Galois' Theory
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Lecture No 58
Primitive Element Theorem

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Ok so we have stated a Primitive Element theorem

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for separable, finite separable extensions and we will prove it now.

So the theorem we will prove is this is Primitive Element theorem.

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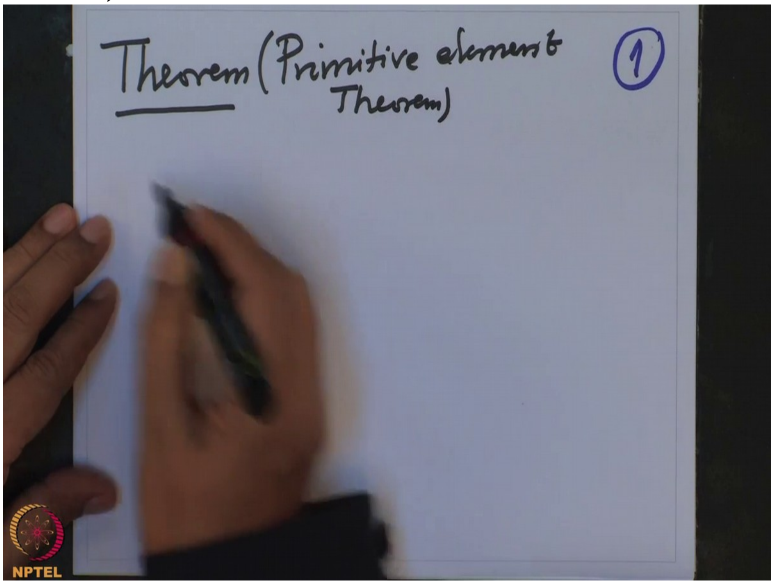

So let L over K be a finite field extension, finite separable field extension.

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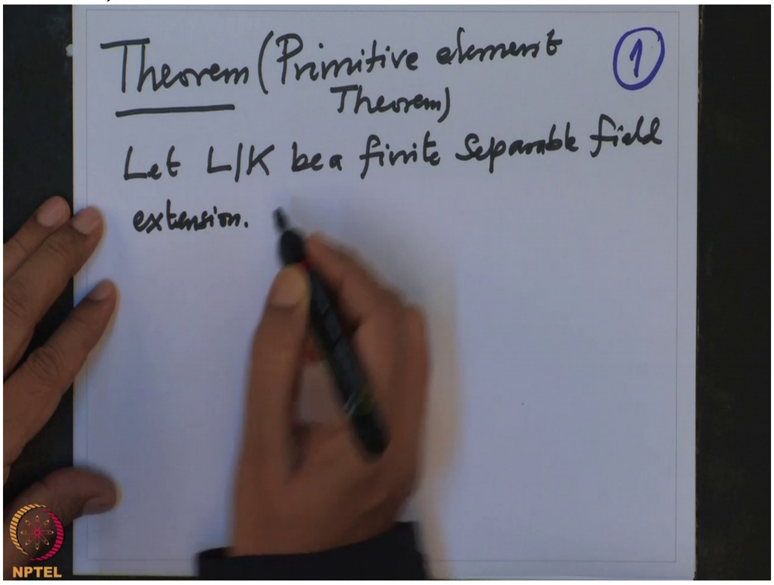

Then L over K is simple. This is what we want to prove.

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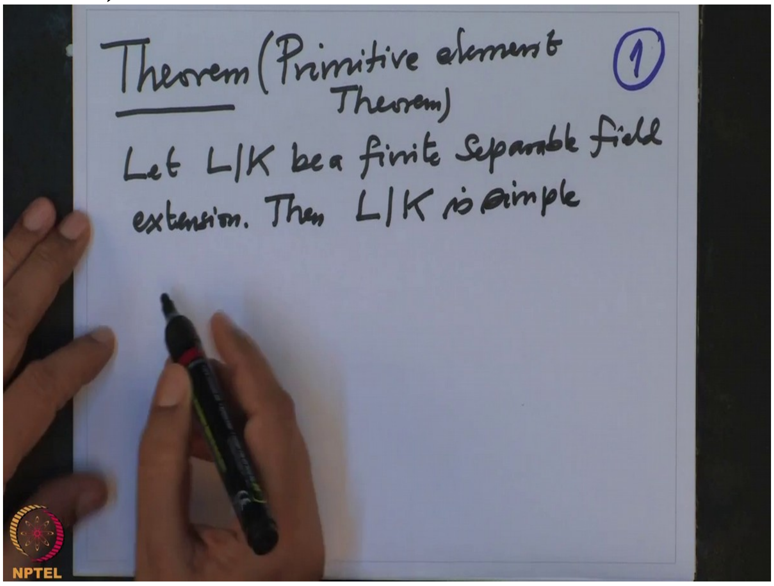

Proof

Let me just mention

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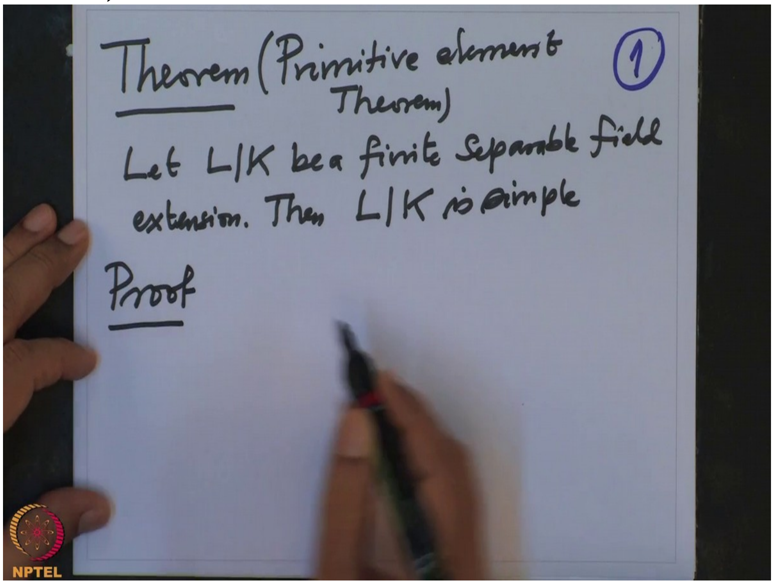

that this was the most important step in Galois Theory. Galois was looking for primitive element for a separable extension.

His, in his time, all extensions considered were characteristic 0 fields. They were characteristic 0 fields therefore all were separable and they were all the time looking for primitive elements.

And that is very, very important step in the Galois Theory.

So, so let E over K be any algebraically closed field extension. Then what we have proved

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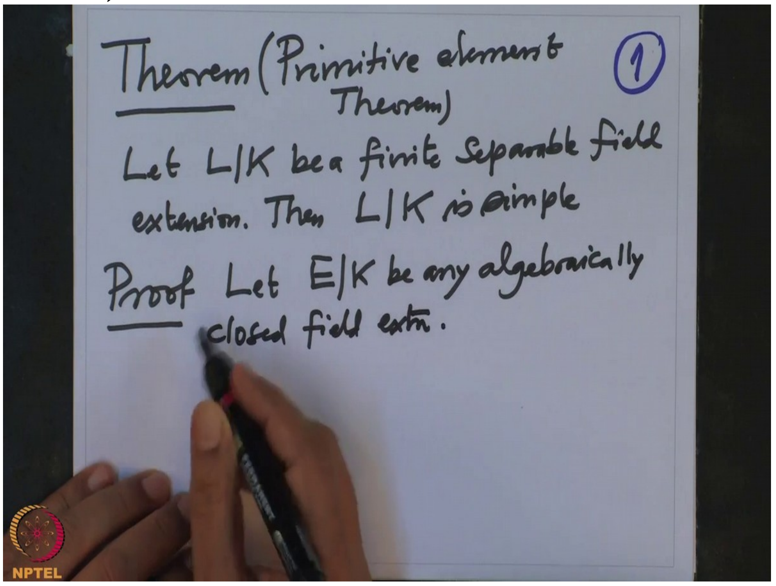

is then since L over K is separable, we know

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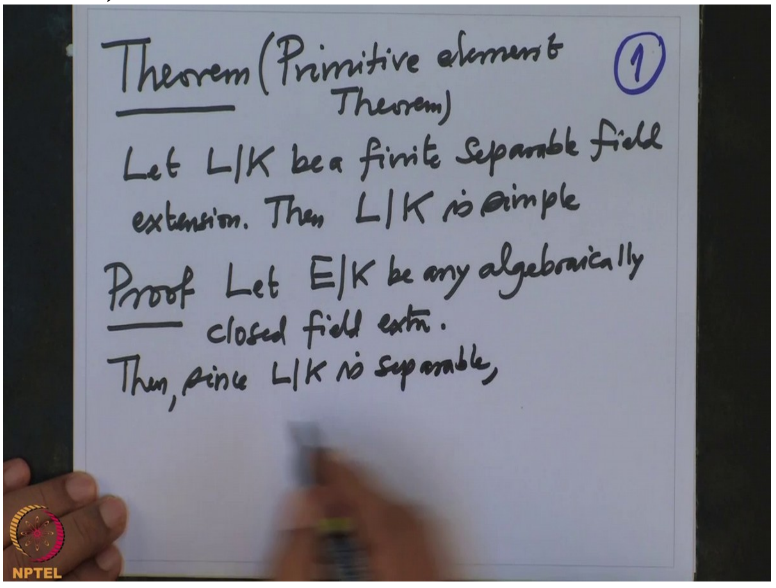

that if n is the degree of the field extension, this degree is also equal to the separable degree L over K S, and

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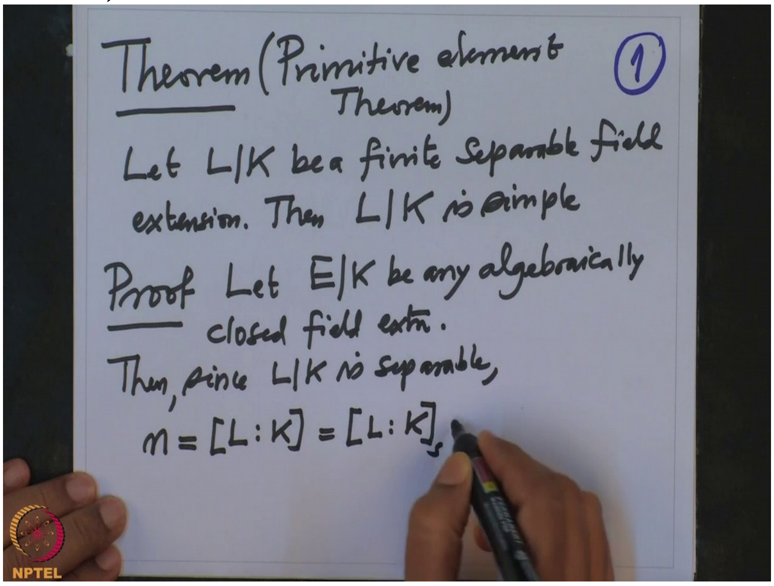

separable degree is by definition, cardinality of the embeddings of L inside K, K embeddings of L inside K.

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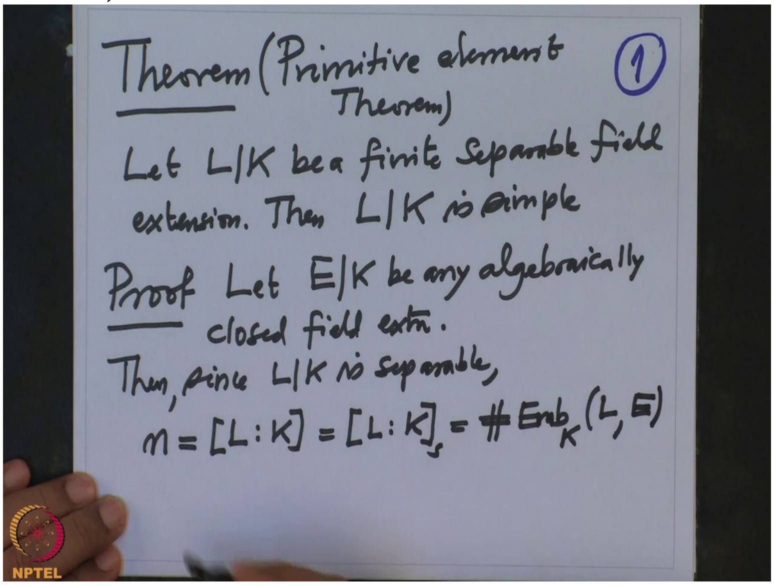

So let us say that, that means these embeddings, there are precisely n embeddings. So let embedding set L E, this be exactly equal to $\sigma_1, \sigma_2, \ldots, \sigma_n$.

There are precisely n K embeddings

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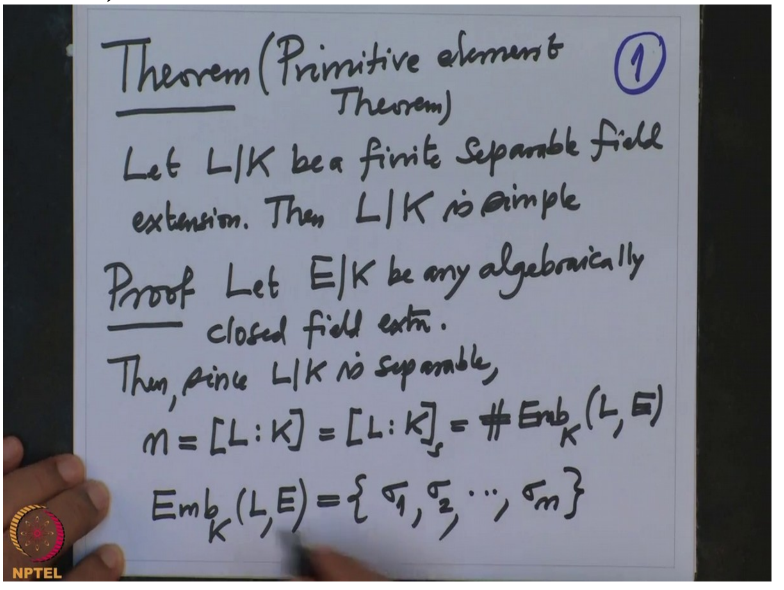

of L into E. And what are we looking for? We are looking for an element z, so we are looking for an element $z \in L$

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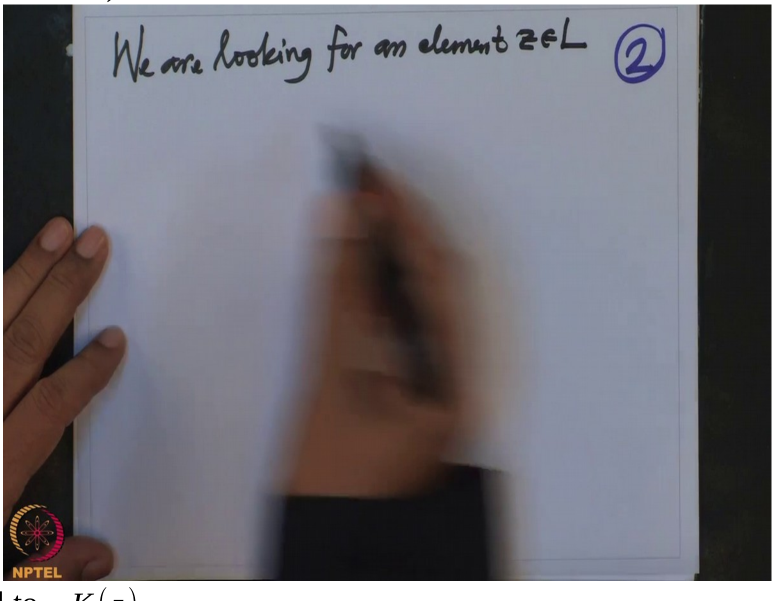

such that L equal to $K(z)$.

