the intersection has a open ball contained in that set, so intersection is an open set, yes? Is it okay?

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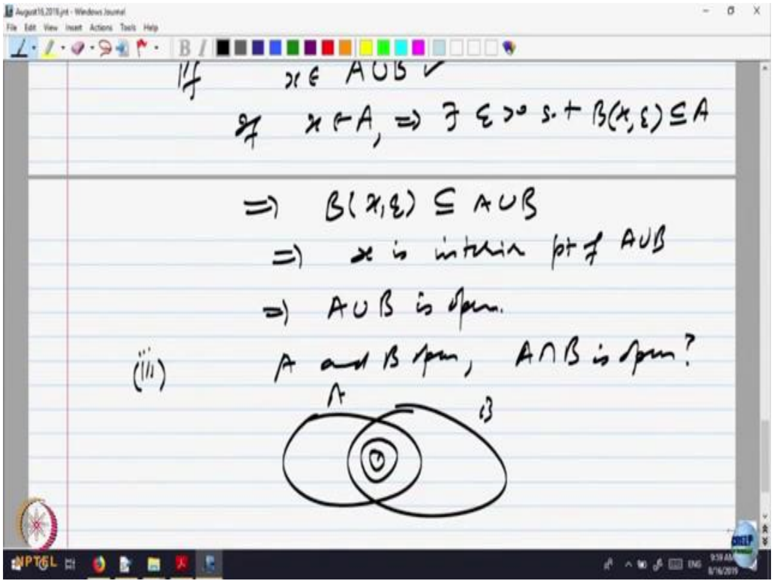

So, here is, if you like here is A, here is B, here is a point, so there is one ball like this, another smaller, so smaller one is inside the intersection anyway. No problem, or if you like to go by our definition, we have already proved a set is closed if and only if its complement is open. So, if A and B open, there complements are closed, finite union of closed sets is closed, we have proved that.

So, A complement union B complement is closed, but what is A complement union B complement? It is A intersection B complement, so that is closed, so A intersection B is open if you want to go that you can that route also you can go, or you can just go by definition, every point of the intersection is a interior point, so hence it is a open set. Once again, does not matter whether I take two sets or three sets or any finite number of them, same proof will work because the minimum of the two I was taking radii any finite need I can take the minimum of that, that will be inside any finite.

So, in finny finite intersection of open sets is also open, question can I say arbitrary intersection of open sets is open or not, so think about this a question. So, this goes to close sets also, we had property of closed sets also looked at.

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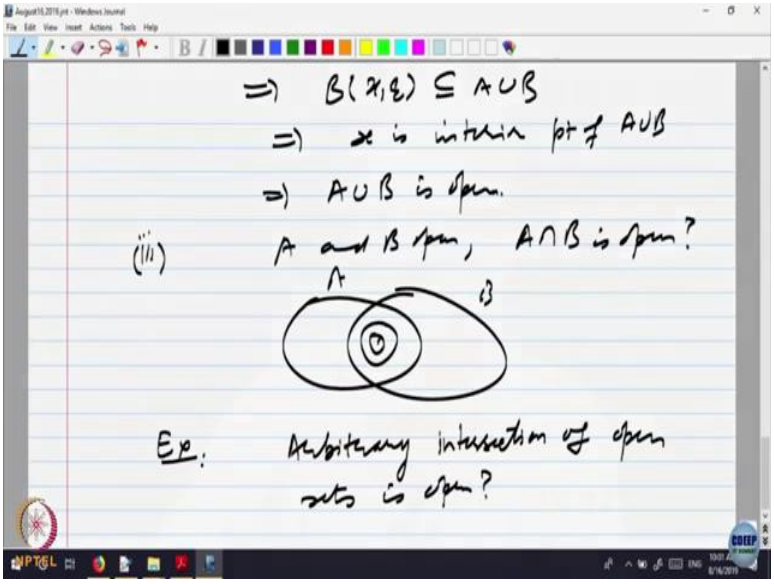

So, I am leaving this as a exercise, so this is okay, exercise arbitrary intersection of open sets is open, if you think it is true, you have to give a proof, if you think it is not true you need produce example of a collection of sets which are open. But there intersection is not open. Okay.

