Basic Real Analysis
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Lecture 12
Topology of Real Numbers: Closed Sets - Part III

A is inside A bar, and A bar is closed, if I look at the whole set Rn, is that closed? A, closeness is of any set, so is the whole set Rn A is subset of Rn, is Rn closed? Yes or no? If I want to show it is closed, what I have to show? I have to show that whenever I take a sequence in Rn, if it converges to a point, that point must belong to Rn, what is to be shown?

If a sequence in Rn is convergent to a point in Rn, then the point is in Rn, so nothing to be shown. So, whole space Rn for every n is a closed set, so let me write that also something nice to observe that, which one this one, okay.

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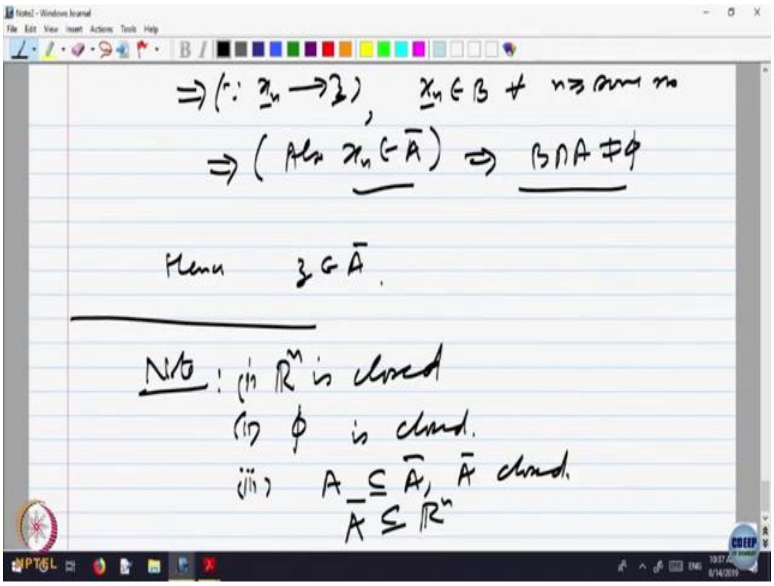

Note, Rn is closed, okay I think so this thought process is okay, Rn is closed. 2, what about empty set? What would you like to call empty set, closed or not closed? Is your choice, you see, you want to say that, for every sequence in the set, if there is no sequence nothing to prove, or if you do not like that you can just say is closed either by definition or by convention you can declare empty set also closed, no problem. A is inside A bar, A bar closed and A bar is subset of Rn.

So, given any set, I can put it inside a closed set, namely A bar, what is so big deal about it we have also got Rn which is closed and includes A. We want to specialize that A bar is the smallest closed set in Rn that includes A, not only A bar is closed, includes A in fact it is a smallest one that can, that is closed and that includes A.

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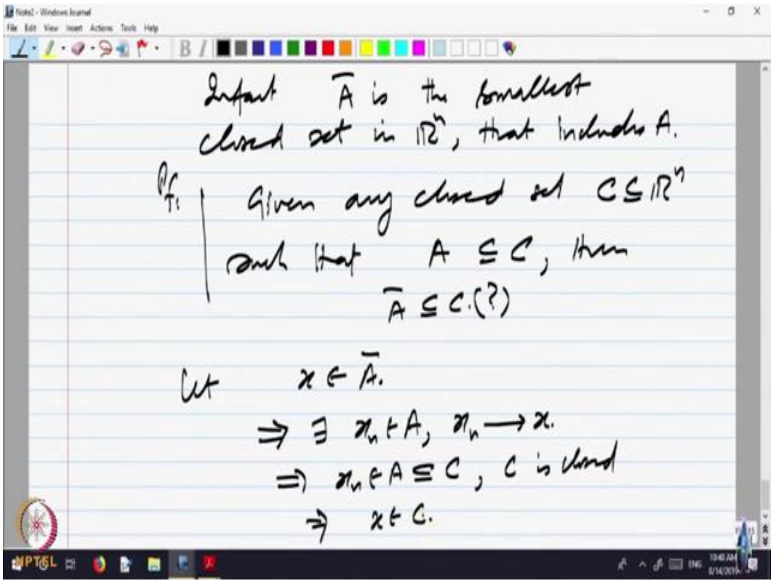