Or equivalently this equality here will also follow from this inequality L over K bigger equal to $K(z)$ over K

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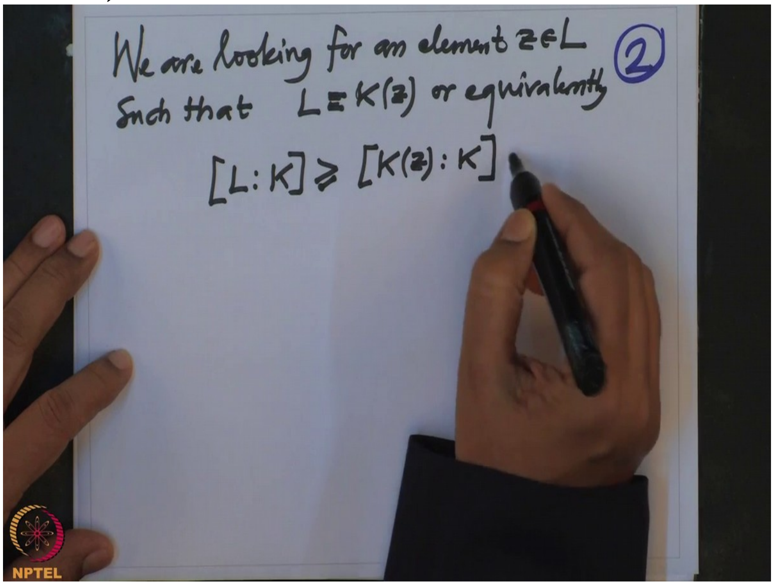

and this is equal to degree of $\mu_{z,K}$,

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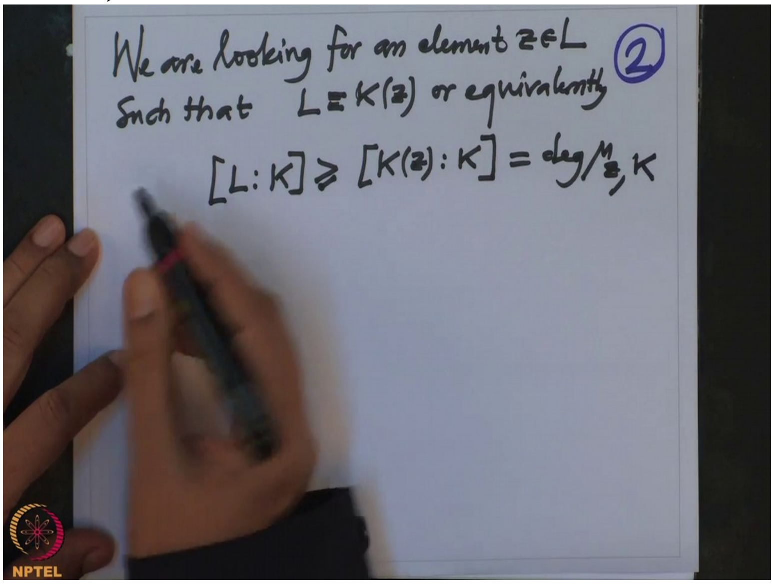

this is n

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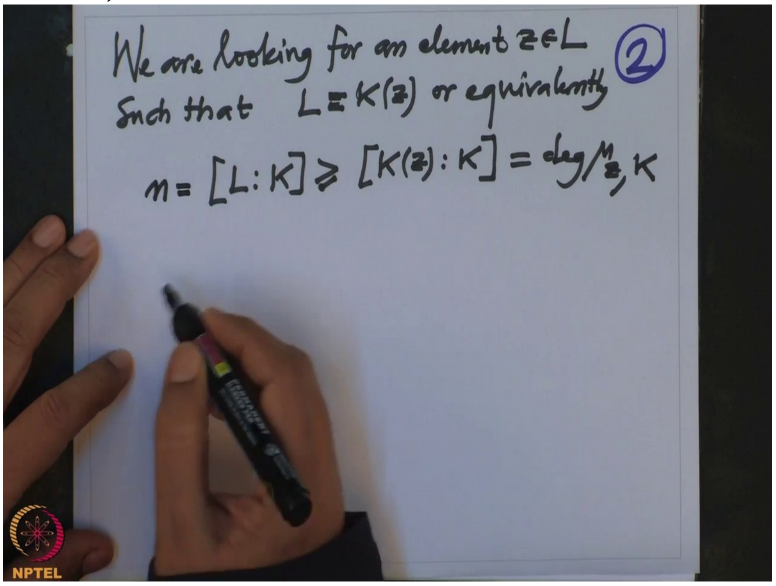

and this is bigger equal to, we have seen this degree is bigger equal to the zeroes, zeroes inside E but $\sigma_1(z),\ldots,\sigma_n(z)$, they are all zeroes

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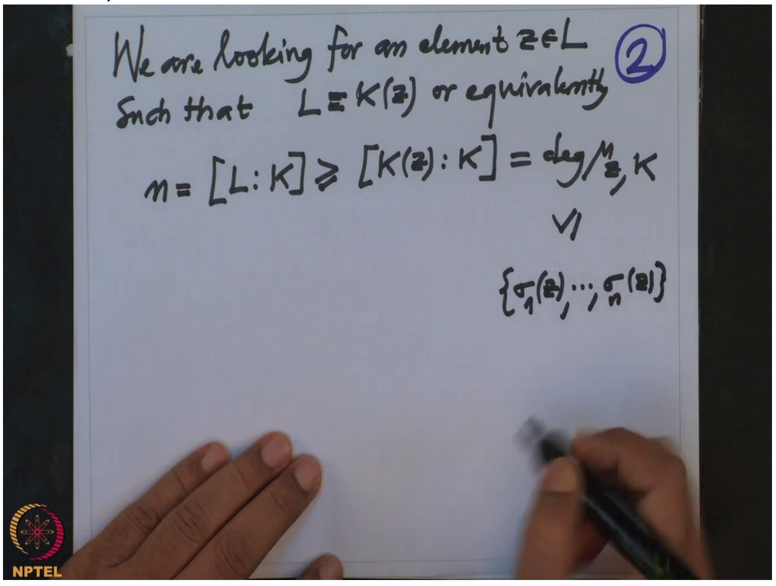

of mu, because the sigmas are K-algebra homomorphisms.

And this, this number,

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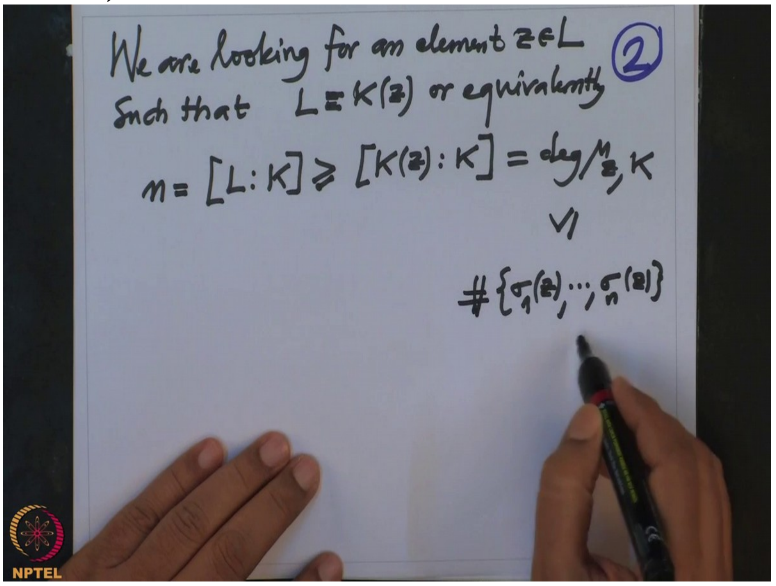

so this number is also n because

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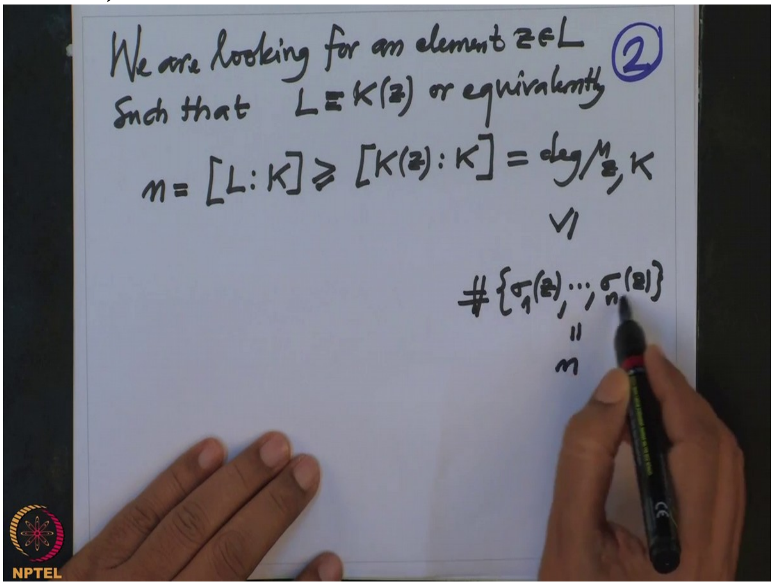

these sigmas are uniquely determined by z

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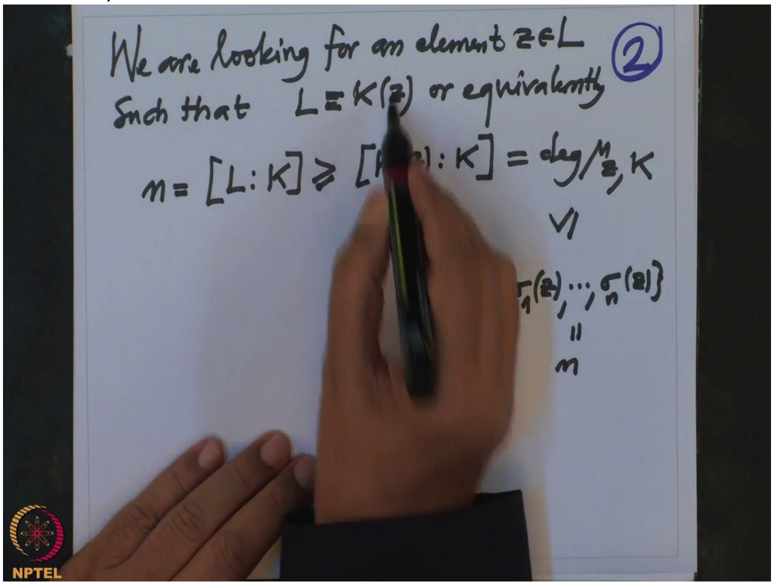

on L.

So therefore this number will be n. So therefore all these inequalities will be equalities and we will get equality here and we will get

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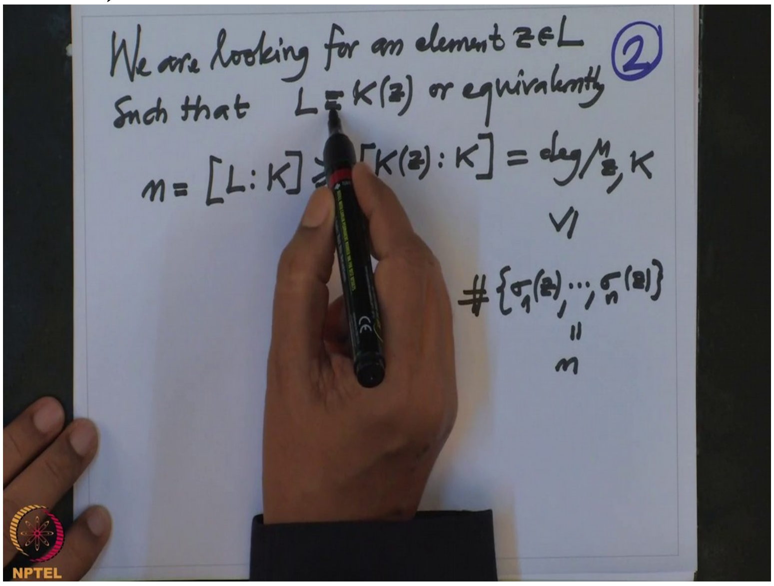

L equal to $K(z)$.

So I want to check that, so in other words I want to find z, to find $z \in L$ such that

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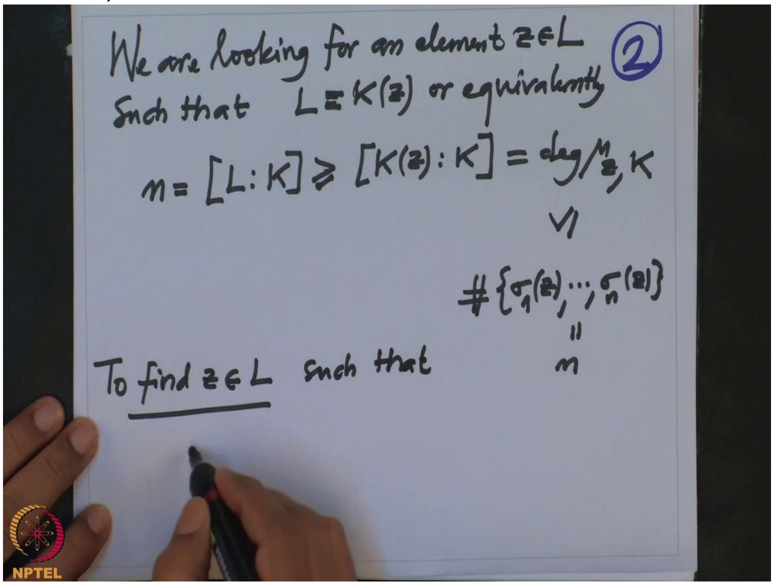

the $\sigma_1(z),\ldots,\sigma_n(z)$ are all distinct elements of E, they are elements of E, but the distinct is important.