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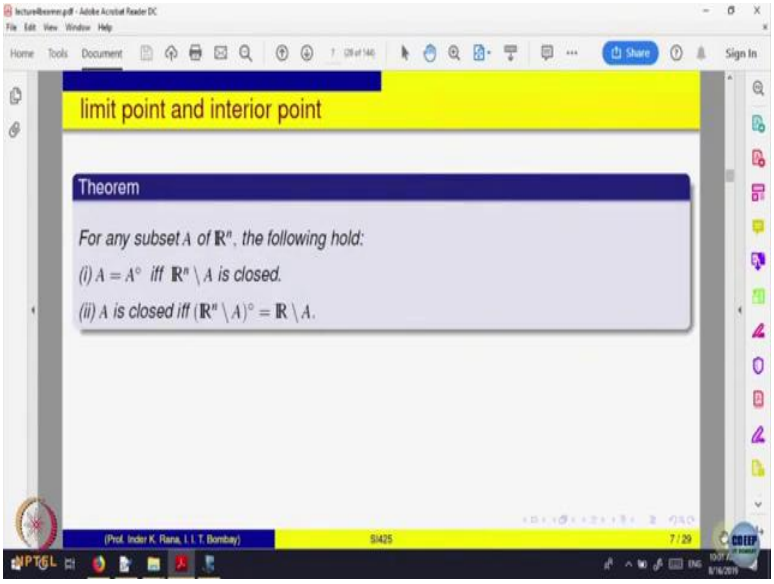

So, these are various properties of open sets, so let me just see if anything else is to be said, then we will let us do it.

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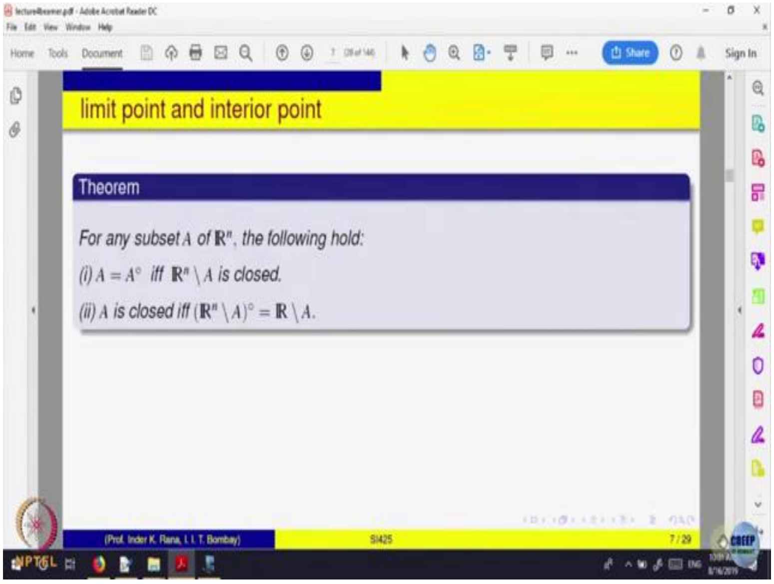

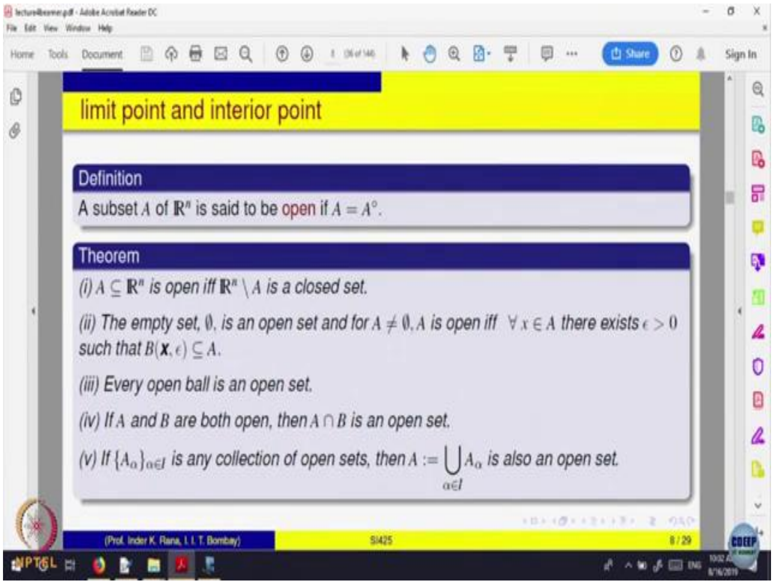

So, close if and only if that we have done, so that is the definition or we defined open sets and that is same as the complement is closed, so that follows from the previous theorem and here are the properties which we have just now proved every open ball is an open set, that is okay.

