Basic Real Analysis
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Lecture 11
Topology of Real Numbers: Closed Sets – Part II

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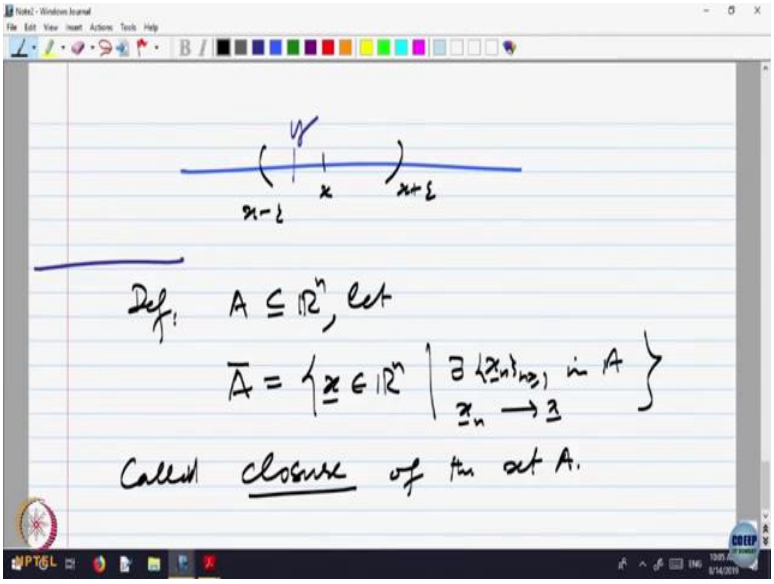

So, let us define such a thing, so definition, a subset A in Rn is said to be closed, no we have already defined closed, so let us for a subset A closed we have already defined, let us define A bar to be the set of all points x belonging to Rn, such that there exists a sequence xn in A, xn converging to x.

So, we are given the set A, look at all possible sequences of elements of A, and their limits. So, all those points in Rn, which are possibly limit of a sequence in the set A, collect them together in a box and call that box as A closure, A bar called, so let us give it a name called, called closure of, so it called closure of the set A.

So, we are given a set A, look at all possible convergence sequences of elements of A, look at their limits. So, all of them put together, all the limits put together, put them in any box call that as set A.

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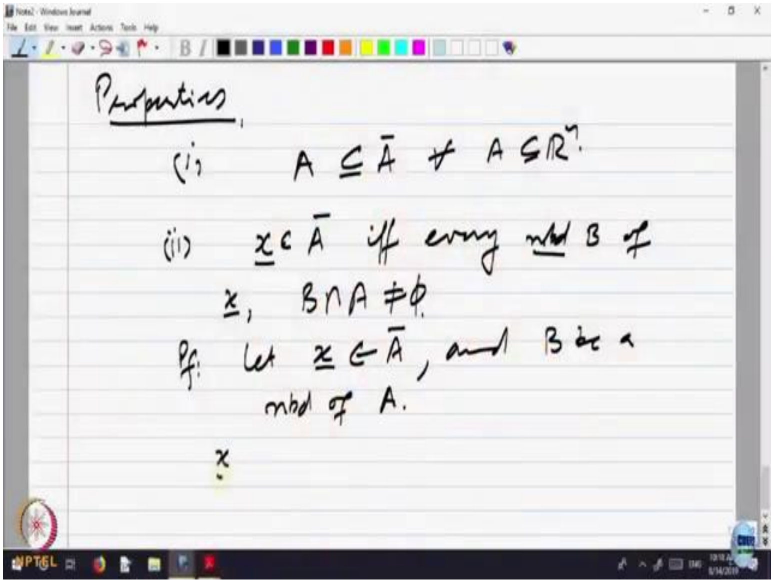

We want to study the properties of this set A, for example, keep in mind say open interval a, b, keep in mind the open interval a, b, if I take a sequence inside the open interval a, b the possibilities are the limit, what can happen to the limit? It can become equal a, it can become equal to b, so positively it will be between a and b, it can be possibly a or b.

