Basic Real Analysis
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Lecture 10
Topology of Real Numbers: Closed Sets – Part I

We looked at last time, the real number system; we looked at, what we called as the nested interval property. Some more concepts about special subjects of real line, they are also common to higher dimension spaces.

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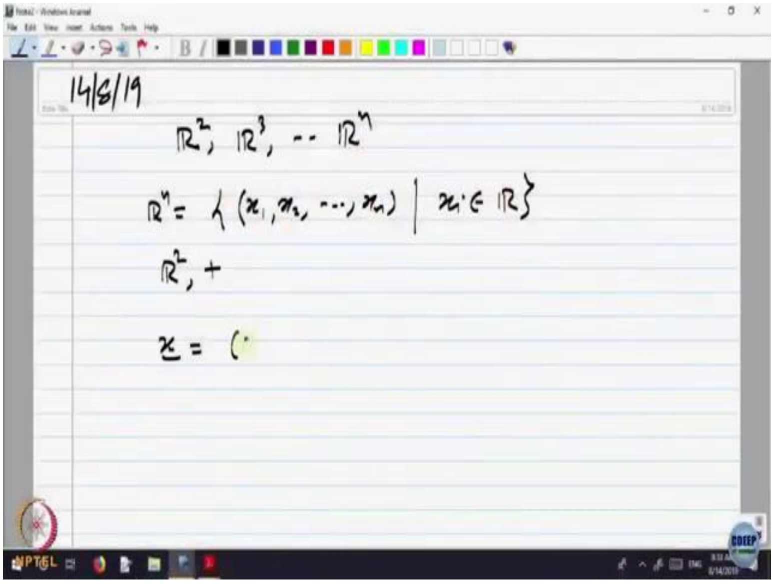

So, let me introduce briefly R2, R3 and in general Rn, I think most of you are familiar with R2 and R3 but the same thing works for Rn also, and we will come back to it later on also. So, the basically idea is that, say for example R2 or Rn is a set of all n tuples, where each component or each coordinate is in real line.

So, you can think it as a vector with n components, x1, x2, xn there is on Rn there is addition, so what is addition of two vectors that is a component wise addition, scalar multiplication you can and multiply each component by the same scalar, so that is scalar multiplication.

There is no multiplication of vectors as such in R2 or Rn, like in real line there was, there is notion of a dot product and cross product, we will not be using them and there is no order on R2, R3 or Rn you cannot compare, you cannot define an order between vectors one vector bigger than the other vector.

So, there are problems about, however, what you can do is, like absolute value you can define what is called the magnitude of a vector.

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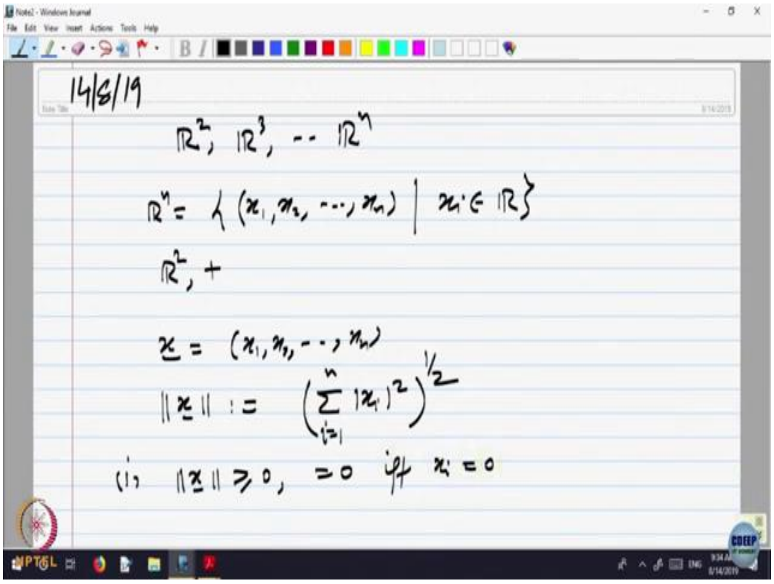

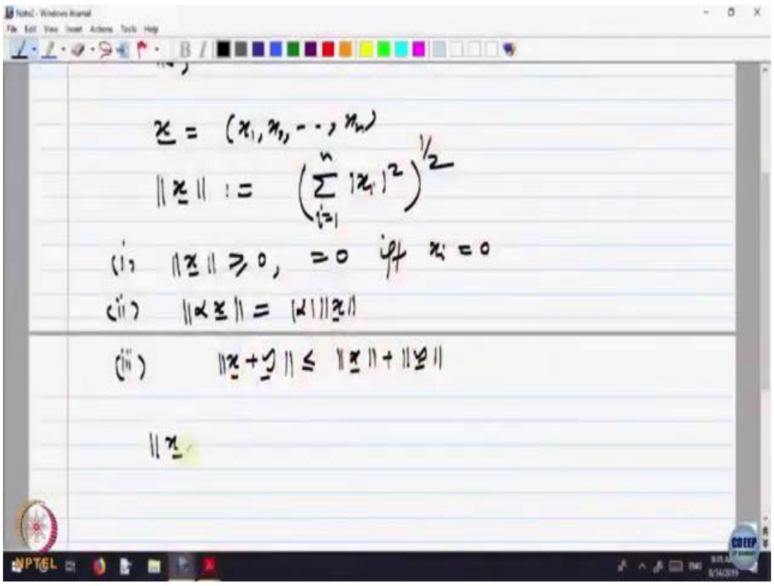

So, for a vector x with components x1, x2, xn, we define what is called the magnitude of the vector or the norm of the vector, that is sigma absolute value xi square raise to power 1 by 2 i equal to 1 to n. In R2, this is just the Pythagoras distance, right angle triangle x1, y1 and this has properties which are very similar to the absolute value function, so properties this is always bigger than or equal to 0, equal to 0 if and only if each xi is equal to 0. So, that is corresponding to the absolute value.