So, let us say that that is a nice thing to observe in fact, A bar is the smallest closed set in Rn that includes A, so let us prove that. So, what is to be proved? That is what the first step should be, English should be translated into mathematics. So, I want to prove A bar is the smallest closed set, we have already proved A bar is closed and A is inside A bar, I want to prove it is a smallest, that means so given any close set C in Rn, such that A is subset of, if C is any other closed set that includes A, I want to prove A bar is a smallest, then what should happen? A bar must be inside C, so that is what is to be shown, that it is a smallest.

So, let us start with any C, so given any set C I want to prove this, if I want to prove A bar is inside C, what is the natural thing to do? Take a point in A bar and show it is also inside C, I want to say it is subset, so let x belongs to A bar, what is to be shown? x belongs to C. Now, x belongs to A bar implies what? At least something is given to us, it is a limit of a sequence in A, so there exist a sequence xn belonging to A, xn converging to x, by definition of A bar, is that okay?

xn belongs to A, and where is A? A is inside C because I have to go to C somehow, A is inside C and C is closed, so what does this imply? x is inside C, C is closed xn converges to x, so implies x belongs to C, is that okay? Nothing very complicated going on, it is just keep what is given, where you want to head to, and use the tools what are given to you, I want to prove that if x belongs to A bar, then x belongs to C, so what is given x belongs to A bar, so that is a sequence in A which converges to C, from there I have to go to C and I am given that if xn belongs to A, A is a subset of C so xn belongs to C and I want to claim xn converges to x I want to claim x belongs to C and that I can claim because x C is given to be a closed set, so over.

So, that proves, let us look at something more, this is also very common in mathematics, we are looking at subsets of real line and we have defined a property of subsets of real line namely a set being called closed if something happens. One would like to know, if I take collection of all subsets of real line, what are the operations possible on all sets? I can take the reunion, I can take intersection, I can take compliments and so on. So, does this property of saying something is closed remains true when I do those operations.

So, this kind of thing will be done very often, you must have done in your calculus courses, look at a function and say f is differentiable, some property f is differentiable, then theorems come if f is differentiable, g is differentiable, f plus g is differentiable, f into g is differentiable and so on, why?

Because if you look at the collection of functions, for a function you have defined a property and functions can be added, functions can be multiplied and so on, so you want to see whether those properties remain true when you do these operations or not. So, here, we have defined a property for a set which is a subset of Rn, and you can take intersection, take a union and so on. So, let us study some of those properties.

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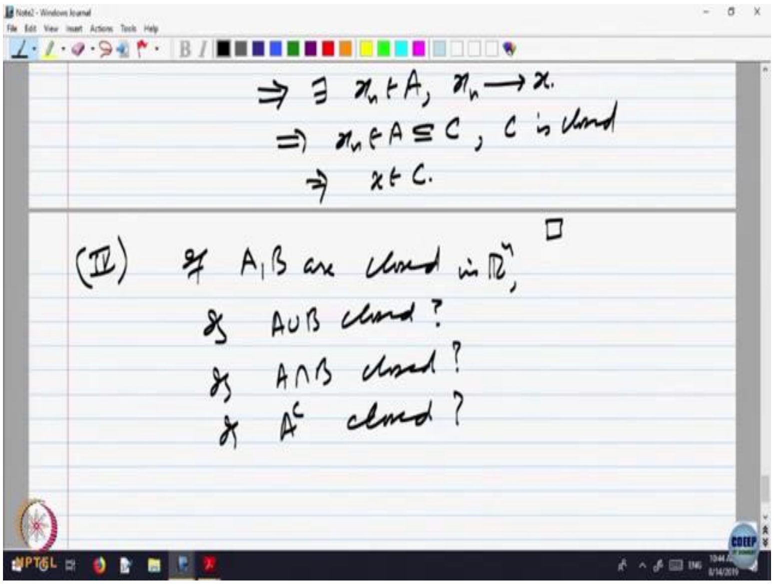

So the question is if A are closed in Rn is A union B closed, so A is A intersection B closed is A complement closed. So, these are natural question that one can ask, so let us analyse some of them, okay. For example, let us look at, see here is George Polya coming back, I am trying to analyse things in Rn, you can try to do these things analyse in Rn equal to 1 because they are simpler to visualize probably and see whether they remain true or not, and if they remain true then probably try to do it in Rn and write the proof in real line itself and try to generalize.