So, it seems the closure of a set includes the set, closure of a set is something bigger than the given set, it is enlargement of the given set. And that is so because, if I give a point in the set A, then I can take the constant sequence. So, it is converging to that point, so that point must belong to a closure.

So, properties 1 an obvious property is A is contained in a closure for every A contained in Rn, is that okay for everybody? It is something bigger, we are enlarging, for the points in A we can look at the constant sequences, they are the limits anyway of the constant sequences, so they belong to A closure, so what A closure? It is all those points, which can be put as limits of sequences in A, which can be approached as limits, so every point of A can be approached by the constant sequence anyway.

So, what is, what do you expect now, as far as the closeness of the set A is concerned, can you say A is closed? Because we are just try to put all the limits inside now, so claim A closure is a closed set or before that, let me just write something which will be useful, x belongs to a closure if and only if every neighbourhood, so I will be shortening neighbourhood for nbd, instead of writing “n, e, i, g, h, b, o, u, r, h, o, o, d”.

So neighbourhood, every neighbourhood, what shall I call, say B of x B intersection A is non-empty. So let us observe this, what we are claiming is we are trying to give a characterizations of points in A closure, the set A closure in terms of a property of the neighbourhoods of that point.

So, let us see why is that, so let us write a proof, so let it belong to A closure and B be a neighbourhood of A. So, to show, what is to be shown? This neighbourhood must intersect A. Now, if B is a neighbourhood of x, what is x in A closure? That means, there is a sequence of points in, it is A closure, so there is a sequence by the definition, there must be a sequence of points in A which is converging to x, but if the sequence xn converges to x, what happens to the tail of the sequence? It must come closer to the point x, so it will come inside that neighbourhood, after some stage onwards.

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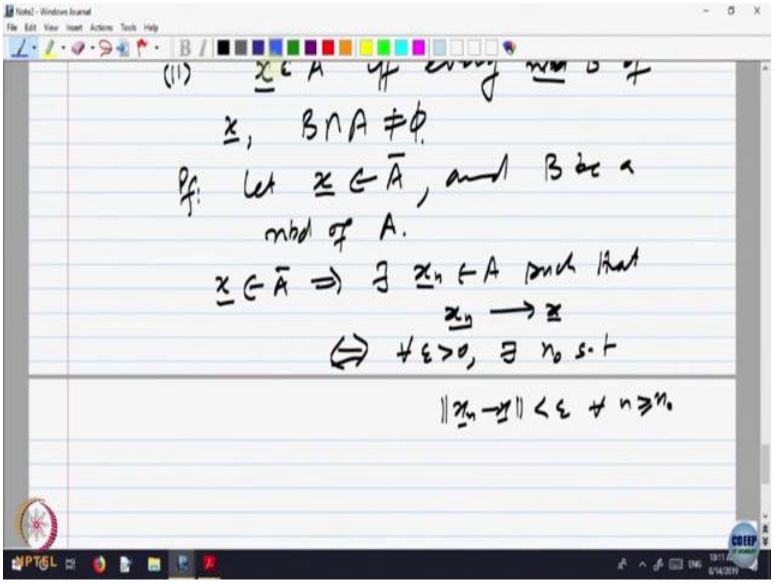

So, let us write that, be a neighbourhood of x belonging to A closure implies there is a sequence xn belonging to A such that xn converges to A, sorry, xn converges to point is x, not A it is x implies, this is actually true if and only if, when do you say xn converges to x? Which is for every epsilon, there exists some n not such that norm of xn minus x must become smaller than epsilon for every n bigger than n naught.