The second property is as far as scalar multiplication is concerned alpha times x is equal to mod alpha times norm of x. So, that is scalar multiplication how does this magnitude behaves and the third is the triangle inequality property, namely absolute value of x plus y is less than or equal to. So, we are basically it behaves very much like the notion of absolute value for real numbers.

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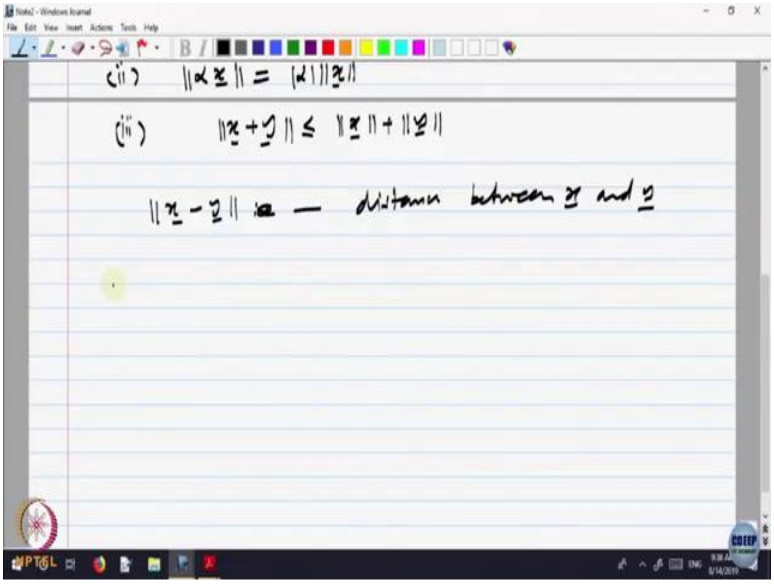

So, as a result it gives rise to a notion of comparison between or the distance between two points, one point is close to another, so we define norm of x minus y, so you can call it as the you can define this as you can call this a distance between x and y. So, basically, the idea is that the notion of limits of sequences, and notion of convergence of a sequence depended upon the notion of distance, we set a sequence xn of real numbers converges to a point x if xn is coming closer to x and that closeness was measured in the term of the distance.

Now, we have a notion of distance available in Rn, so we can consider notion of sequences in Rn and notion of convergence of sequences in Rn.

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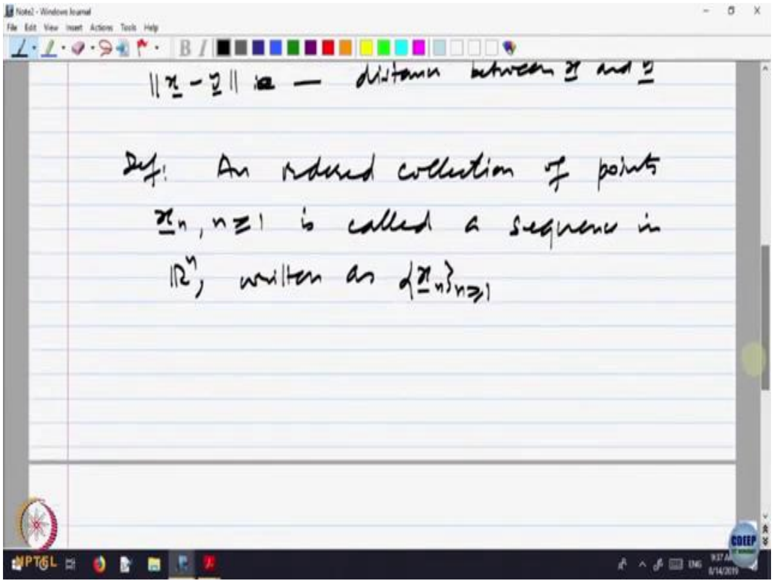