Say, for example, let us look at last one, if A is closed I am looking at the complement of it. So, let us look at various examples of closed sets in real line, we saw if I look at the interval including end points A and B then that is a close set, what is the complement of that? What is the complement of the closed set? Which is close interval, so I am looking at the last one, I am looking at this is A, this is B, what is the complement? That is open.

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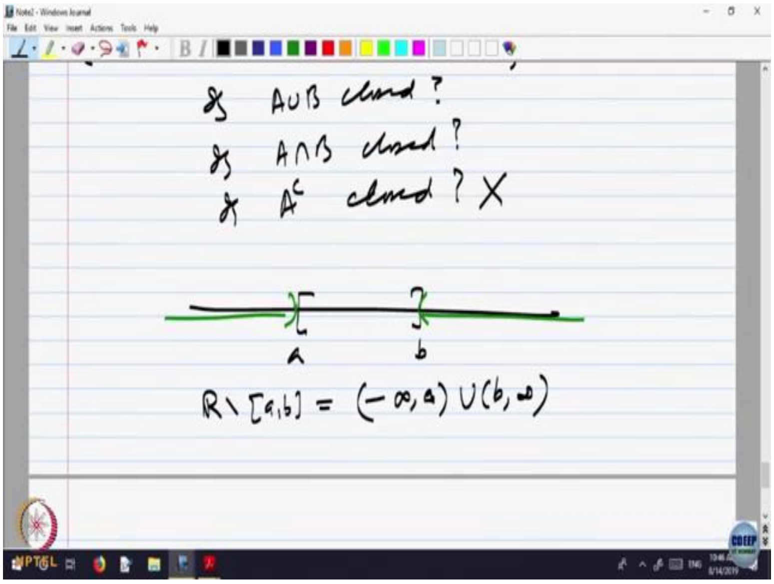

So, complement is going to be this side and this side, so the complement of this, so R minus AB is minus infinity to A union B to infinity, is that closed? We discussed today itself, at least minus infinity to A is not closed, that is good enough to say that the union is also not closed because I can have a sequence converging to A, inside minus infinity to A, but A is not inside the set complement.

So, that is good enough to say it is not a closed set, so last one goes bad, this is not true. What about A intersection B? I want to look at A intersection B, if A and B both are closed, I want to look at A intersection B.

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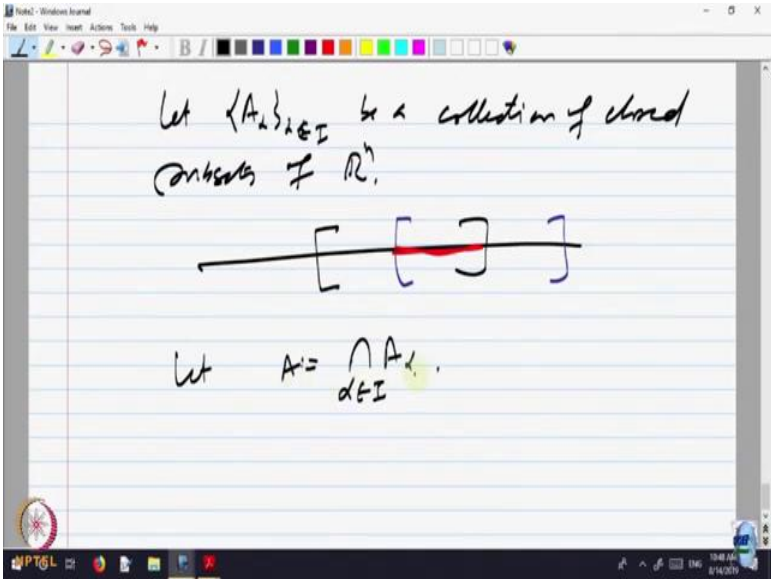

So, we can look at real line, if we take two closed intervals, and take the intersections, what happens? This is one and this is another, so what is the common thing? So, this is the common thing, there is a close interval, so looks like probably it is true but we have to prove it.