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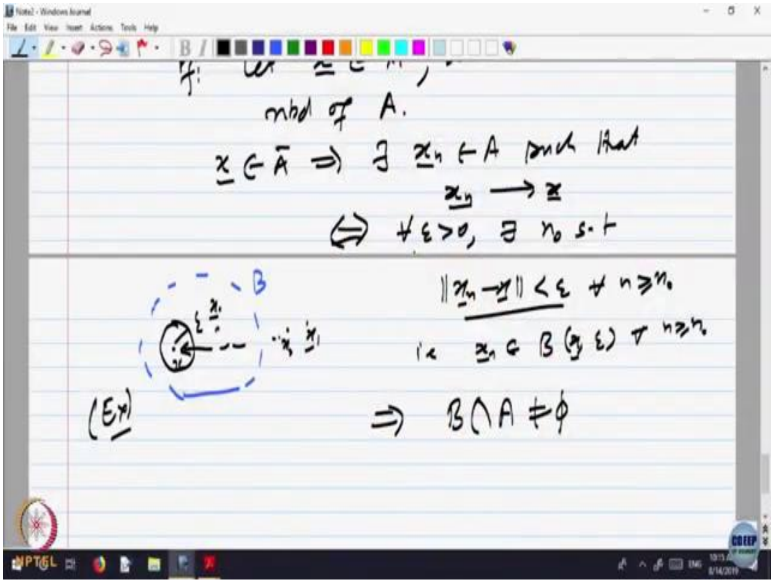

So, let me if you like, let me draw a picture, so here is a neighbourhood B and here is probably the point x, and I have got a sequence x1, x2 which is converging to x. So, this says this is less than, can I say that, this says the distance of, so here is a, for every epsilon, is it clear? This says for every epsilon there is a, what is the meaning of this? That means xn belongs to, so that is same as saying xn belongs to ball centred at x of radius epsilon.

So, keep in mind the geometric and analytical thing, the distance between xn and x naught and x smaller than epsilon means that belongs to the ball. Now, the only question is, I am given a neighbourhood B, so what is a neighbourhood? It is some open ball centred at some point, we do not know where. So, can I say that, for every neighbourhood of a point, there is small ball inside the neighbourhood, that is the only thing required.

Given a neighbourhood, given this neighbourhood, and a point inside x, can I say there is a epsilon such that the ball centred, see this ball bigger ball may not be centred at x we do not know where it is centred at, it may be centred at some other point, say x naught, but what it is saying for every point inside it, I can have a small epsilon such that there is a open ball which is inside the neighbourhood, is that okay for everybody?

Requires a bit of writing down, so let me write this also as exercise because geometrically very obvious, if the point x lies inside, then its distance from the centre will be strictly less than. So, you can always go very close so that all the points are inside. So, that is saying implying that the ball intersection A is non-empty, is that okay?

All the points xn’s will be inside the ball now, after some stage onwards, after some stage onwards all the points xn’s will be inside the ball of radius epsilon at x, but that is inside the bigger ball B, bigger neighbourhood, so there will be points xn’s of A which are inside B, that is same as saying the intersection is not empty

Basically saying that, convergence means it is coming closer. So, it must come inside a ball, centred at that point, very very small, so epsilon is a arbitrary, but this is happening for every epsilon. So, you can always choose epsilon suitable enough so that the ball centred at x of radius epsilon is inside B, for that ball everything is okay, is it clear to everybody? So, what we are saying is, point x belongs to a closure implies every neighbourhood of that point x must intersect A.

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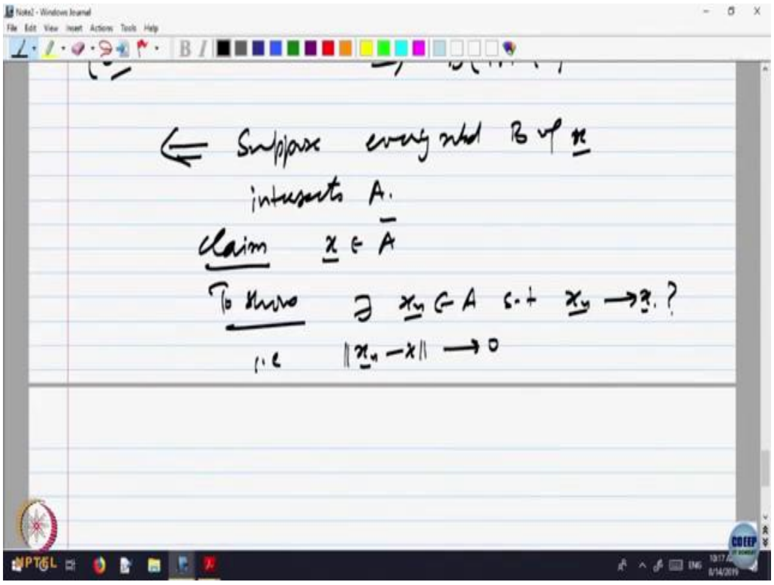