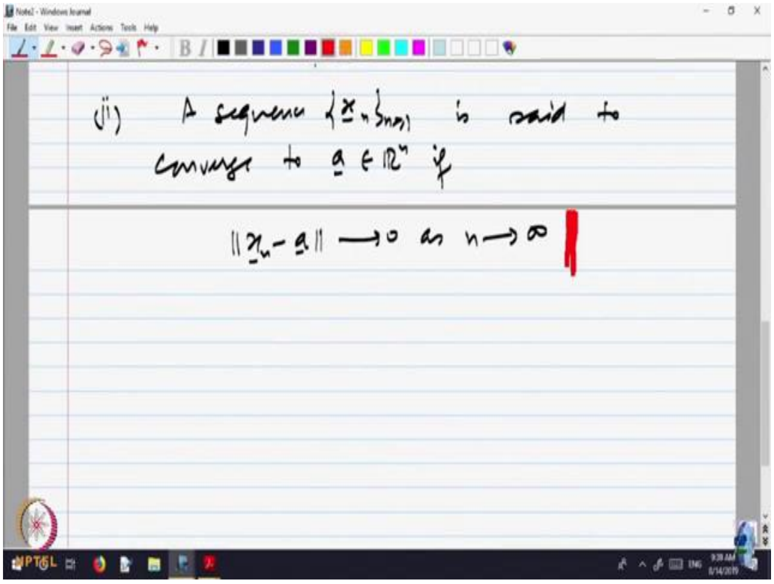

So, let us just write because that will be useful, so let us write definition of a sequence, an ordered collection of points xn and bigger than or equal to 1 is called, also we will put a underscore to indicate this is a vector, okay is called a sequence in Rn. And as usual written as, we write it simply as xn same notation n bigger than or equal to 1.

So, let us say definition 1, definition 2 we can define now convergence, a sequence xn is said to converge to, say a vector a belonging to Rn if, what should be definition? The distance between xn and a becomes smaller and smaller, that is same as saying norm of xn minus a goes to 0 as n goes to infinity.

Now, keep in mind, this quantity, norm of xn minus a is a number, is a real number, so for every n you are getting a non-negative sequence of real numbers, is a sequence of numbers which are non-negative and real, so we can ask whether the sequence converges to 0 or not. So, this part, this is precisely convergence of real numbers, so we are using convergence of real numbers to define convergence of vectors in Rn, so keep that in mind.

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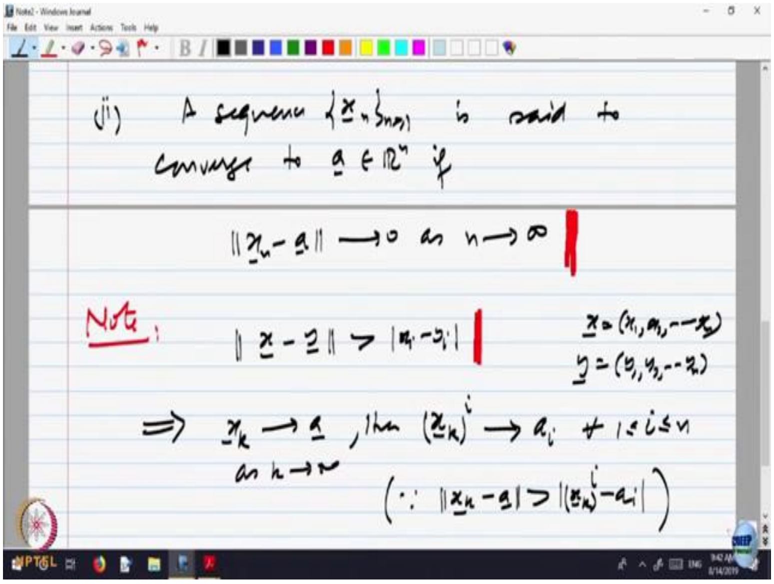

Now, here is a small observation, write it as a note. So, supposing our vector, let us note something namely x minus y, if I look at the distance between two vectors x and y, this is always bigger than, x has got components, y has got components, so let us say x is x1, x2 and so on xn and y as components y1, y2, yn, then this is always bigger than or equal to xi minus y, is that okay? Because what is the distance? Norm of x minus y that is xi minus yi absolute value square raise to power 1 by 2, so that summation is 1 to n, so that is always bigger than each term and this is, is it okay?

So, this implies, that if xn converges to a, then, okay I am using n so I should probably because I am using n for Rn also, so let me write xk, if xk converges to a as k goes to infinity, then this is same this implies, then this implies that look at, so x vector k, let me write this as i, what is that?

That is ith component of the vector xk, will converge to the ith component ai for every i between 1 and n, this inequality just now, we are saying that if a vector is convergent, sandwich theorem, mod of xi minus yi is less than this. So, if this becomes smaller, then this becomes smaller sandwich theorem, is it okay for everybody?

If not get clear, so let me write here, because is bigger than xk ith component minus ai, each component distance is dominated by the distance between the vectors, so sandwich theorem implies that if that goes to 0, then this goes to 0. So, what we are saying is if a sequence of vectors is convergent, then each component converges to the corresponding component.

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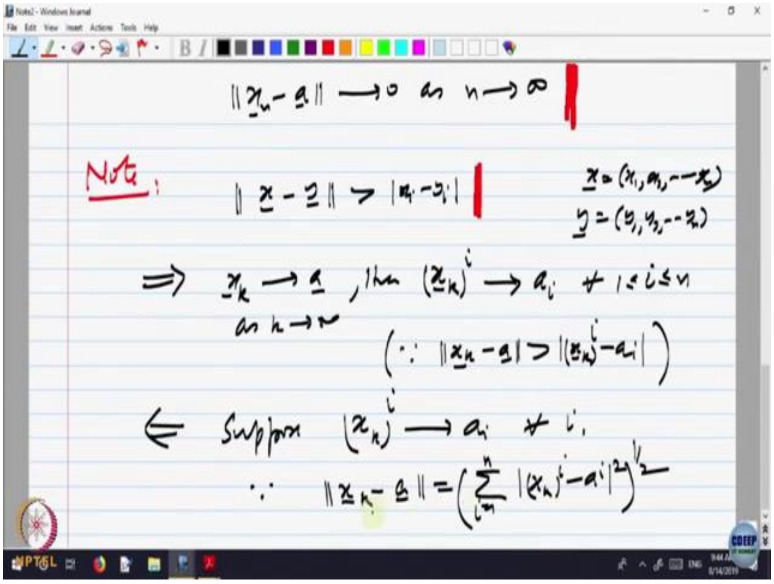