Like every open interval is open set, every open ball, every point is a interior point we discussed that, A and B are open intersection is open that you can extend it to finite number of sets, so property this you can extend it to finite number of sets, whether you can extend to arbitrary intersection or not that is the question left for you to analyze, union is okay, that we have proved. So, this is properties of open sets.

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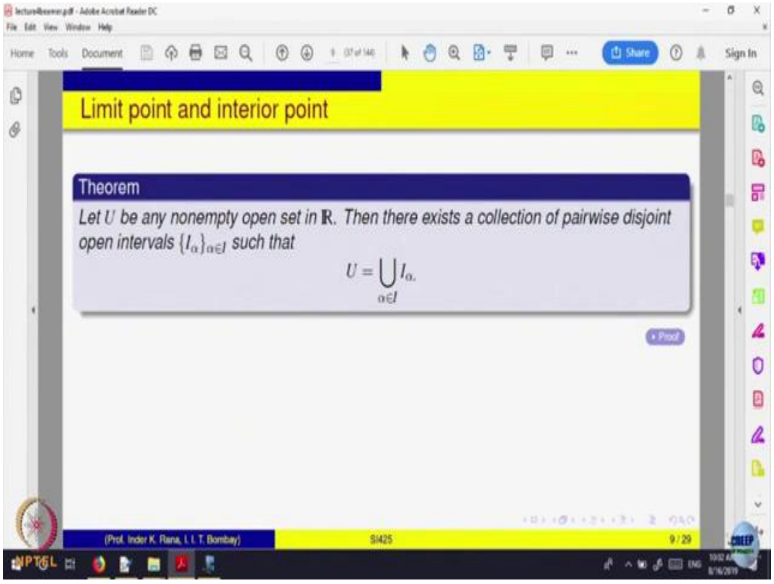

Here is something I will not prove that, but it is quite nice property to have it handy, let us say as if it characterizes open sets in the real line, it says a set is open in R, then it can be written as a disjoint union of open intervals, every open set it need not be a interval, every open set can be written as a disjoint union of open intervals. That means, open intervals are in some sense give you everything, all open sets by taking countable, by taking union.

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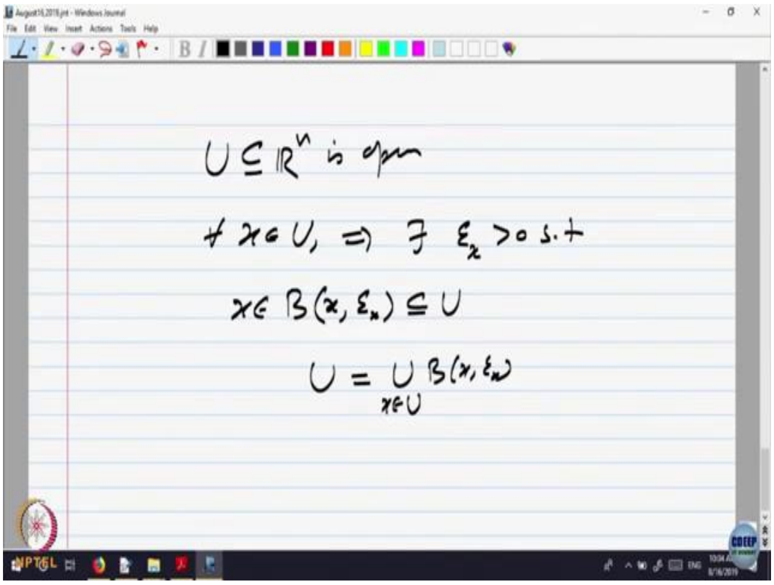

Anyways, why this is important or useful, let me just, say supposing U contained in let us say Rn is open, let us take any open set in Rn, okay? Then for every x belonging to U it is an interior point, so there must be a ball around that point inside, so implies there is some radius, let us call it epsilon x bigger than 0 such that the ball centered at x of radius epsilon x is inside U, x belongs to it. That is a center.