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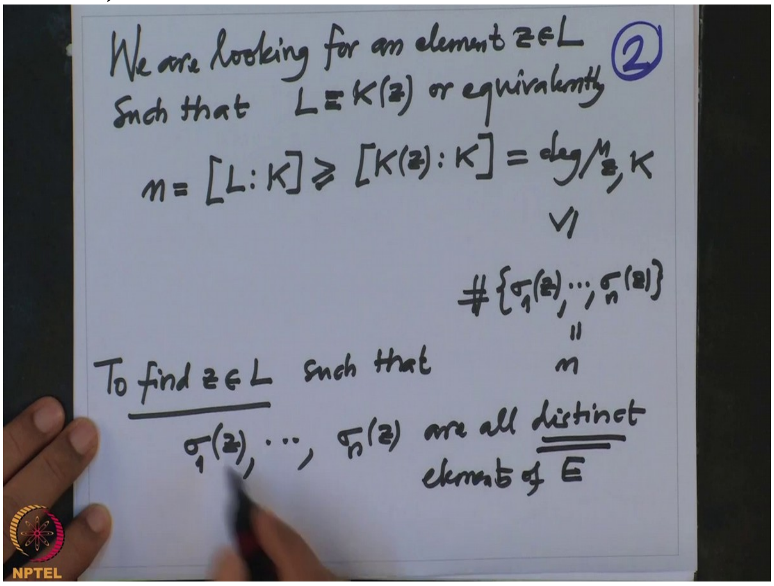

So once we achieve this then this number will be n and this μ, they are all zeroes of the minimal

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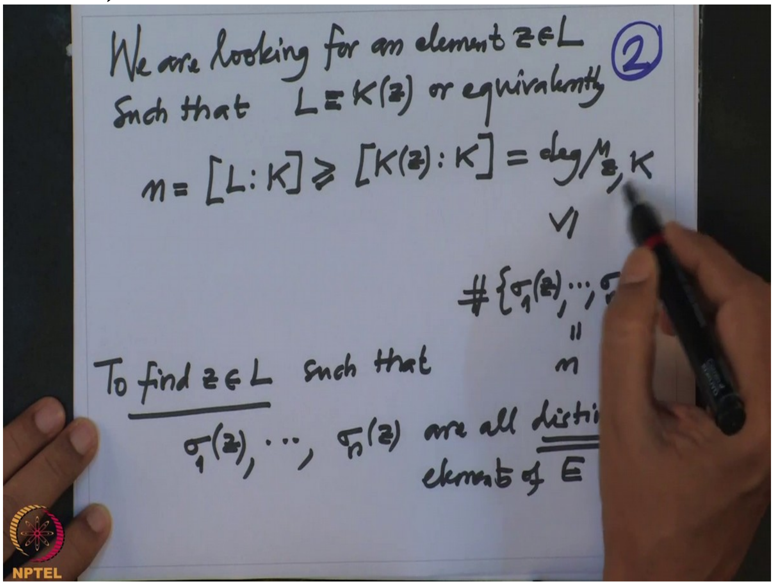

polynomial, therefore the degree of the minimal polynomial will be bigger equal to this. Degree of the minimal polynomial

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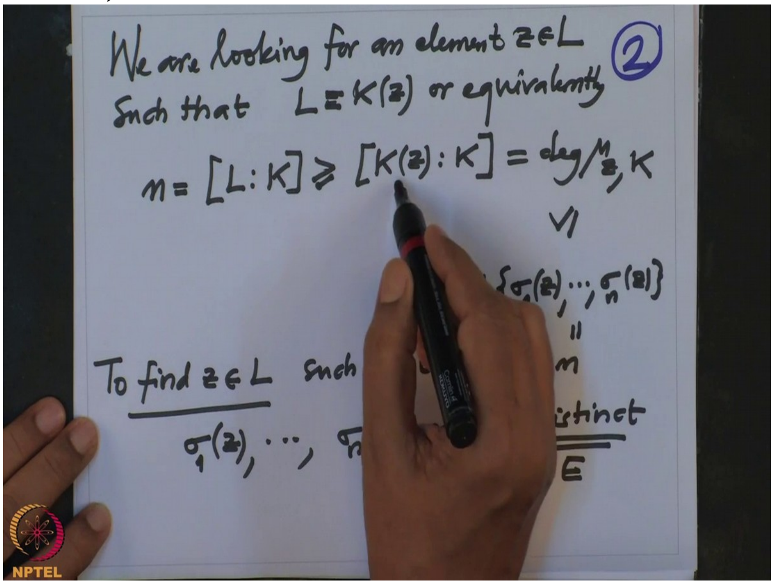

is the degree of $K(z)$ over K and z is contained in L therefore this degree is smaller

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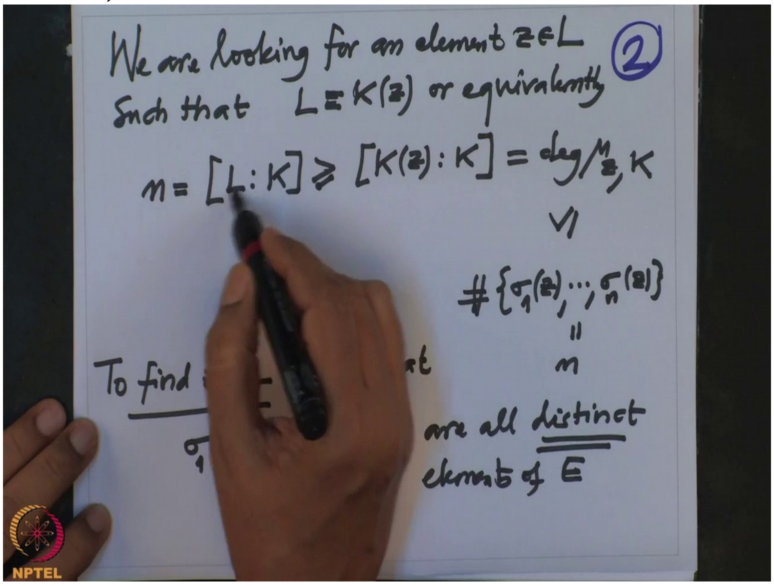

equal to degree of L over K.

But this is precisely n. Therefore if I prove that these are distinct then all will be equalities here and this will follow,

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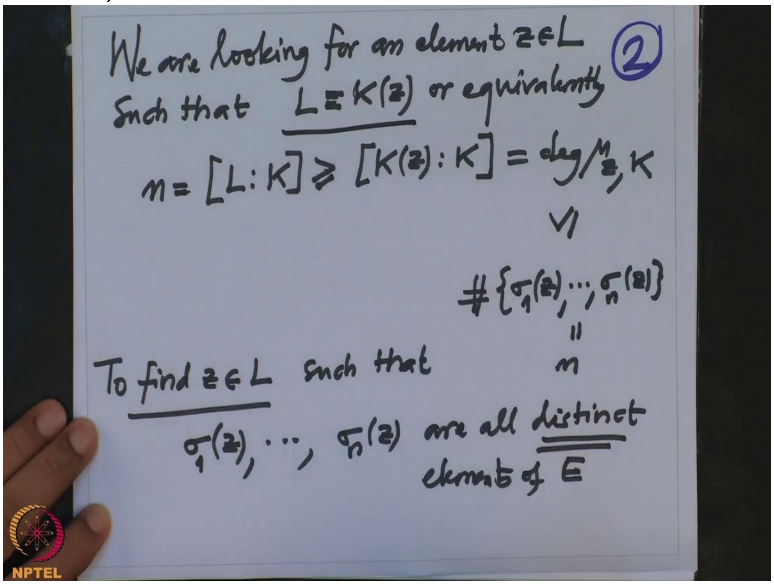

clear.

Therefore our problem is to find z

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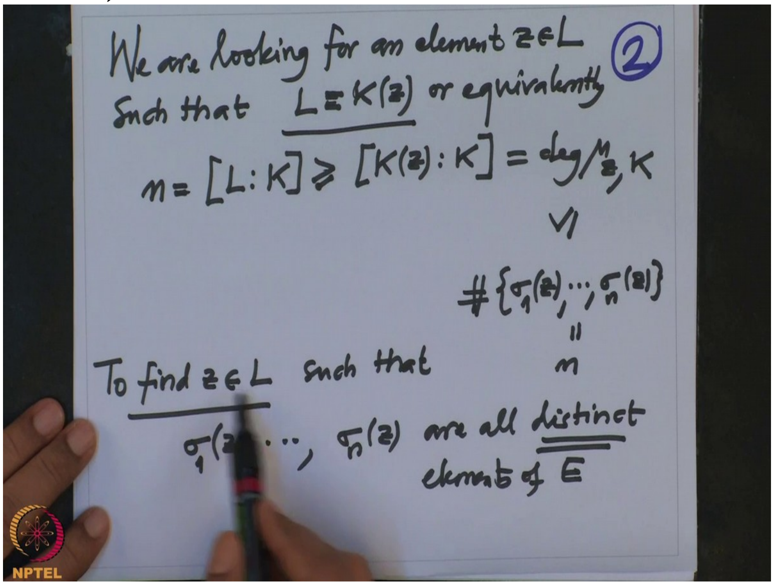

in L such that if I take, if I evaluate all embeddings of L into E at z the number should be different. Then all these elements should be different elements of E.

That is what we are looking for.

So this will imply L equal to $K(z)$.

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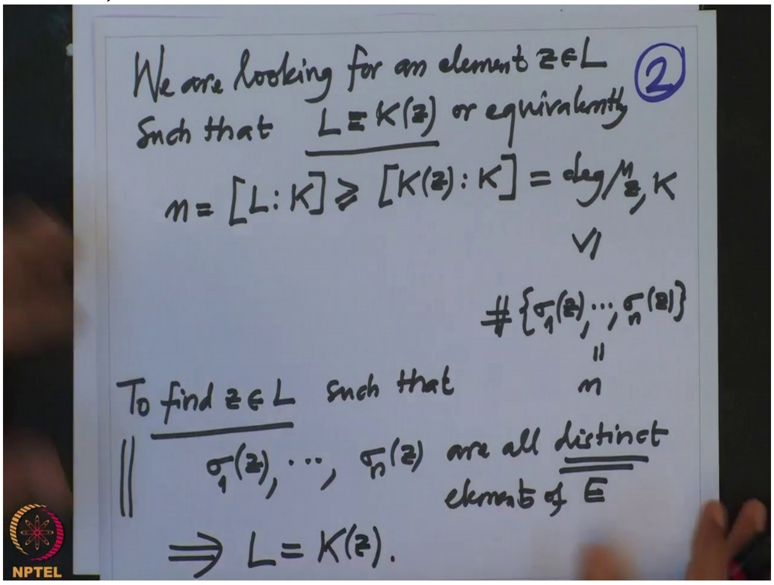

And we would finish our proof, Ok.

So how do we achieve that? I should have said in the beginning itself we may assume K is infinite.

Because if K

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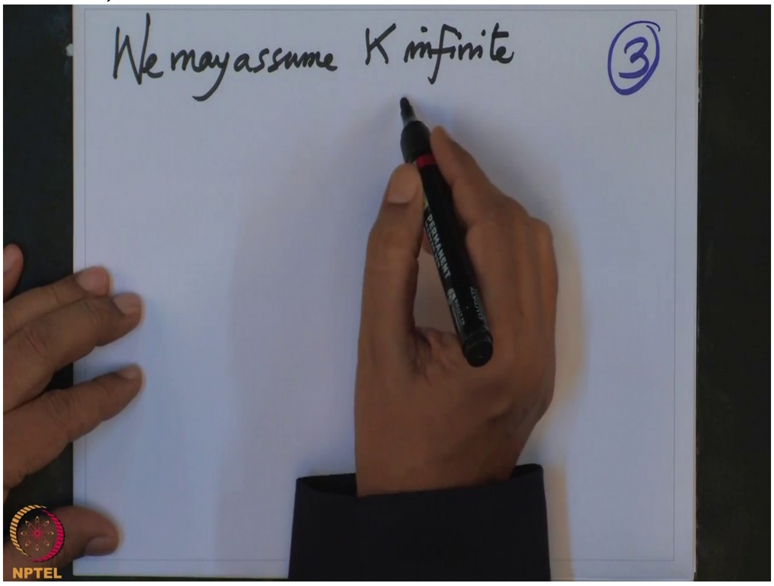

is finite and L over K is a finite field extension of a finite field then you already know that L cross is cyclic group.

And therefore L will be simple extension. So that is not a big deal.

So we are assuming

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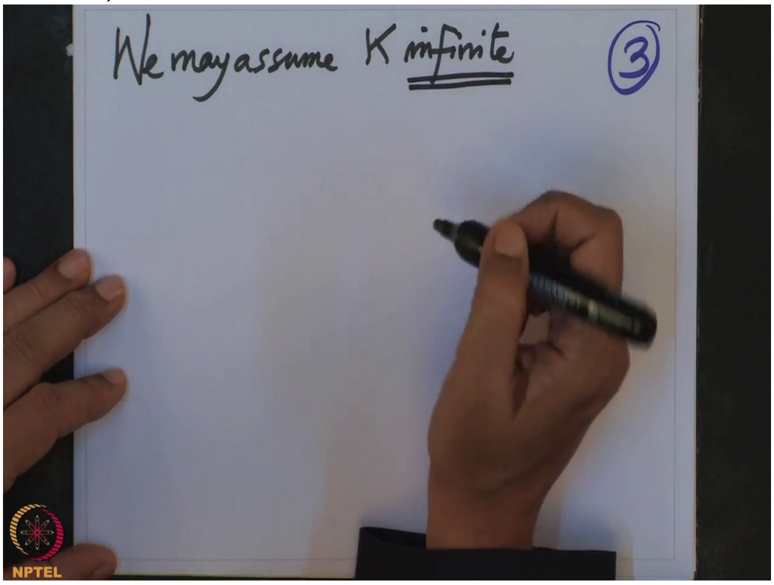

K is infinite field. And we will proceed the proof, so therefore L is a finite extension of K.

So L will generated by $x_1, \ldots, x_r$.