Now, let us look at the converse of this, suppose the ith component converges to the ith component, for every i, can I say that the vectors also converge? Clearly yes, because what is this distance between xn and a, that is summation of the distance is, summation here now, it is (())(11:33). Each one is going to 0, so sum goes to 0, by the limit theorems for real numbers, because norm of xn minus a is equal to summation i equal to 1 to n, norm of, I am writing k here, so xki minus ai, not no it is absolute value because they are real numbers, square raise to power 1 by 2, that is the definition of distance.

Here each term is going to 0, so square goes to 0 by limit theorems, the summation goes to 0 non-negative square root goes to 0. So, by the limit theorems for sequences of real numbers, this for each i goes to 0 implies the distance between the vector also goes to 0. So, what we are saying is, a sequence of vector is convergent if and only if each component converges to the corresponding component.

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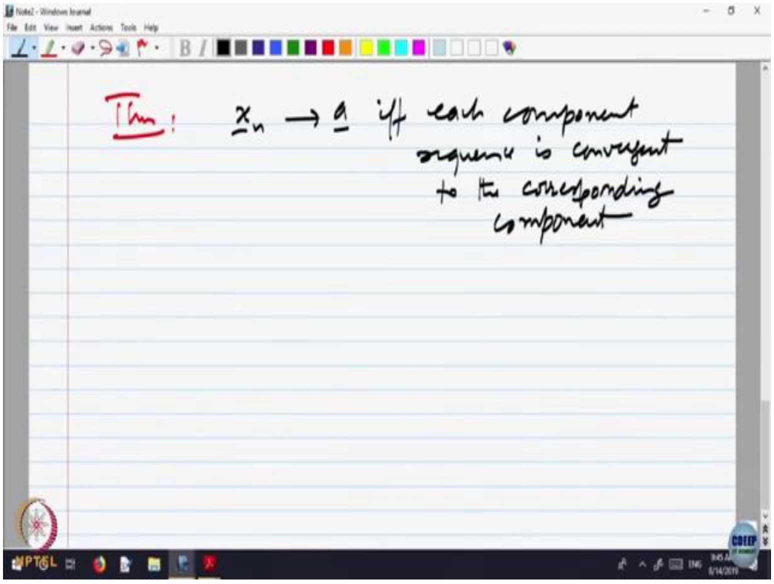

So, let us write it as a theorem if you like can converges to a if and only if each component sequence is convergent, to where to the corresponding component. So, as far as convergence of sequence is of vectors is concerned, it is same as analysing convergence of each component, so not a big deal, so that is okay.

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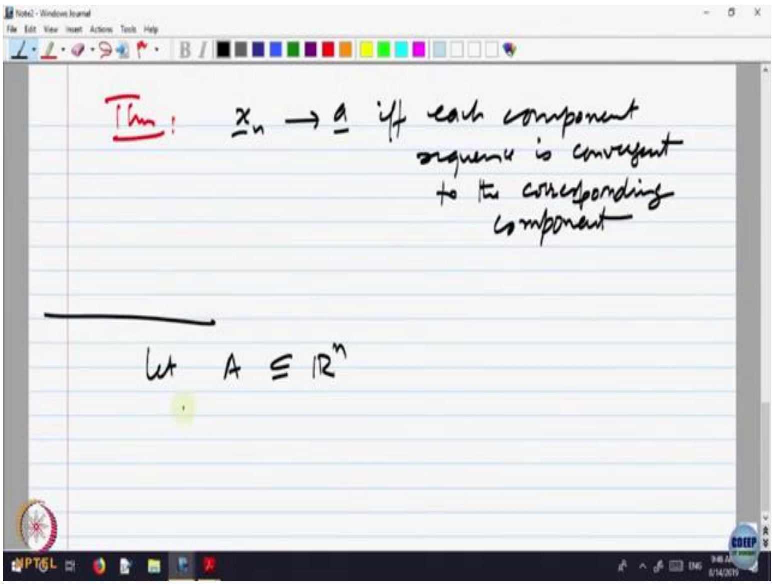

So, now we will be using this concept, to define something, so let us start with, let us take a set A contained in Rn, A is a subset in Rn. So, given a sequence in A, a sequence such that each element is in A, supposing it is convergent, then the limit of that sequence may or may not belong to A.