Then what is U, can I say what is U equal to, in terms of this balls? Do you agree that I can say that this is same as x epsilon x union over all x belonging to U, all balls are inside U so the union is inside U, but all points are inside the balls, so all U also is inside the union, so both are equal, is it okay for everybody? That this implies this, obviously because if I take union, union singleton's x that is the whole space U, whole set U.

So, U is inside the union of balls which is inside U again, so everything must be a equality. So, U is equal to. So, this says, an open set can be written as a union of open balls, reinterpret this, every open set is a union of open balls. So, if you interpret in the real line, that says every open interval, for example every open set is a union of open intervals. Balls are open intervals, now given two intervals, you can always make them disjoint kind of, by taking intersections, so that is the basic idea.

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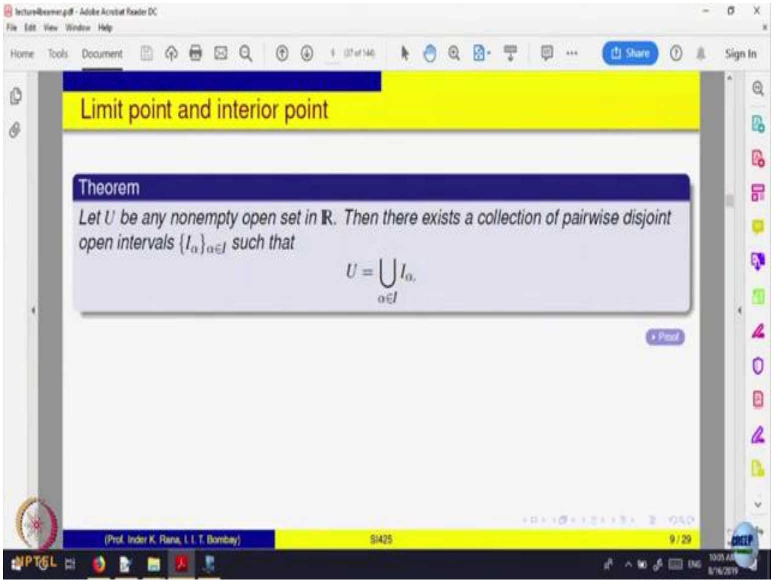

So, we will not prove that, so it says not only that, so in the real line you can write, not only as a disjoint union of open intervals, you can write as a disjoint union of open intervals, so that is a very useful thing. And if you do not, this indexing set I is not saying it is finite or countably infinite or what, some collection, actually one can show if you for go disjointness, then you can write every open set is a countable union of open intervals, then you can make it countable.

That property, you will not be using it but that is a beginning of something called a subject called, matrix species and you say real line is where everything comes from open intervals, every countable union, so you say it is a second countable space and such kind of thing.

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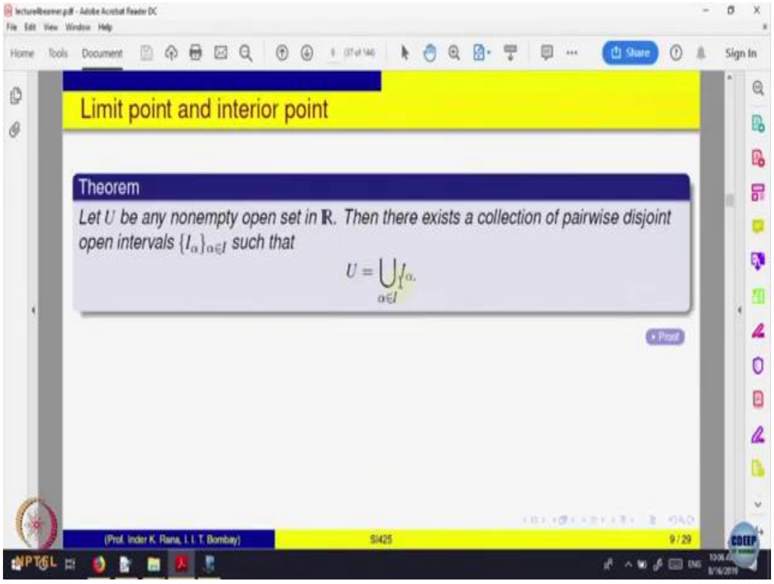

So, that is something separate, but this is useful sometimes or it is a beautiful result in itself saying that every open set can be decomposed into a union of disjoint union of intervals. So, just keep that result in mind sometimes you may use it in your courses somewhere else.