So, let us look at to prove try to prove, because what is happening is if I take a sequence inside the intersection, that will remain inside that box only. So, let us try to prove that in fact the prove works for any intersection, so let A alpha, alpha belonging to some indexing set I be a collection of closed subsets of Rn.

Let A be equal to intersection of all these alpha belonging to I, okay? I am looking at wheatear A is closed or not, so what are the possible tools available to me, either I can take a neighbourhood of points in A and try to do something or I can look at limits of sequences in A and try to show that the limit is inside A, but if I take a sequence in A, it belongs to the intersection, that means that is also a sequence in each A alpha and A alpha is close, so limit will be in A alpha for every alpha, so it will be in the intersection so that seems the easier route for me to follow.

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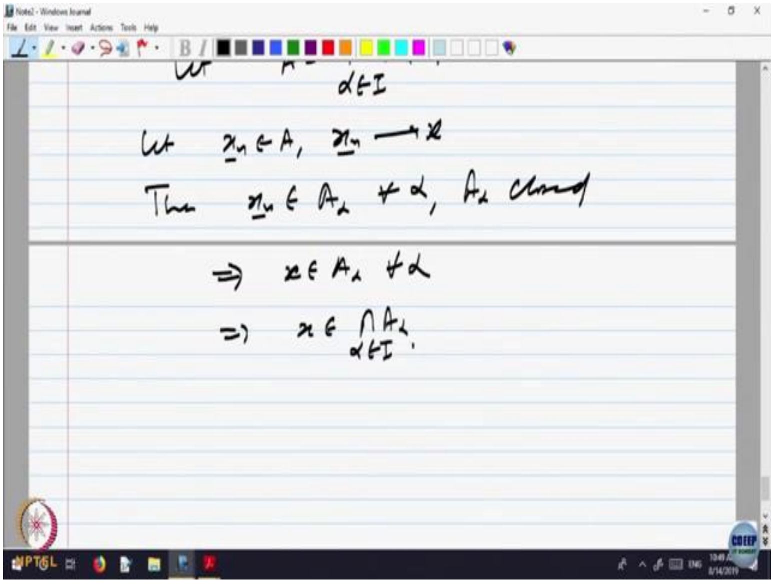

So, let xn belong to A, xn converge to some x, then xn belongs to A alpha for every alpha implies A alpha closed implies x belongs to A alpha for every alpha, and hence implies x belongs to intersection A alpha. So, not only intersection of two closed sets is closed, intersection of any family of closed sets is also closed.

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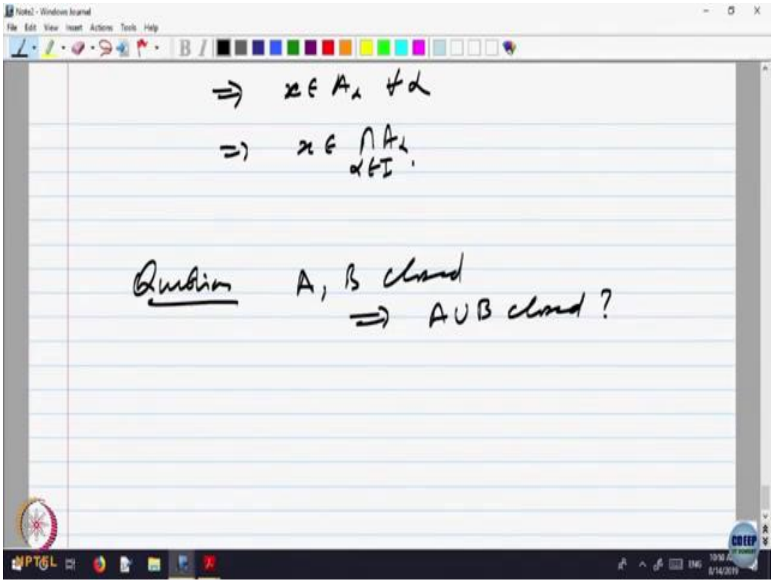