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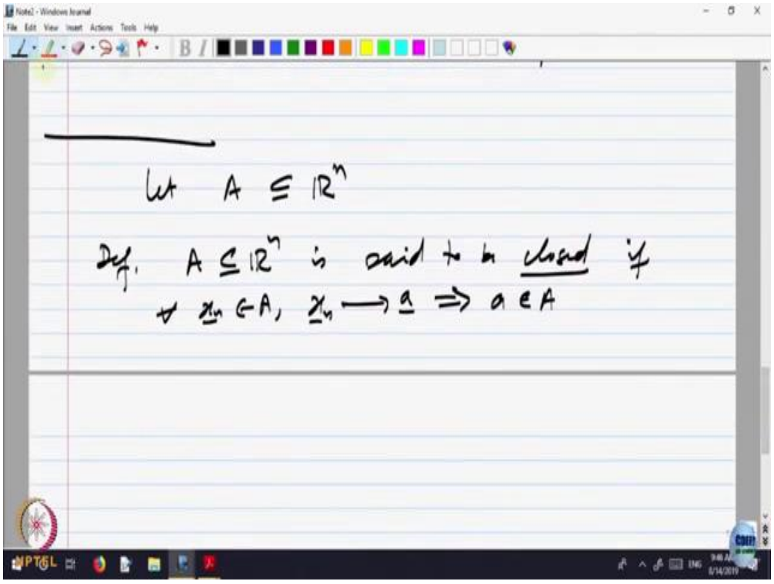

So, we are trying to now specialise special subsets of Rn, so we say definition, a subset A in Rn is said to be closed it is called a closed set if for every sequence xn belonging to A xn converging to A implies A belongs to A. So, what we are saying is, a set which includes all limits of sequences of its elements, we will say it is a, nothing goes out kind of thing, so it is called a closed set, the nomenclature is very clear.

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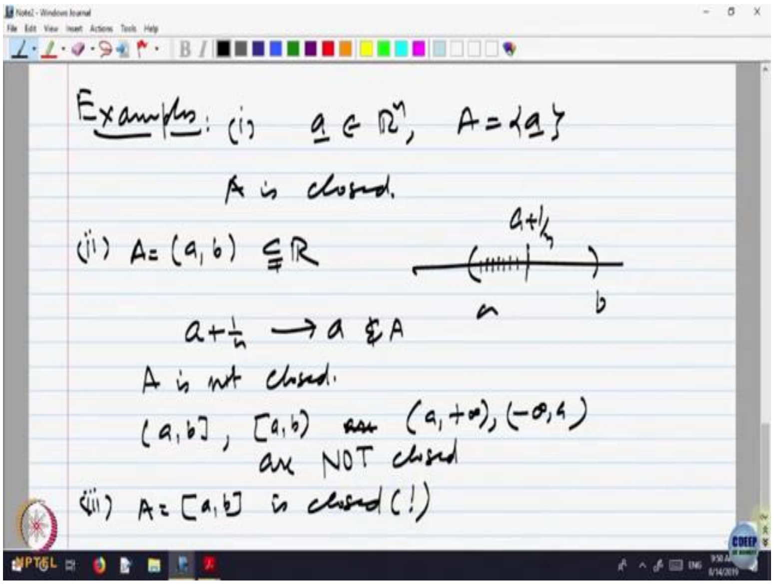

So, let us look at some examples, so let us look at a belonging to Rn, and the set A is singleton a, is this set closed? Is the set consisting of single point closed? Well obviously, because the only possible sequence is that you can consider of elements of the set R the constant sequence, and constant sequence converges to the constant, and that belongs to A, so A is closed. Let us look at when n is equal to 1, some special cases so A is equal to that is a subset of R.

An open interval, the name itself says open interval, so it should not be closed, but anyway that is not the reason, we want so if we want to prove this set is not closed, then what should I prove? There is a sequence of elements of the set, it converges somewhere, but the limit is not inside that set. So, precisely if this is my a and this is my b, I can take a sequence sought of coming from the right side to a, the limit will be a, which is not inside the set. So, what could be such a sequence?

Students is answering: (())(17:05).

So, you can look at a plus 1 by n, n suitably large enough so that you are inside, so here is a plus where a plus 1 may be outside b may be bigger than, but anyway let us, from some stage onwards a plus 1 over n will be inside the interval a, b for some n large enough. So, this will to converge to a, which is not part of A.

So, this is a A is not closed for the same reason, you can take as a, b it does not matter one point, one counter example is good enough, or you can take a, b these are all are or you can even take a to plus infinity and minus infinity to a are not closed, all these are not closed subsets.

Obviously, if I include here, can I say a, b is closed? Why is that closed? It is not because a sequence converging to a that means a is inside a, to say is close I have to show for every sequence in the set if it converges that point limit must be inside the set.

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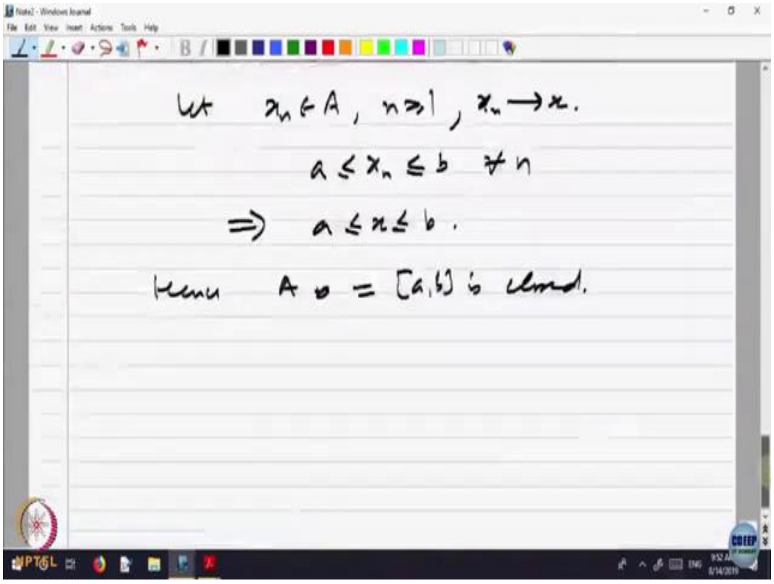