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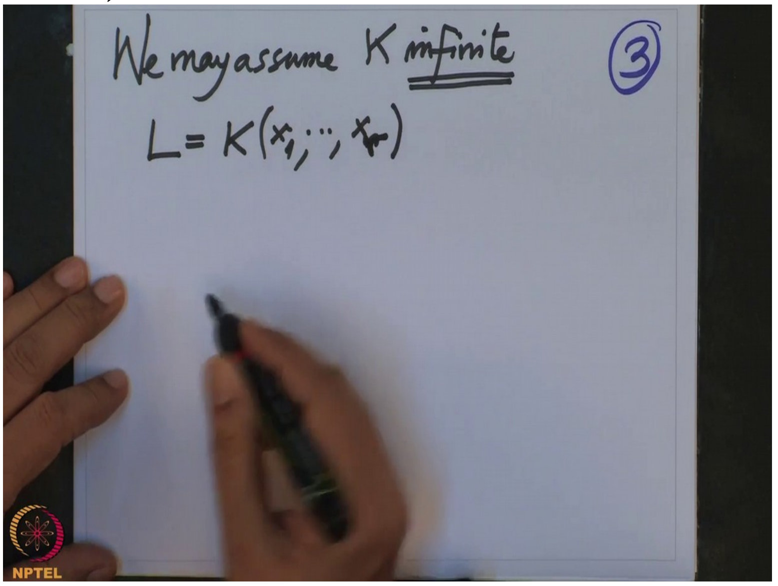

And by induction on r, induction on r, enough to prove that, enough to prove the case r equal to 2.

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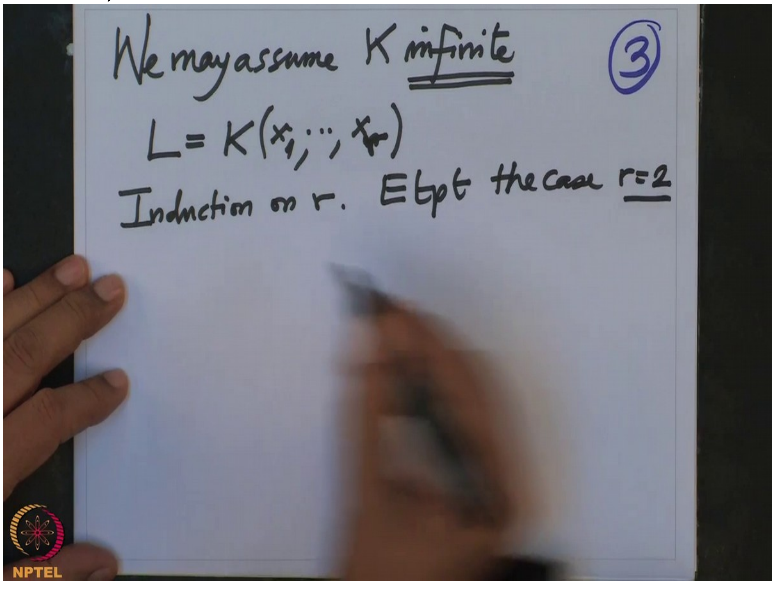

So that means what? We may assume L equal to, L is generated by 2 elements, now I will call them x and y.

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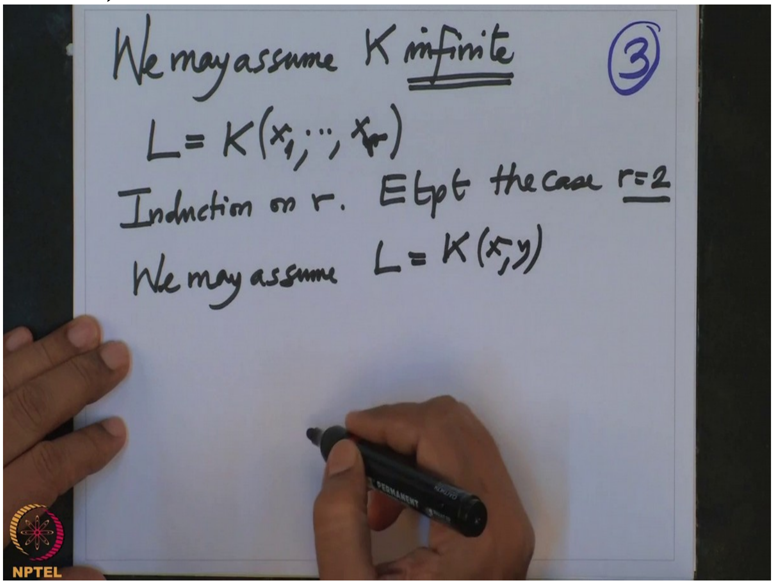

And I want to find an element z so that all values of different embeddings are different.

So then, for any indices i and j, i not equal to j but in-between n and 1,

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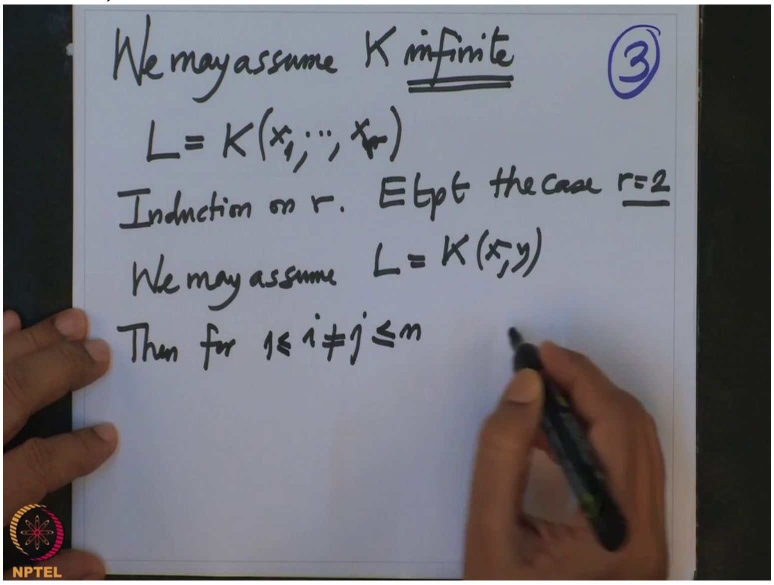

these embeddings we know they are different.

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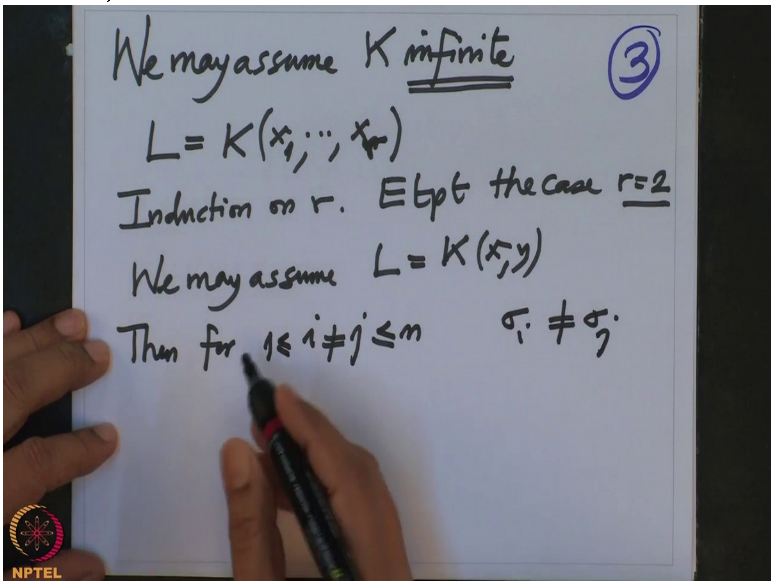

Therefore the values, these embeddings they are uniquely determined

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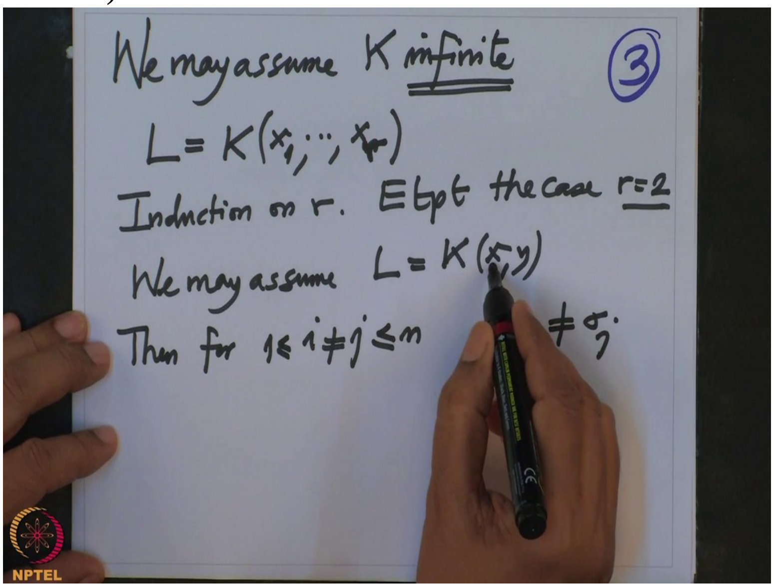

by x and y, their values and x and y.

If I know $\sigma(x)$ and $\sigma(y)$ then that is uniquely determined because this L is generated by x and y.

So if I look at $(\sigma_i(x),\sigma_i(y))$, this pair, this is a pair of elements in E.

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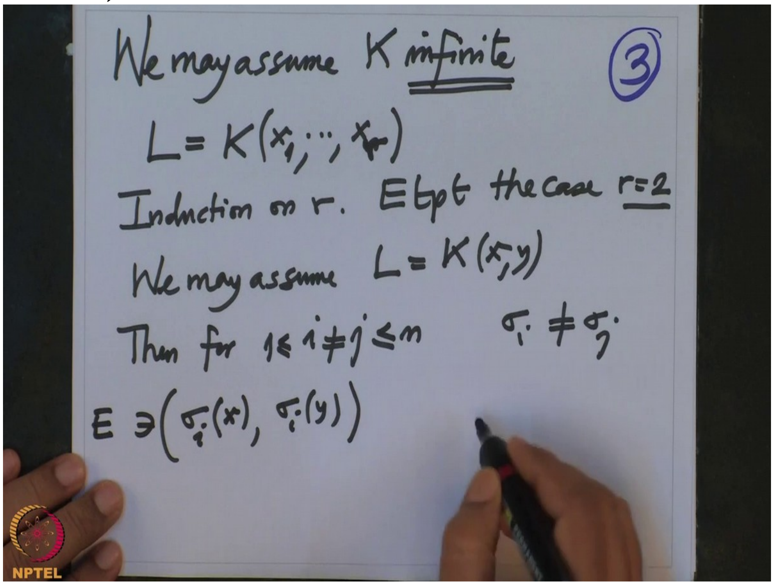

And we have this $(\sigma_j(x), \sigma_j(y))$, this is also pair in E.

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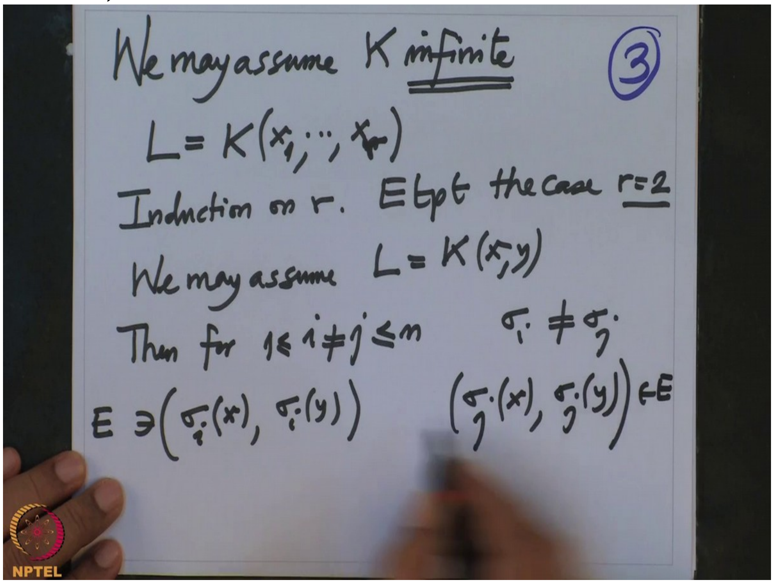

So if they are different

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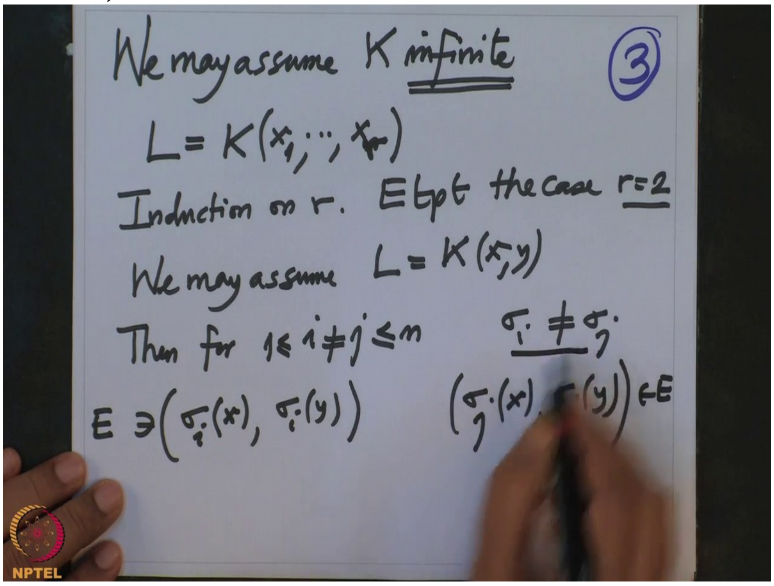

these are different.