What are the third one? Union, so let us look at the union. So, question A and B closed, does it imply A union B closed? So, let us look at I want to prove A union B is closed or not, so let us look at a point which is a limit of sequence in A union B, and I want to show that xn is in A union B, x is a limit of xn I want to show that x also belongs to A union B.

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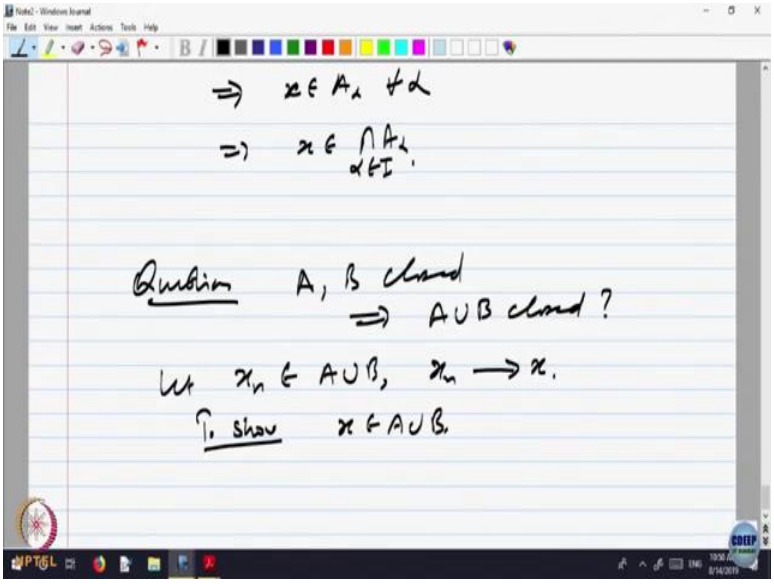

So, let us look at that, so let xn belong to A union B, xn converging to x to show possibly x belongs to A union B, and what is available to me? A and B are closed, but xn is in A union B, so I cannot say xn belongs to A for every n because it is in A union B, some may belong, some may not belong.

So, let us try to analyse for how many n's xn does not belong to A, possibly all xn's belong to A, then I am through because A is closed, so limit will be inside A so it is inside A union B, or similarly if it is in B, but sometime it may be in A, sometime it may be in A or sometime in B, then what happens?

How do I analyse such kind of things, if xn is a sequence in A union B, then for infinite terms of the sequence, either it is in A or it is in B, so let us analyse the statement. Supposing, if you do not feel very happy about my statement this way, let me put it this way. Look at x1, probably x1 belongs to A but x2 may not.

So, we go on looking at some time point probably that term also belongs to A, that means what, at the second stage there is some stage n2, such that xn2 belongs to A, then I go on looking, possibly there is some stage n3, at a third time say that it comes inside A again, then what I will have? I will have a subsequence of xn, which is inside A, so given a sequence xn in A union B, either there is a subsequence of that which is inside A, xn converges to A, so subsequence also must converge to A, so it will belong to the set A.

If there is no subsequence means what? That means after that stage everything is inside B itself, nothing comes to A after that stage. So, after that, the sequence itself is in B, so limit will be in B, so you are through either way.