Let us look at the converse part of it, suppose every neighbourhood B of A intersects A. Claim, every neighbourhood B of, sorry, of the point x intersects A claim x belongs to A closure. If every neighbourhood of the point x intersects the set A, we want to claim that point x is in the closure of A.

So, what is to be shown? To show x belongs to A closure, what is to be shown? What is a definition of a closure? I have to produce a sequence in A, which converges to x. So, there exists some xn belonging to A such that xn converges to x. And when will xn converge to x?

Now, you see keep target in mind what you want to show, I want xn in A of course, but that xn will converge to A only when the distance between xn and A goes to 0. So, basic requirement is norm of the distance or the magnitude of xn and A, xn and x, so that is should go to 0. And saying it goes to 0 means what? After some stage, becomes less than some epsilon given any epsilon.

So let us write, so given, but I have to choose xn, I do not know what is xn, I have to find xn such that this happens, but what is given to me is for every, this becomes that means, so let us write what does that means, that means this will become less than epsilon, that means the point xn should be in the ball centred at x of radius epsilon, what is given to me, every ball intersects, what is given to me is every ball or every neighbourhood of the point, of the point x intersects.

So, somehow I have to bring everything to the neighbourhoods, then only I can use the given thing. So, what I do is because there is sequence xn converging to x, this we have to produce and that will happen if this goes to 0, and this can happen if I take this less than 1 over n, if you can choose xn's such that the distance between xn and x is less than 1 by n, then I will be through, and that same as saying xn should be in a ball centred at x of radius 1 over n, so I should specialise that neighbourhoods now.

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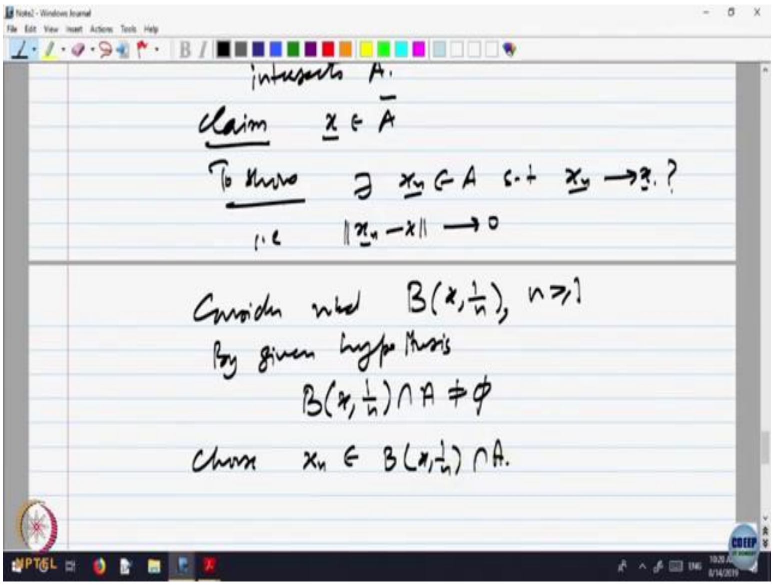

So, consider neighbourhood ball centred at x of radius 1 over n, n bigger than or equal to 1, we are given that every neighbourhood must intersect A. So, in particular by given hypothesis, the ball centred at x of radius 1 over n, this is a neighbourhood of the point x, it is the special neighbourhood, we are given for every neighbourhood something is happening.