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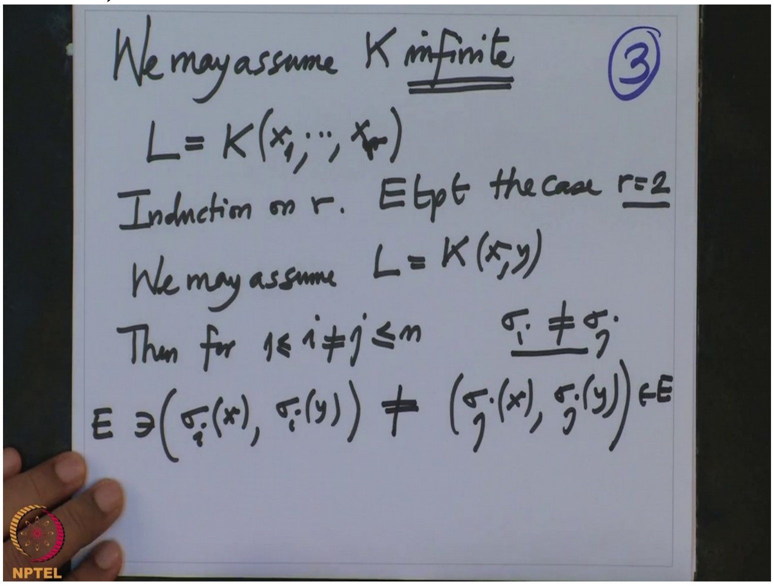

Because if they are equal, each, each element of L

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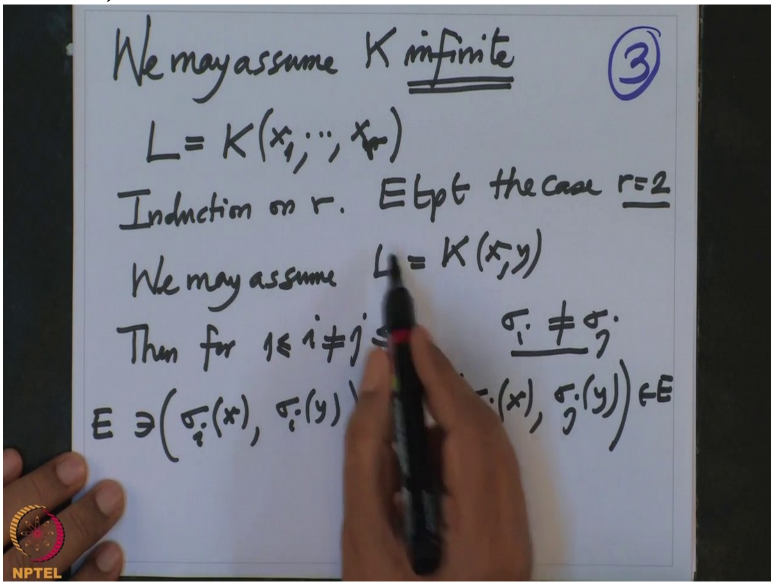

is a combination

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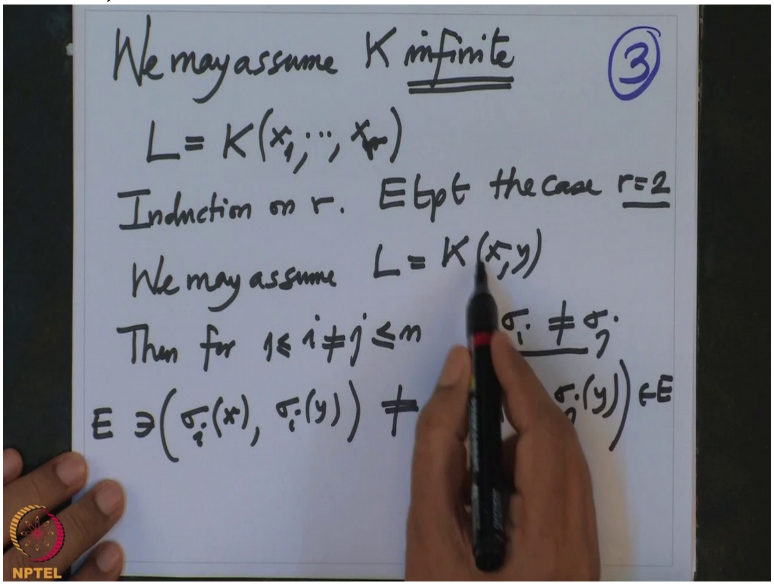

of x and y and therefore these two numbers will be equal, so they will be

So therefore for different i and j these pair of elements are different.

So this is since, $(\sigma_i(x), \sigma_i(y))$, this pair uniquely determines σ_i ,

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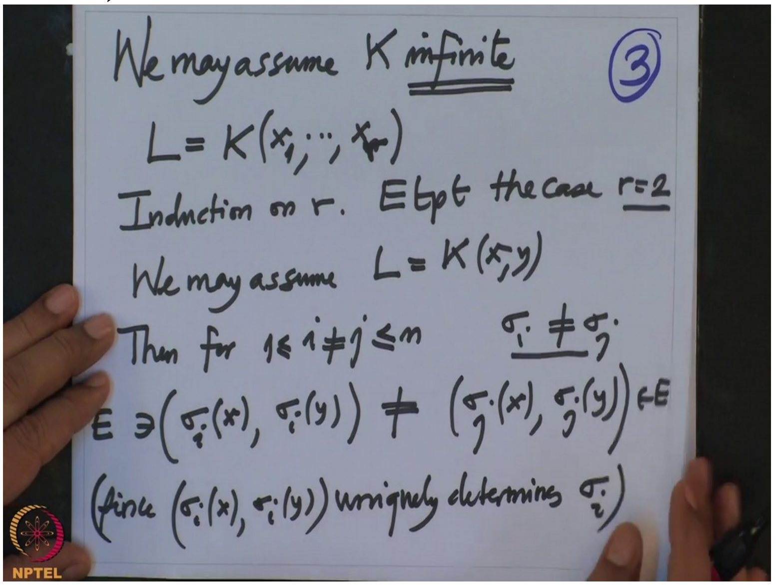

alright.

So now we have, to each embedding we have a pair. And now we consider a polynomial.

So consider the polynomial

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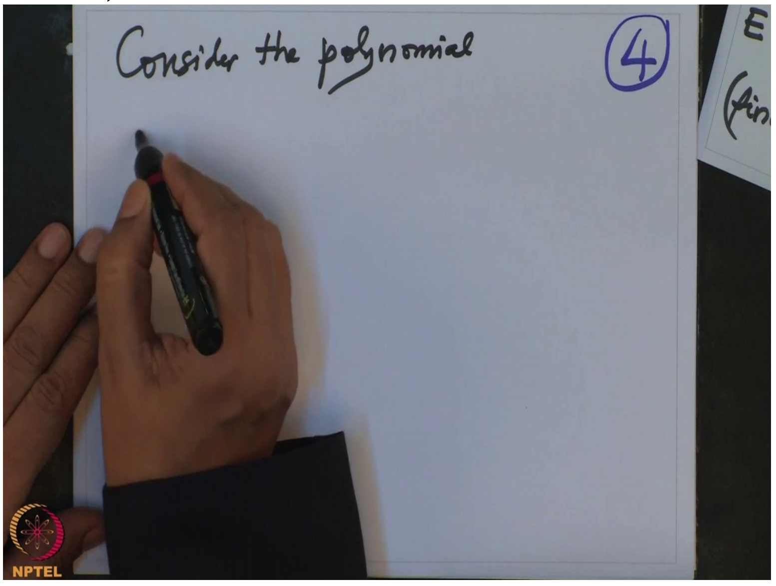

f T, this polynomial is product, product is running over pairs i comma j such that i is less than j. x is equal to n $\left(\sigma_i(x)+\sigma_i(y)T\right)$, this minus,

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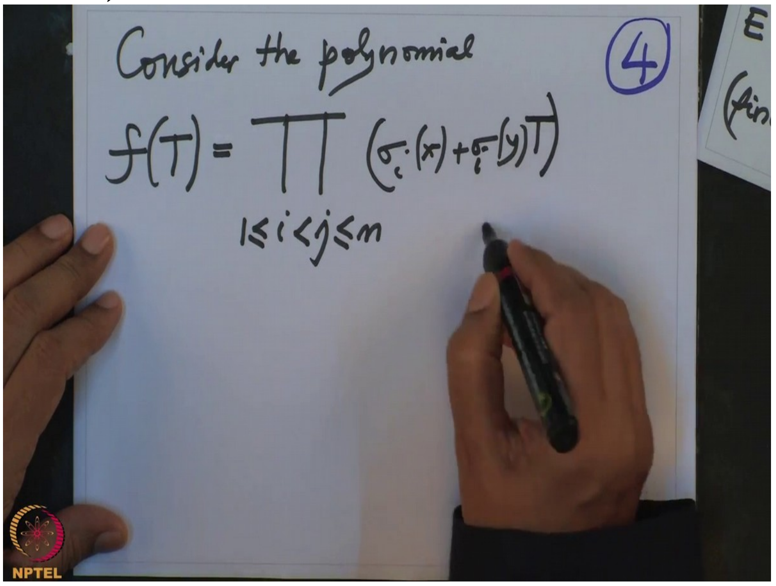

- $\left(\sigma_j(x)+\sigma_j(y)T\right)$.

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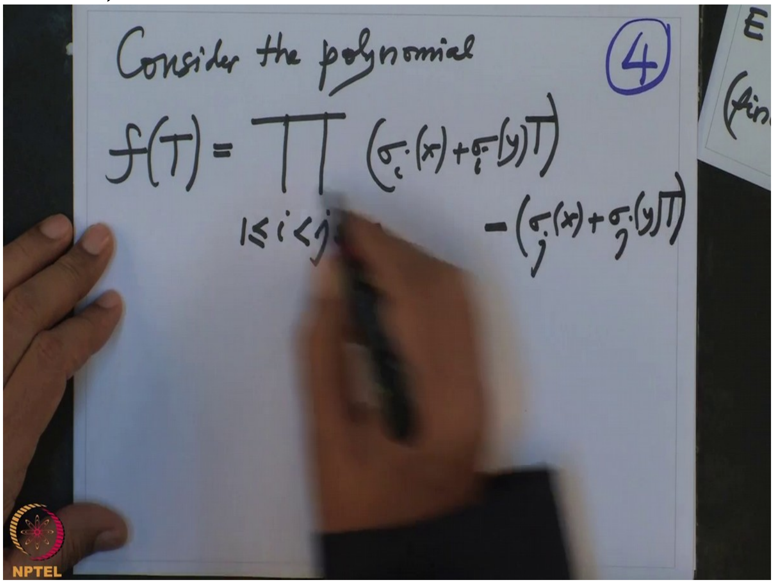

And then this is running over the product, this is like this.

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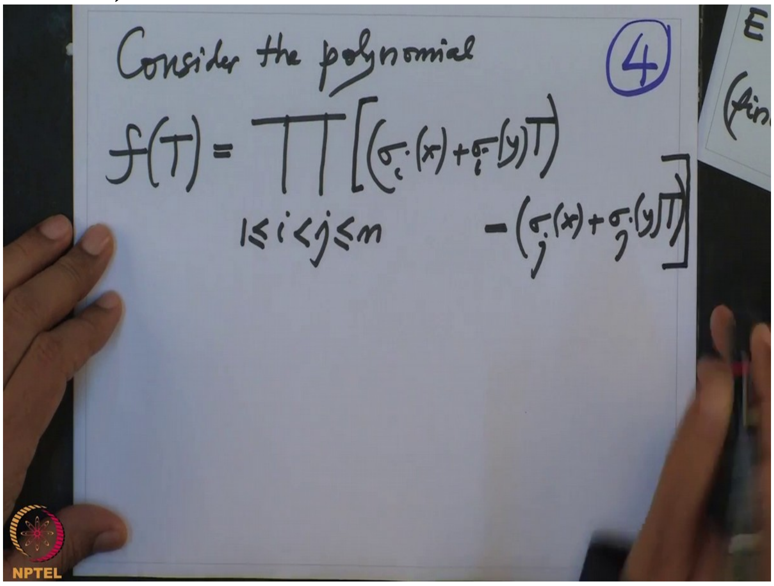

So just stare at this polynomial.

Obviously this polynomial f is non zero first of all.

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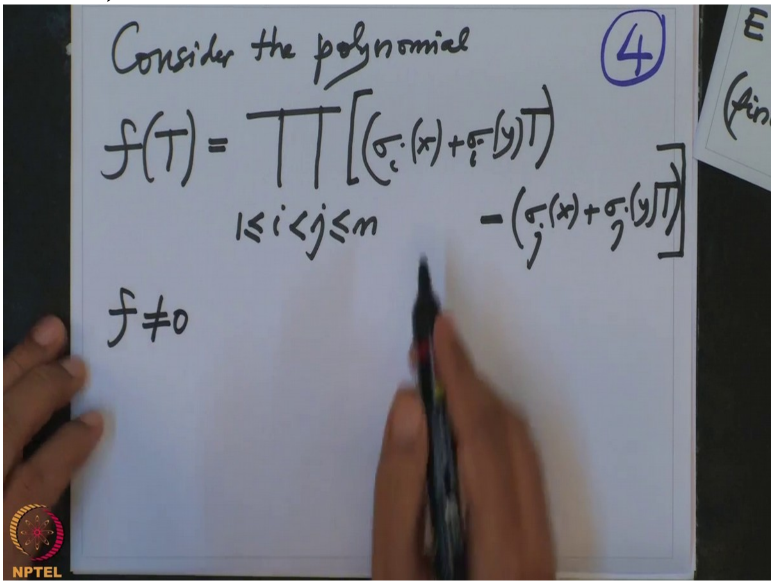

Because f 0 means what, at least one of the element in the product will be 0.

But if the element in the product is 0, that means this $\sigma_i(x)+\sigma_i(y)T$

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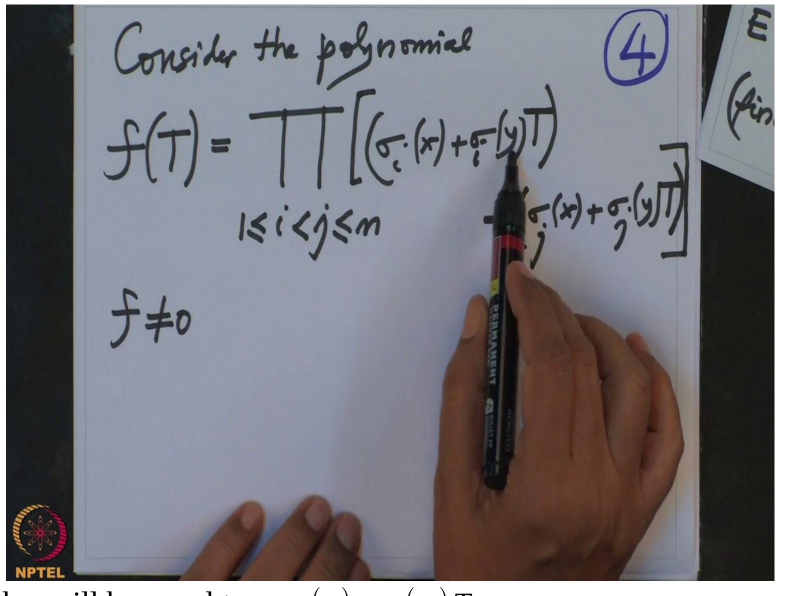

will be equal to be, will be equal to $\sigma_j(x)+\sigma_j(y)T$

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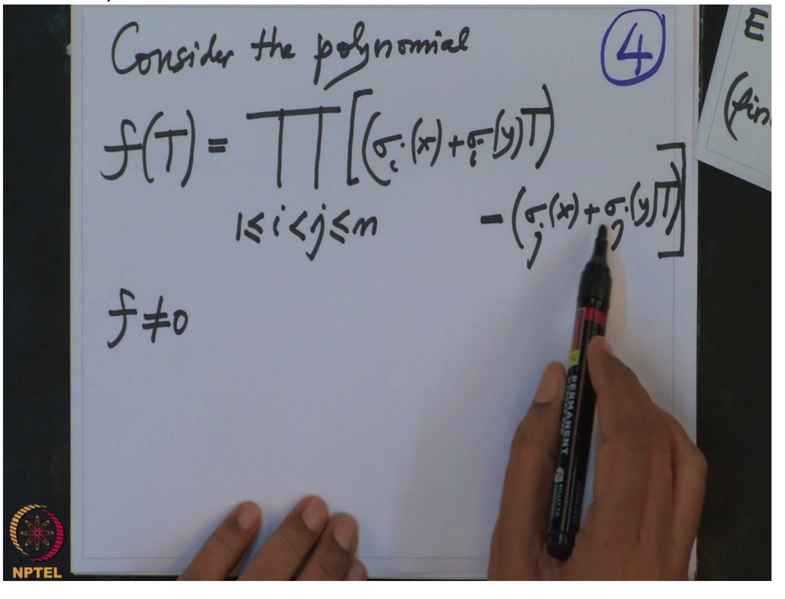

But as the polynomial in T, therefore $\sigma_i(x)=\sigma_j(x)$ and $\sigma_i(y)=\sigma_j(y)$ but i is different from j.