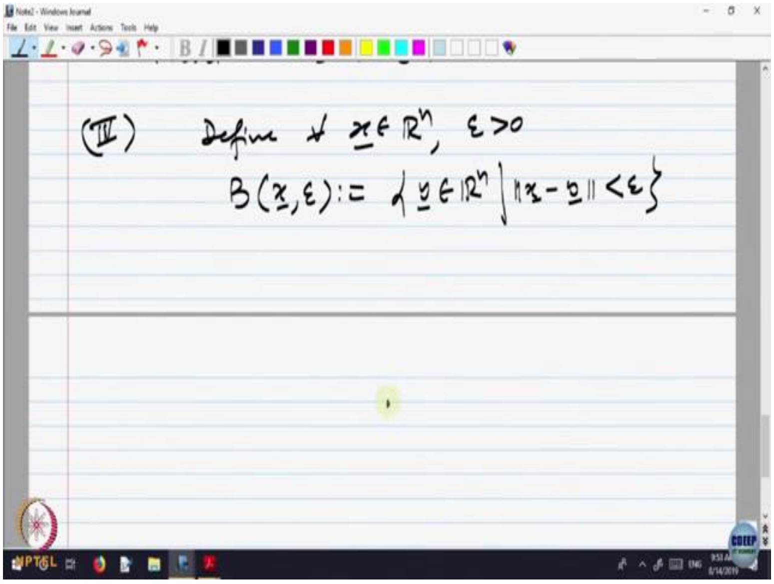

So, let us take, so let xn belong to A, n bigger than equal to 1 and xn converge to some x, then what can I say about this xn's? They are inside the set A and A is the interval a, b that means this is true for every n, is it clear? Because it belongs to A, A is the interval a, b which is closed interval.

So, a less than or equal to, and that implies if it converges xn converges, what can you say about the limit? If xn's are bigger than or equal to a, the limit cannot go below a, we have seen that. So, a less than or equal to x less then equal to b, but they were open if it was a strict inequality, then the limit could become equal to a, that we have seen in the examples, so then that will not be true.

So, less than or equal to a is okay. The limit can become equal to, even though each an is strictly bigger than A, the limit can become equal to A and that is okay for us, because we want A less than or equal to x less than or equal to b. So, hence, A which is equal to a, b is closed.

Let us, similar thing probably will be nice to, let us define something which is going to be useful for every x belonging to Rn and epsilon a number B x epsilon, I am going to define a set, this is all vectors y belonging to Rn, such that the norm of x minus y is less than epsilon, all points, all vectors in Rn such that is distance from x, x is a point which is fixed.

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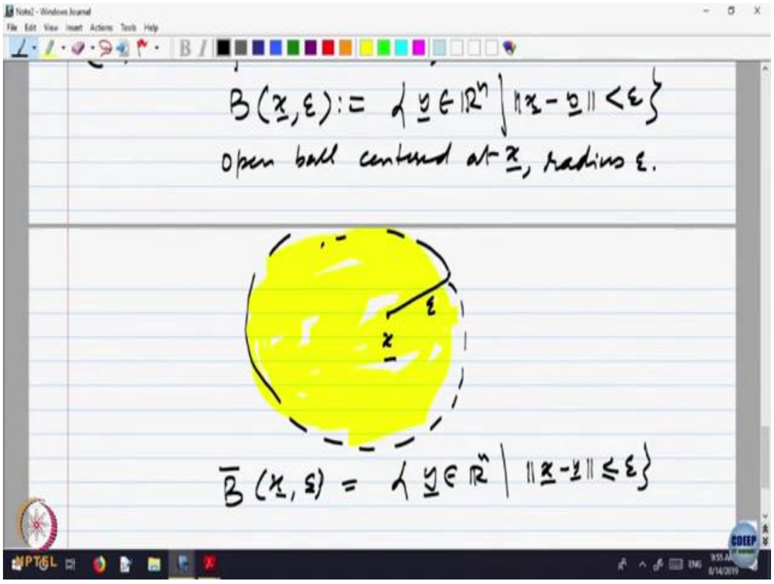