So, let me take this special neighbourhood, this intersection A must be non-empty, and this is motivated by the fact I want a point xn at a distance at the most 1 over n, then I will have xn converging automatically. So, this is non-empty so choose xn belonging to ball x 1 over n intersection A, this is non-empty so choose any point inside that.

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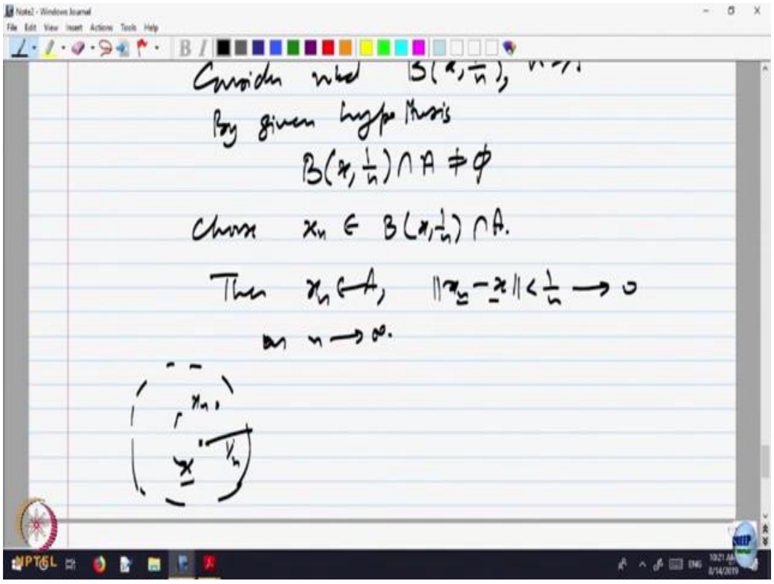

So, then xn belongs to A and norm of xn minus x goes to 0 as because this is less than 1 over n, so let me probably write, make it more specific it is less than 1 over n goes to 0 as n goes to infinity.

So, essentially there is nothing very great about the proof, but it is intuitively very clear that given a point x, let us take the neighbourhood of radius 1 over n, this must intersect a, so there must be a point xn inside it and make it smaller and smaller, shrink it so that the sequence xn comes closer and closer to x.

And this is happening because we are given that for every neighbourhood something is happening, so we are specializing that. So, what we have shown is the following that, so that proves that a point is in the closure if and only if every neighbourhood of that point intersects the set.

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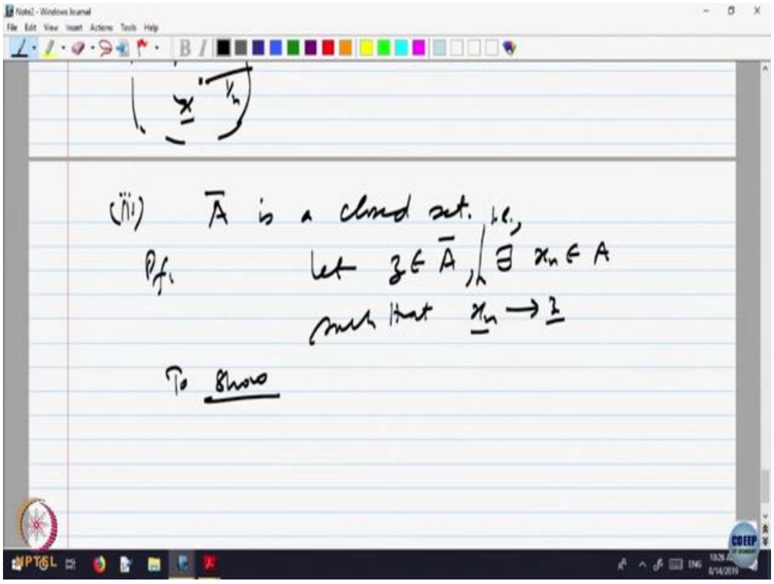

So, now let us look at the next property, that A closure is a closed set, A may not be closed but a closure is a closed set, so, what is to be shown? What is given and what is to be shown, is always very clear, things become clear when you write down what is given and what is to be. So, proof, given a closure I want to show it, we are not given anything, we are just given A is a set, A closure.