But that cannot happen. Because σ_i is different from σ_j.

Therefore f is non-zero.

Where are the coefficients of f? They are in E T.

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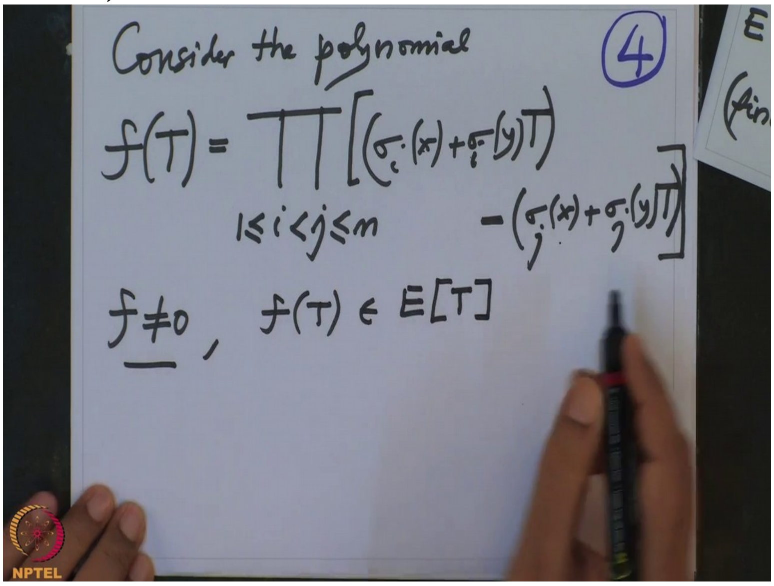

Coefficients of f

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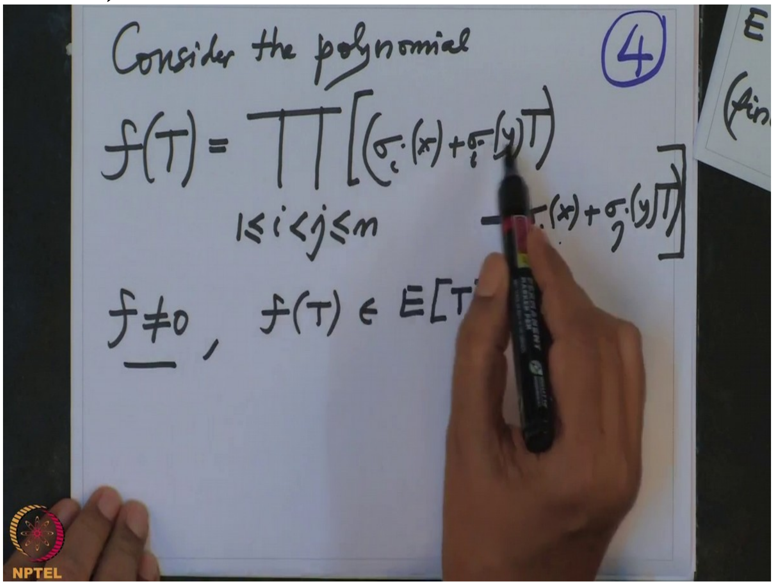

are this, they are elements, combinations of $\sigma_i \sigma(y)$ and $\sigma_j \sigma(y)$, all are elements of E therefore they are elements of T.

E is algebraically closed

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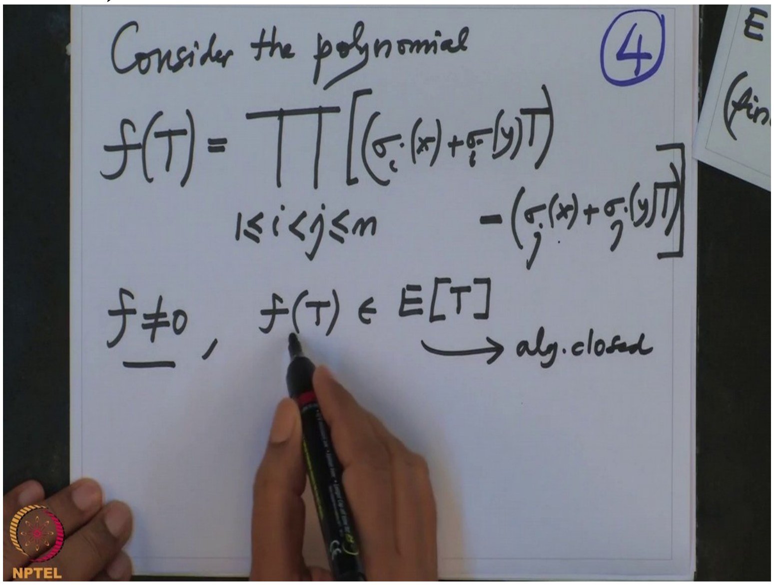

and we have a non-zero polynomial in algebraically closed field.

Therefore, therefore so in particular it has finitely many zeroes and all the zeroes are there.

V K, how many zeroes are in, now I look at the zeroes of this in K but this is contained in $V_E(f(T))$

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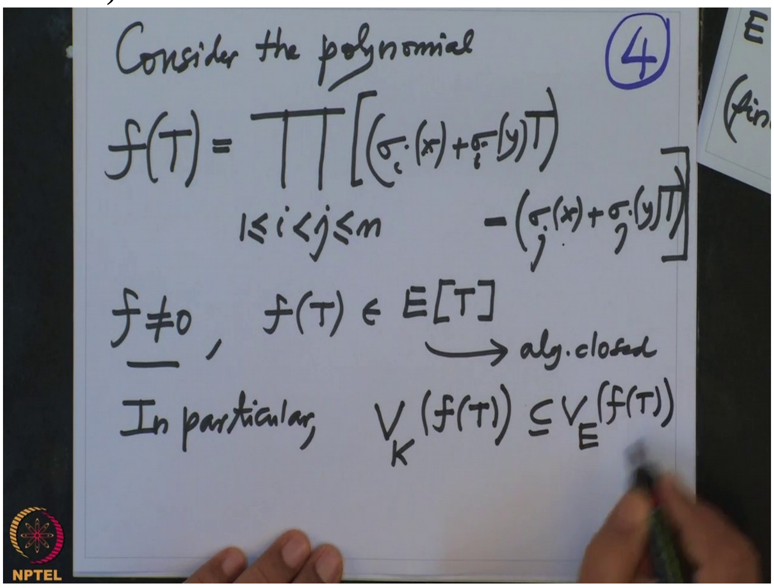

and this is finite,

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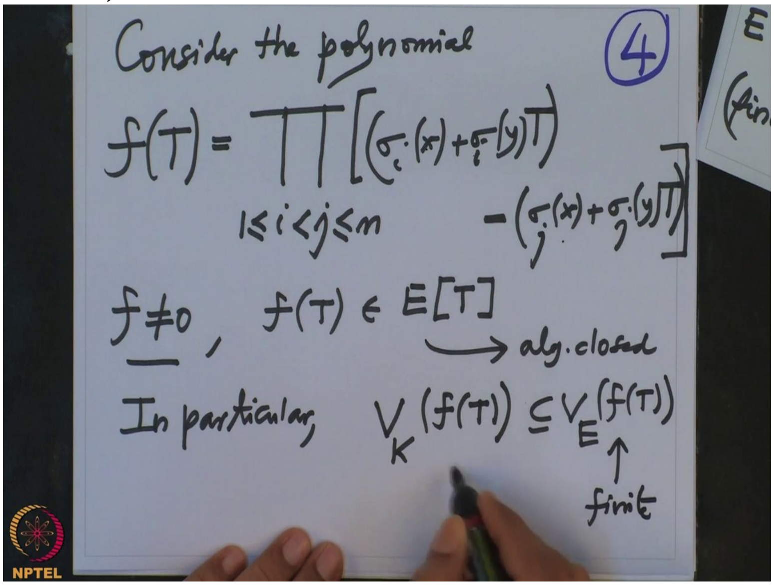

therefore this is also finite.

It may or may not have a 0,

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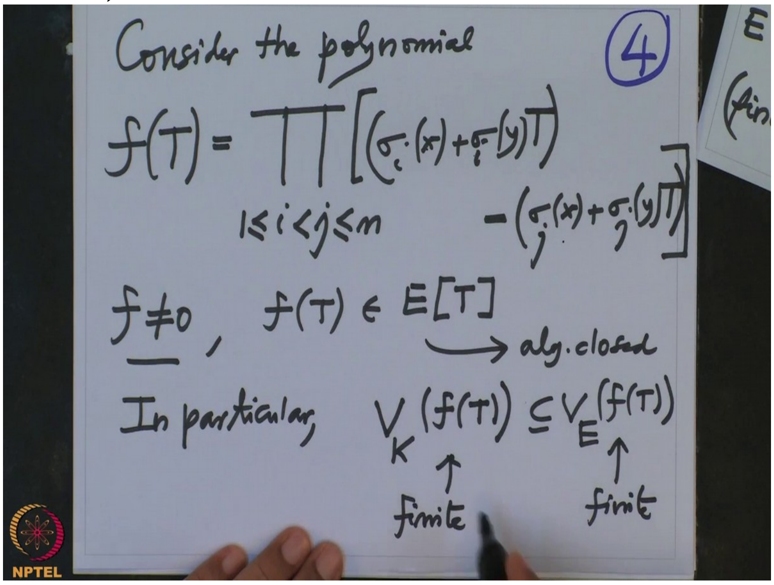

but in any case it is a finite set. It may be empty set

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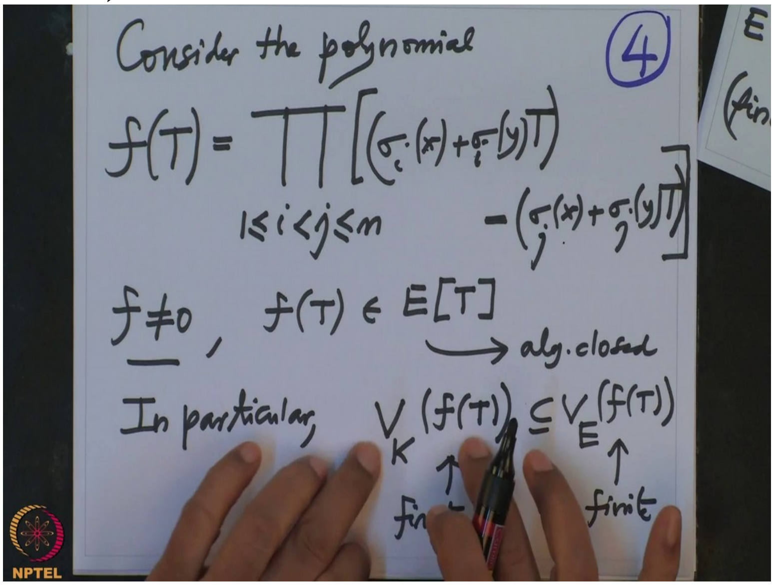

but it is finite,

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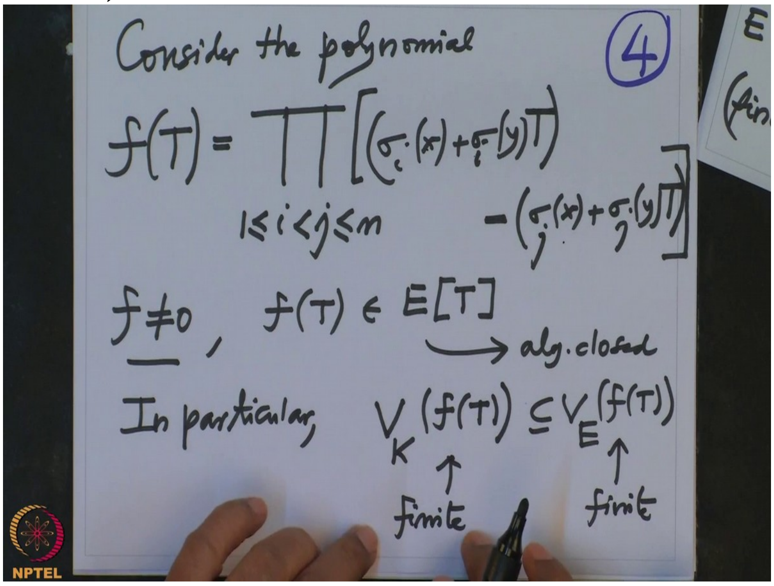

finite is important.

Now it is a finite set. So I can find an element which is outside that.

So therefore there exists an element t in K such that f of t is non-zero

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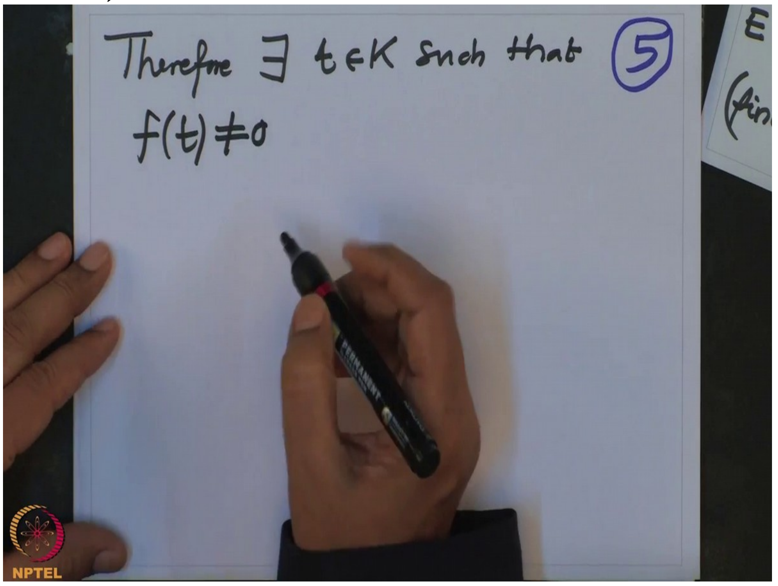

because since K is infinite.