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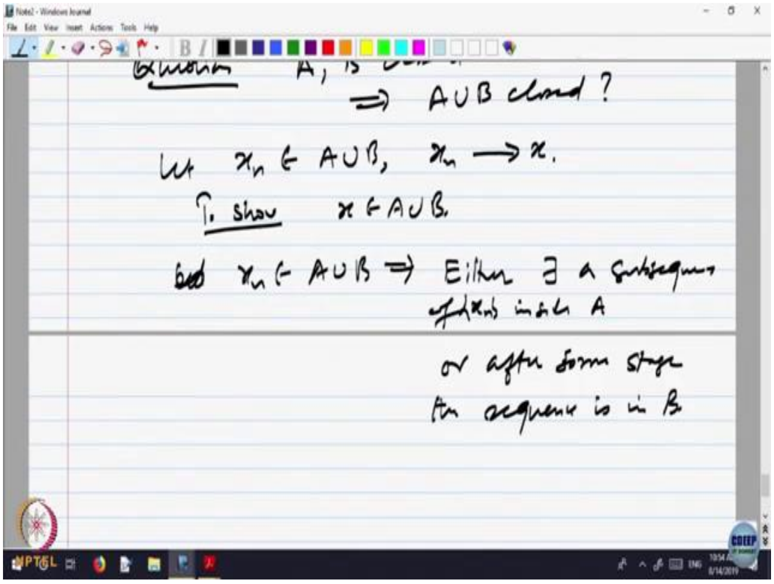

So, given, so I am just writing that convince yourself why it is, so let xn belongs to, okay. So, xn belonging to A union B implies either, the two possibilities, either there exist a subsequence of xn inside A or after some stage the sequence is in B. Saying that there is a subsequence, that means wherever I, 1 billion stage after that also there is some term which comes inside A.

If not, then after 1 billion everything is inside B. So after every n, or after every k there is a stage called nk, such that xnk is inside A, then that is a subsequence in A, happy. If not, if after some k this does not happen that there is any other element of coming to A of the sequence, that means all the elements are inside B only, then after the stage k, xk is inside B even then we are happy because we are only looking at the tale, looking at the limit which is a tale.

So, either way, we find that this is true so implying x belongs to A or B, so it belongs to, if there is a subsequence, then it belongs to A because the sequence converges, the subsequence will converge. If it belongs to B itself then the sequence is converging so no problem at all, so either way.

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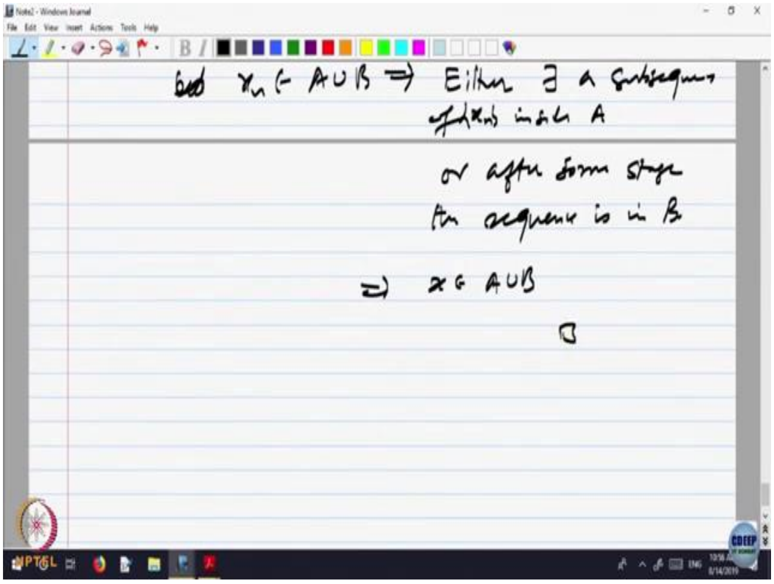

So, if A and B is closed, then A union B also is closed. So, these are properties of closed sets, I think there is no time to do something more. So, let us just stop here today by that we have looked at something very special, namely special property of sets in Rn, Rn as such with the notion of what is called Euclidean distance or the Pythagorean distance whatever you want to call that is gives us notion of closeness, a sequence is convergent if and only if every component sequence is convergent we looked at the notion of what is called closed sets.

And then we looked at the notion of a set may not be closed but look at what is called a closure of a set, that is a smallest closed set including the given set, and looked at various properties, okay.