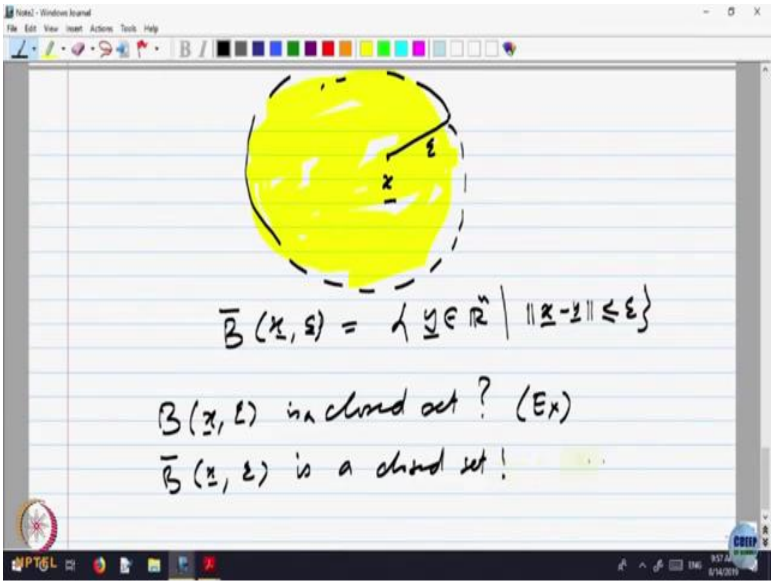

So, because we are looking at a Pythagorean distance or Pythagorean way of looking at, so if I take it R2, it is very easy to visualise here is the point x, a point y will be inside if its distance is less than at the most epsilon. So, it will be all points inside what we visualised as the ball.

So, you can think it as, so this is a visualisation of this thing, sometimes it is good but if I look at a point, geometric kind of thing on the boundary, then its distance will be equal to epsilon, so that will not be a part of the set. So, that is why it is called a open ball centred at x radius epsilon, because geometrically it looks like that, so we call you as open ball centre.

Let me also define because we are defining this thing, let me also define B bar x epsilon to be the set of all y, so the distance x minus y is less than or equal to epsilon, and we are also including the geometric boundary kind of thing also equal to also or included. So, let look at can I say this is closed? So, which of them are closed, can I say the ball x epsilon is closed is a closed set?

No, so but that means you have to construct an example of a sequence, so I will leave it as exercise, it is not difficult to, just intuitively is very clear. You can go towards boundary kind of thing, take a sequence going towards boundary, like we did for the open interval a, b something similar you can do, slightly we have to do slight more kind of slightly more careful about, because there are many possible directions here, in real line only one left or right they are possible directions.

So, but you can go any direction, so what about this thing, why this is a closed set? Now, I have to prove that for every sequence like in the interval closed interval, where the same proof works basically, if I take any sequence xn in this closed ball, and converging to A, which is, that means what?

The distance between xn and a will become, xn goes to a. So, what is the distance of xn from x? Because, if I want to show it is closed, I should show that the distance of the limit from x is less than or equal to epsilon, but the distance of each point xn from x, because xn is inside is less than or equal to, so can I say that.

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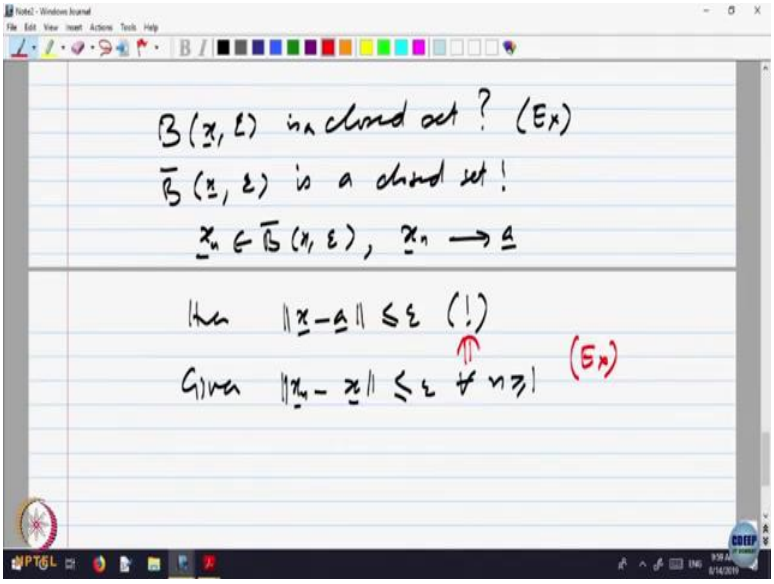

So, here is something I am giving you hint which is true, if xn belong to ball x epsilon and xn converges to a, then x minus a is less than or equal to epsilon, so that is what is to be done, and what is given? Because xn's belong, given norm of xn minus x is less than or equal to epsilon, for is less than or equal to for every n, because xn's belong to xn's belong to the ball, closed ball so less than or equal to.

We want to show that this implies, so that is a exercise, justify y norm of xn minus x implies that thing. Idea is similar to that of the closed interval a, b, if xn is in the closed interval a, b, then xn minus a they remain there is order here, here is that order of distance actually, so try that, so that will also prove module of this part of exercise that is also is a.