Infinite field

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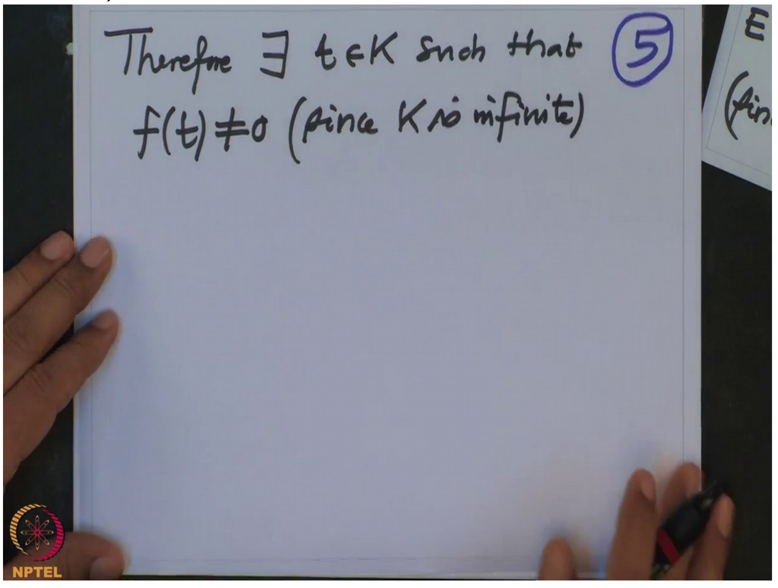

and a polynomial is non-zero therefore it has zero, not all elements it is zero so at least element it is non-zero.

Now let $z = x + yt$. This is obviously an element in E,

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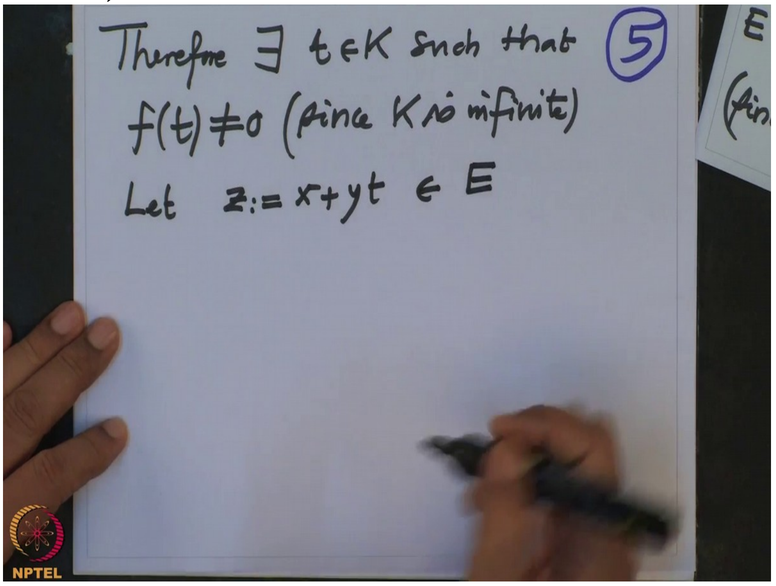

because

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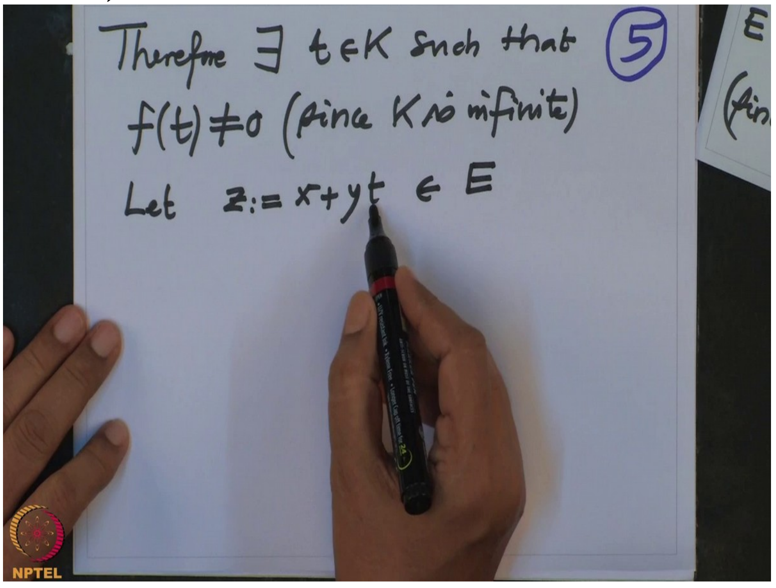

t is in K, K is a subfield of E and x and y are in L and therefore they are, there is embedding so they are in elements of E also.

So these are the elements in, no what I am saying is actually they are elements in L

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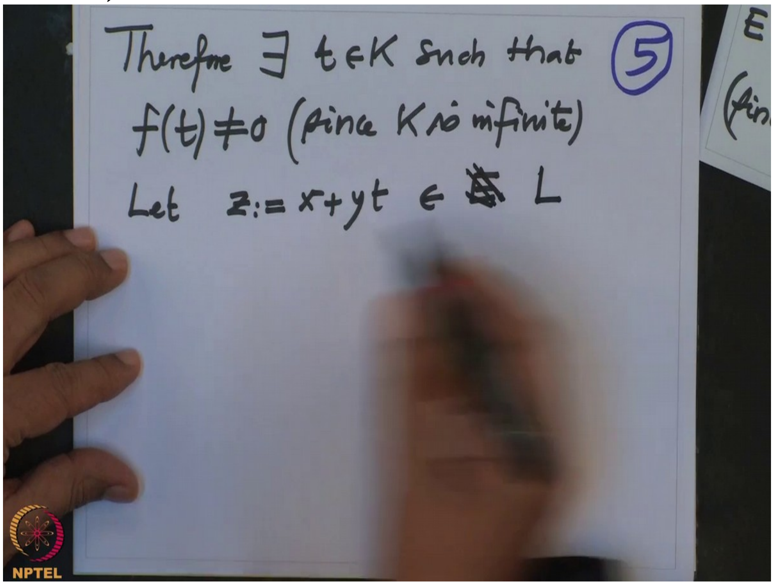

because x is in L, y is in L and t is in K therefore they are elements in L.

Now I want to check that this z is

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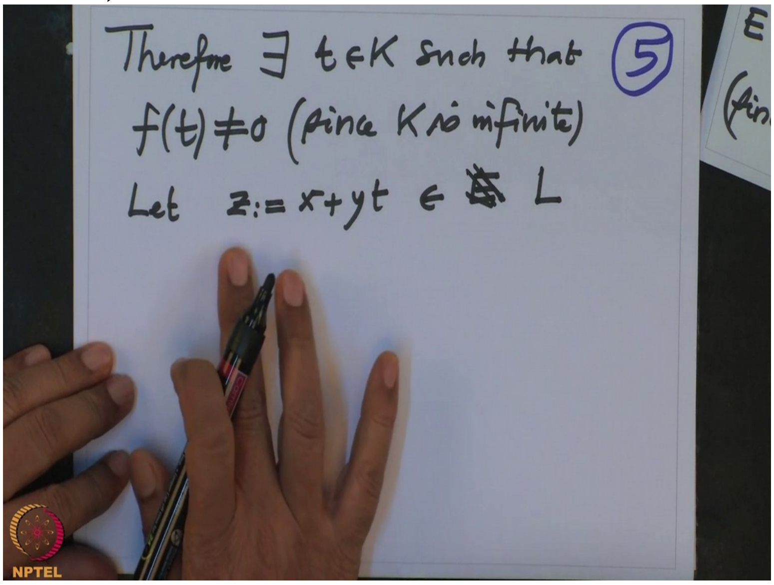

a required element.

So we may want to check, want to check that L equal to $K(z)$. And that

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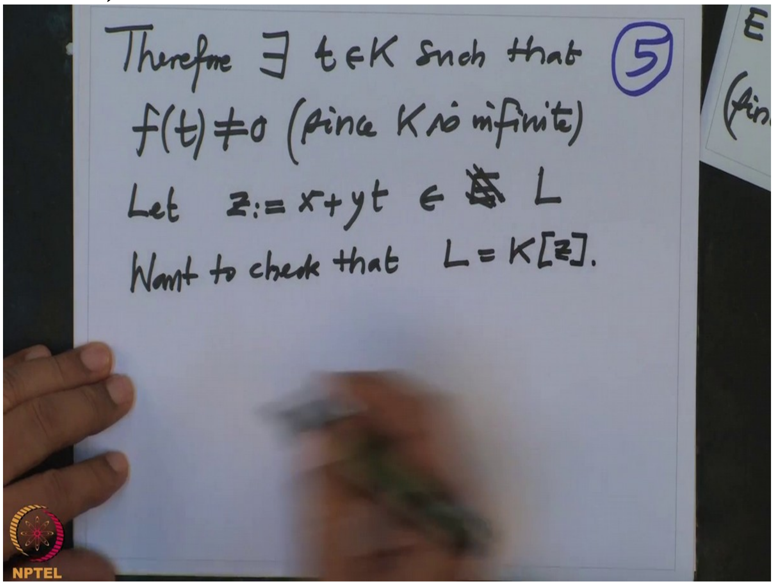

will finish our proof, alright.

So look at $\sigma_i(z)$ is, apply σ to this.

Because sigma is a K-algebra homomorphism so this is $\sigma(x)+\sigma_i(yt)$

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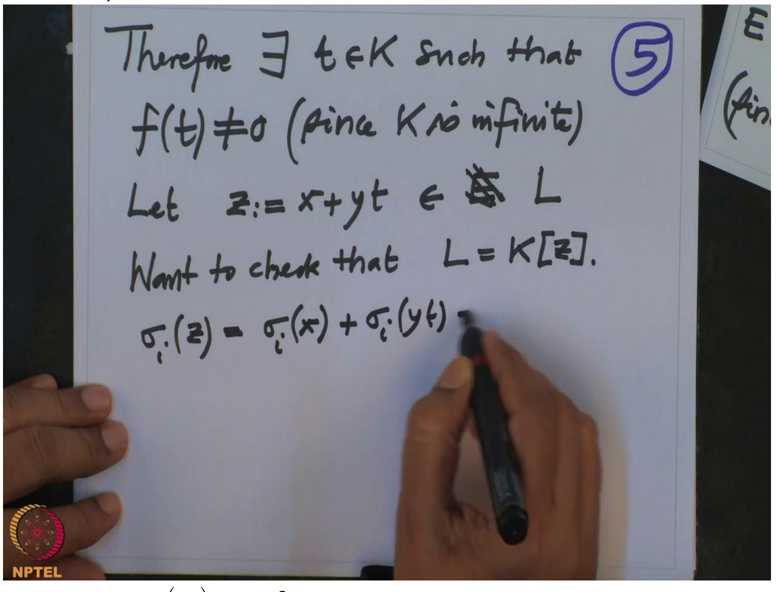

but $\sigma_i(yt)$ is same as $\sigma_i(y)$ and

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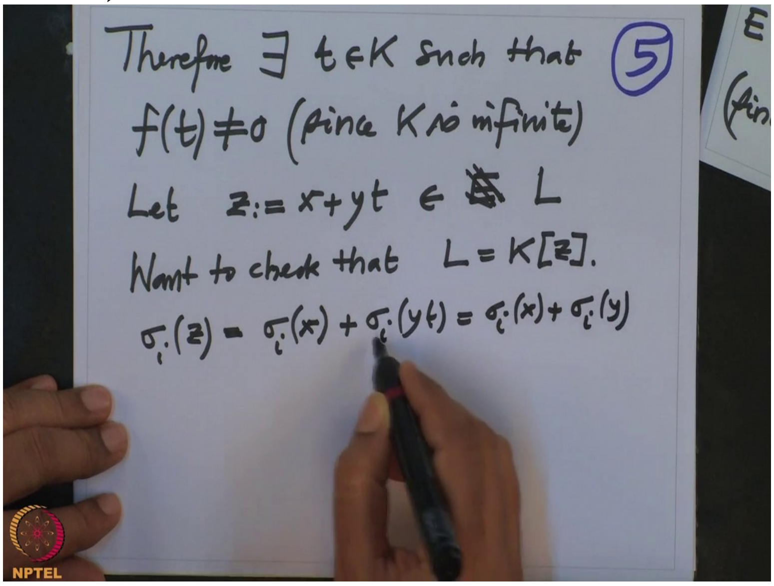

$\sigma_i(t)$ but σ is identity

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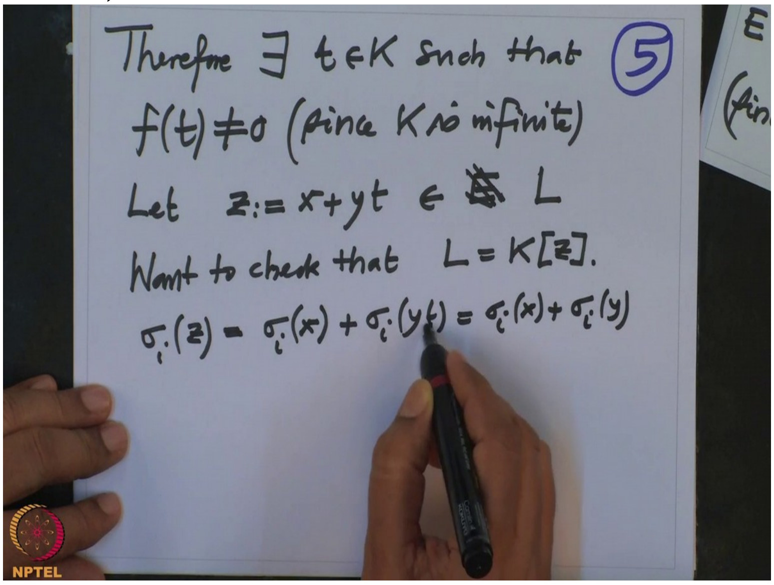

on K, therefore this is just t.

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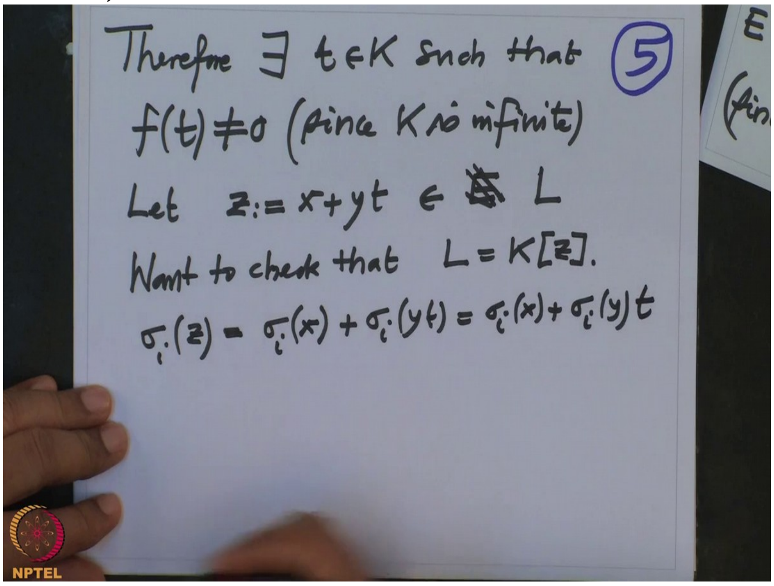

Ok, on the other hand, Ok when I apply $\sigma_j(t)$, $\sigma_j(z)$, what do I get? I get this is $\sigma_j(x)+\sigma_j(yt)$.