So, let, if I want to show it is a closed set, what is to be shown? So, let us write to show, see this is English, A closure is a closed set, what is the mathematics behind it? Every point, what we wish to be shown? To show for every z belonging to A closure, there is a sequence, there is xn belonging to A, such that xn converges to z, no no is that values to be shown? Or is that the definition A closure?

What is meaning of, z belongs to A closure, so let us go back and just recall because since looking at, where is z belongs to A closure means what? There is a sequence that is given to me, so let us so this is given, so let us write this is not, so let so let us write, let so that means, so let z belong to A closure, to show or to show A is closed, so what is the meaning of A is closed?

Now, look what is A is closed, I want to show that, that means if a sequence xn belonging to A and xn converges to A, then A must belong to, so what is to be shown here? What is to be shown?

Students is answering: (())(21:30).

Professor: z belongs to?

Students is answering: (()(21:32).

Professor: That is given, so we are given z belongs to A bar.

Students is answering: (())(21:37)

Professor: No, that is not to be shown, that is not given. So, let us write, see I want to prove A bar is closed, so that means what? If I take a sequence in A bar and it converges to some point, that point must belong to A bar, A bar is closed, that means every sequence in that set, if it converges the limit must be inside the set.

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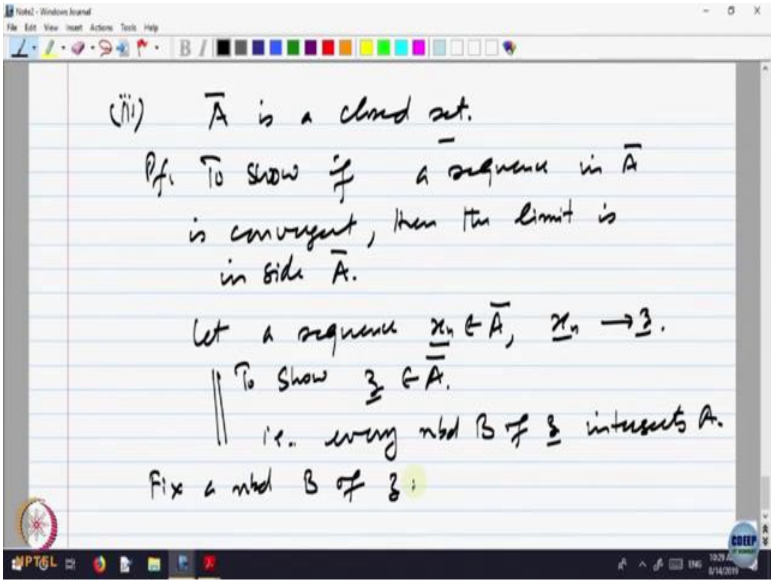