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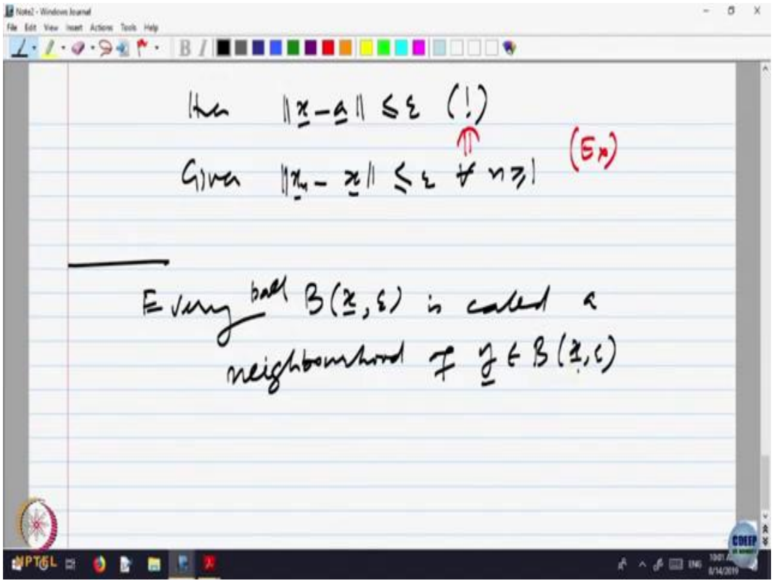

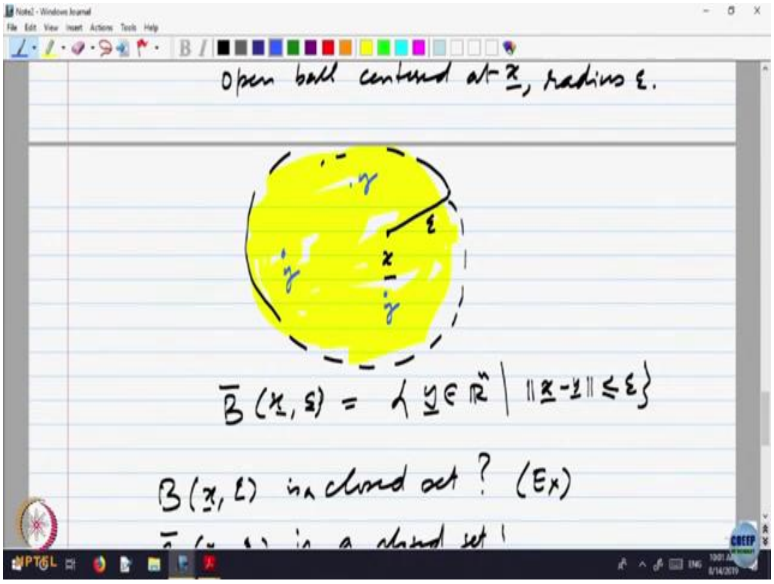

So by the way, this kind of things, let us give it a name because they are going to be useful, for every point y belonging to ball centred at any x and radius epsilon is called, no I should write it properly, every ball every open ball is called a neighbourhood of y belonging to be x epsilon.

So, what you are saying is, if it is an open ball of radius epsilon centred anywhere, does not matter, then for every point y inside that ball the ball is called a neighbourhood of the point y, the ball is called a, so that means for this, y could be anywhere, so here is a point y or it could be anywhere y here. For all elements of that open ball, this open ball is called a neighbourhood of that point, so obviously it is neighbourhood of the centre also anyway.

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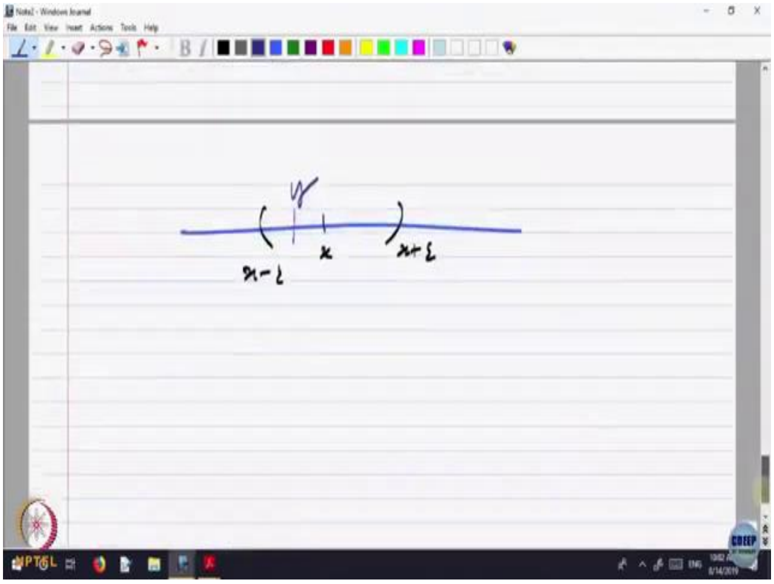

So, it is something similar to, let us look at what happens in the real line, so in the real line, if you have got a, so here is point x, what is an open ball of radius epsilon? So, it will be x minus epsilon and that is open interval simply. So, what we are saying is for every point inside this is a neighbourhood, it is a neighbourhood of the point y or this is called okay. So, I think I will elaborate it slightly more later on. So, is the concept of neighbourhood okay for everybody? An open ball is neighbourhood of every point inside it, that is a definition.

So now, let us look at closed sets, so we have looked at many examples of closed sets, so we have seen that there are sets which are not closed and there are some sets which are closed. So, can we do one thing, take a subset of Rn, some of the limits may go out, let us throw them in and make a bigger set, so what do you expect of the bigger set? The bigger set should have the property that it contains limits of all sequences in that.