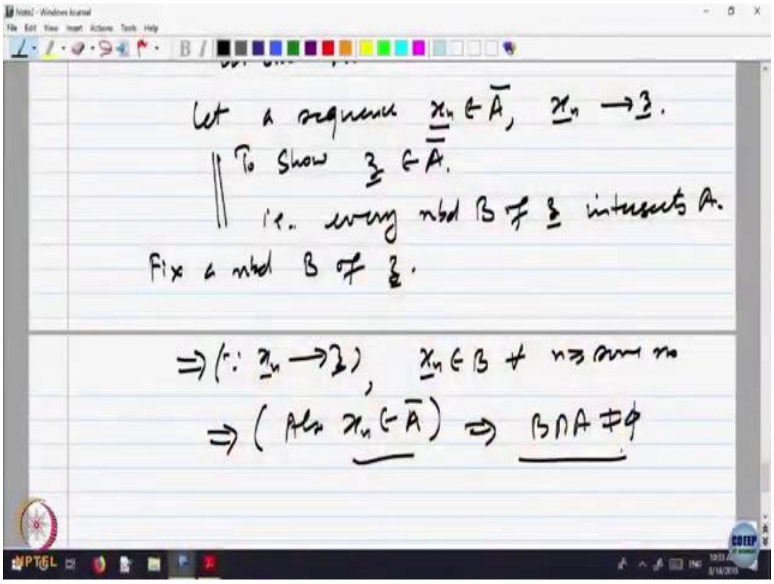

So, let us write what is A bar is closed, so that means, to show if a sequence in A bar is convergent then the limit is inside A. For every sequence in the set, if it converges the limit must be inside, that is what it says. So, let, so how do you prove it? So, let us take a sequence, let a sequence xn belong to a closure xn converge to z to show z belong to A closure, is that okay that is to be shown?

Now, xn belongs to a closure, we want to show, this means this, so that is every neighbourhood that is same as saying, every neighbourhood of B of z intersects A, so this is same, we want to show z belongs to a closure that is same as saying every neighbourhood of z should intersect A because we want to show the point is in A closure.

So, let us fix, fix a neighbourhood B of z, fix a neighbourhood. Now, what is given to us, xn belonging to A bar, converges to z, and B is a neighbourhood of z, so what should happen? Xn is converging to z, so xn’s must come inside B, so implies because xn converges to z, xn belong to B, xn’s belong to B for every n greater than some n naught. So xn’s belong to B, because xn is converging to z and B is.

So, that implies what? xn belongs to B, and what is B? Where is xn? xn is in A closure and z belongs to B, so what does this imply? See, here is a very small thing, the idea xn belongs to B, also xn belongs to A bar, so that means what? If xn belongs to B, B is also a neighbourhood of xn, if xn belongs to B, then B is a neighbourhood of xn and xn belongs to A bar, that means what? Implies B intersection A is non-empty.

If xn belongs to A bar and xn also belongs to B, so B can also be treated as a neighbourhood of x, though it was a neighbourhood of z, but x belongs so it can be treated both as a neighbourhood of x as well as a neighbourhood of z. So, if I treat it as a neighbourhood of x, then Bn belongs to a A bar, so xn belongs to A bar, xn belongs to B, so B is a neighbourhood of xn, so it must intersect A, because xn belongs to A bar by the definition of A bar, yes?

Very small point, that is all, xn belongs to B and z also belongs to B, so B can be treated as a neighbour of both of xn as well as z. If you treat it as a neighbourhood of xn, xn belongs to A closure that implies xn belongs to A closure implies it must be non-empty.

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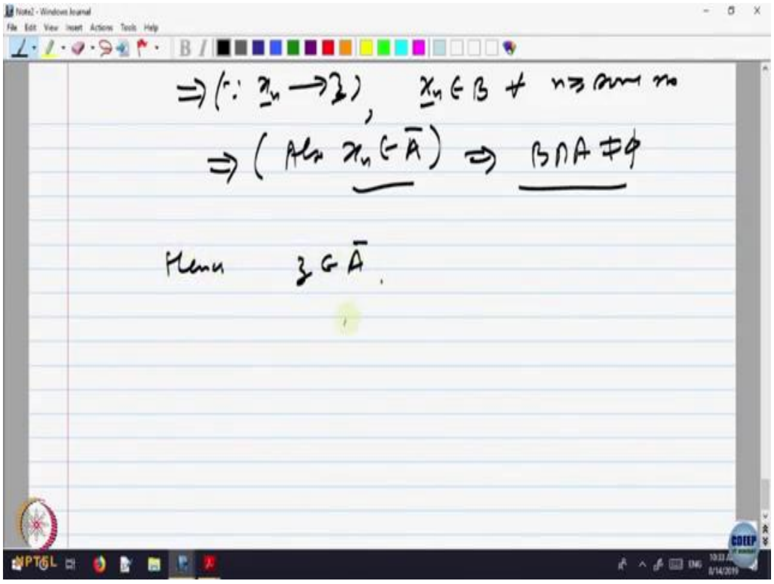

So, this gives us hence, z belongs to A bar. So, we have proved A bar is a closed set, A bar is a closed set, that we have proved just now, A bar is a closed set.