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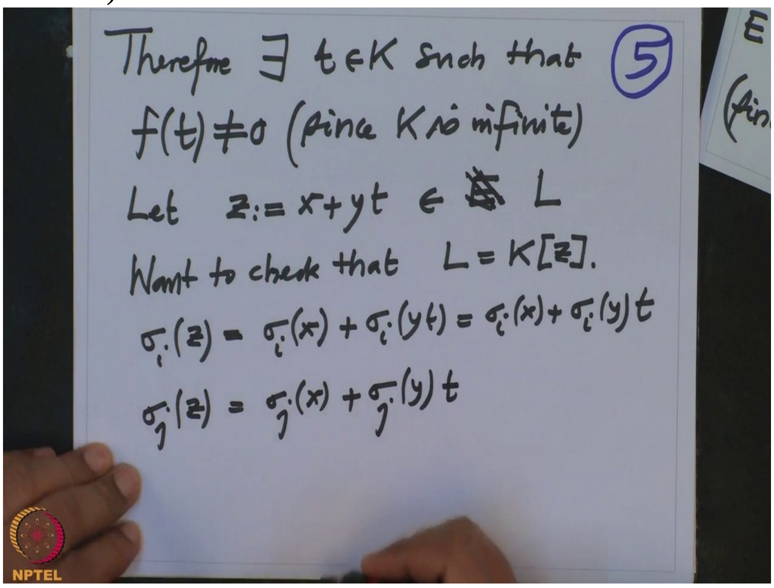

So i not equal to j.

Ok, but now note that i is, if i is different from j

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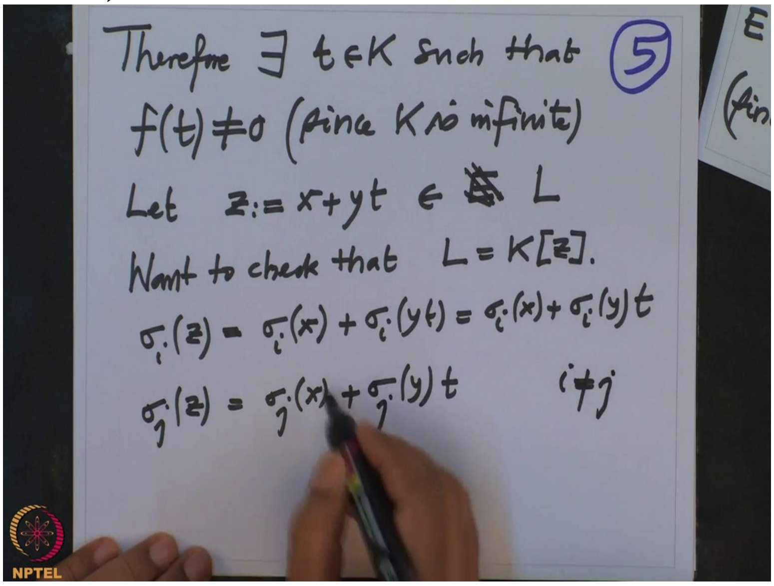

this cannot be equal to this. Both this cannot be equal to, so I claim that this is not equal,

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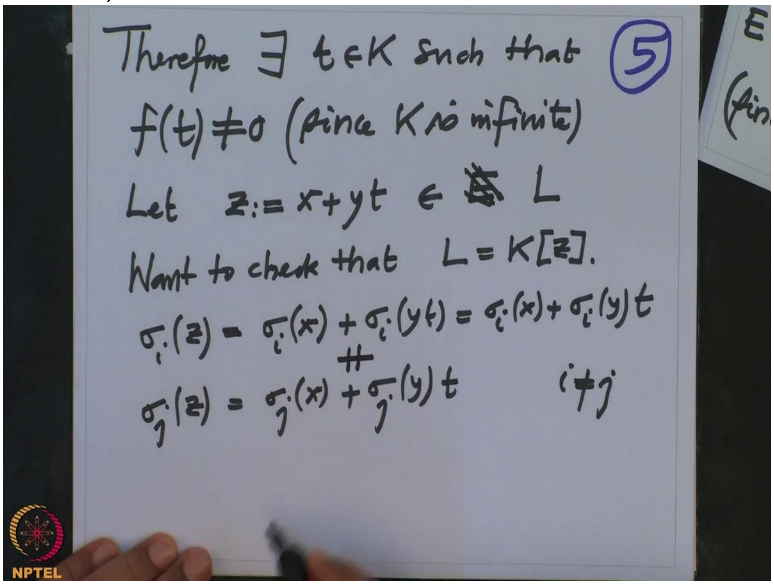

because if it is equal then, if it is equal here, if equality so I want to write the another color, if equality here, then what happens?

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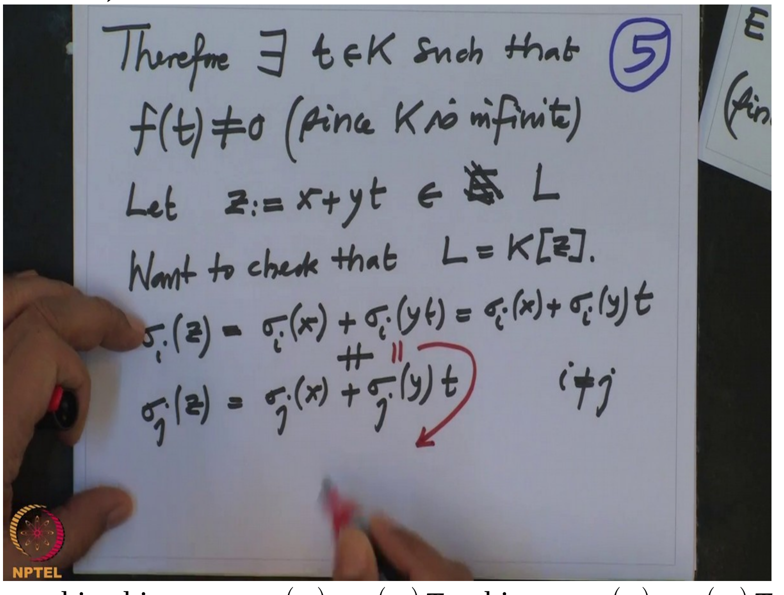

Then we get, we get this, this guy, $\sigma_i(x)+\sigma_i(y)T$ this $-\sigma_j(x)+\sigma_j(y)T$.

This is one of the

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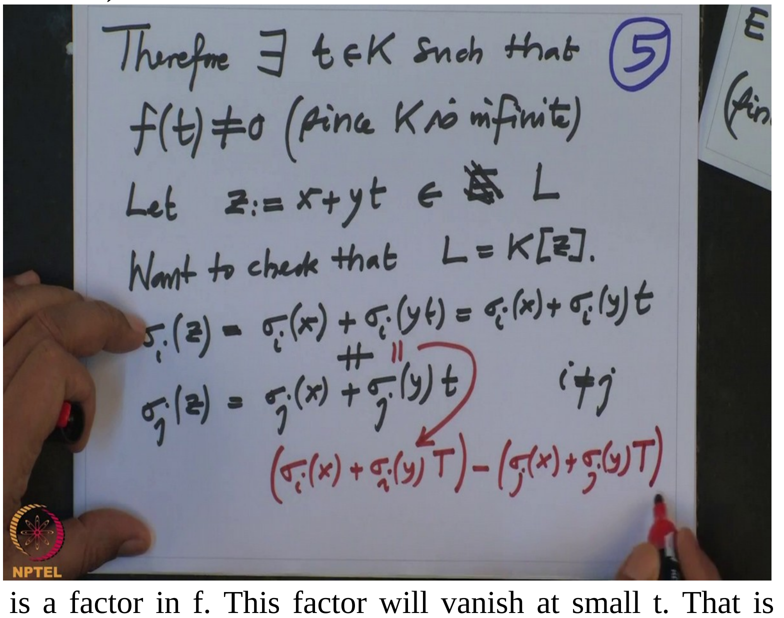

factor in f. This is a factor in f. This factor will vanish at small t. That is the meaning of equality here.

So this equality means the factor of f, $f(T)$ vanish, vanishes at t,

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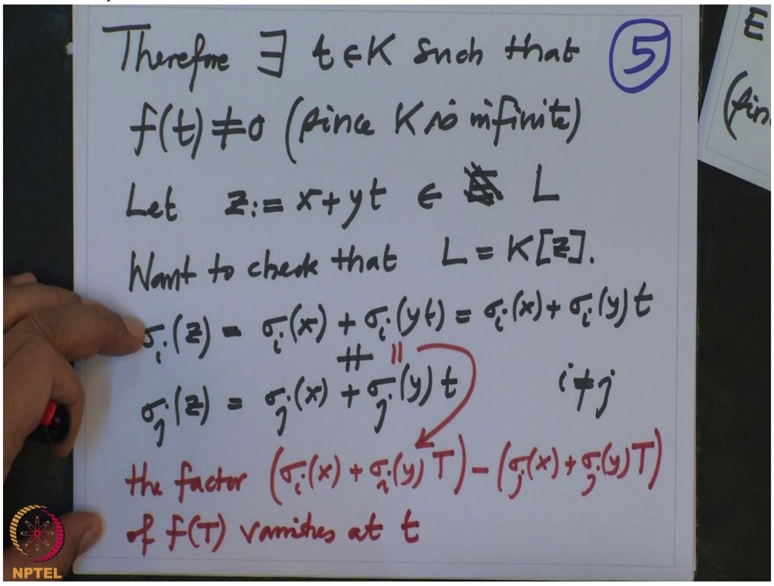

because equality here means when I evaluate it at T; that is equal here, so it is equal.

But that will mean if a factor vanish then f of t will be 0.

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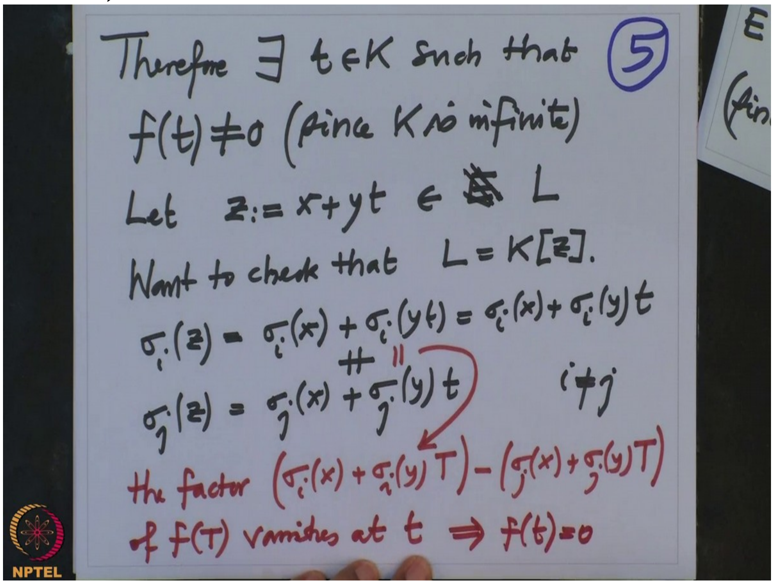

But I have chosen f of t, so that

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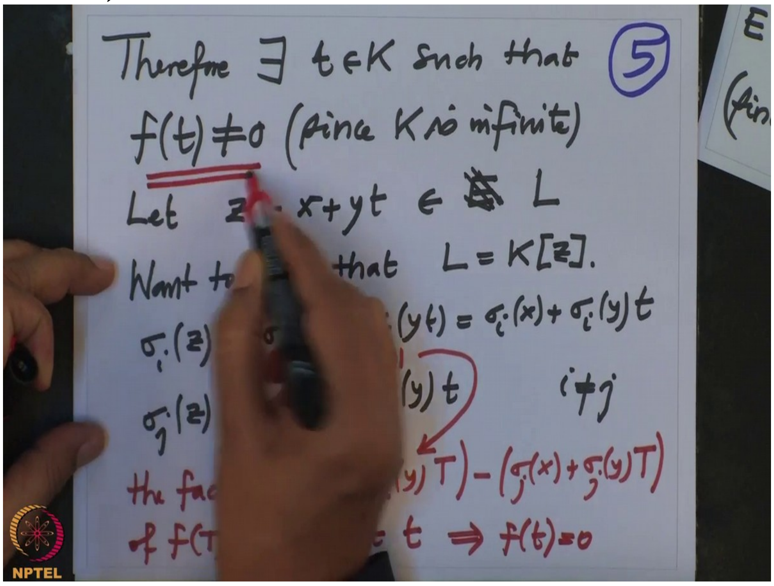

I have chosen t

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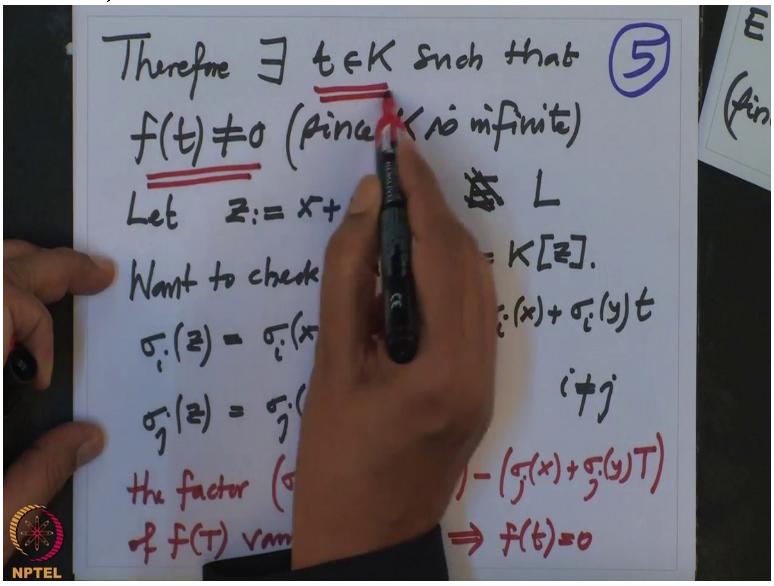

so that f of t is non-zero. So therefore this

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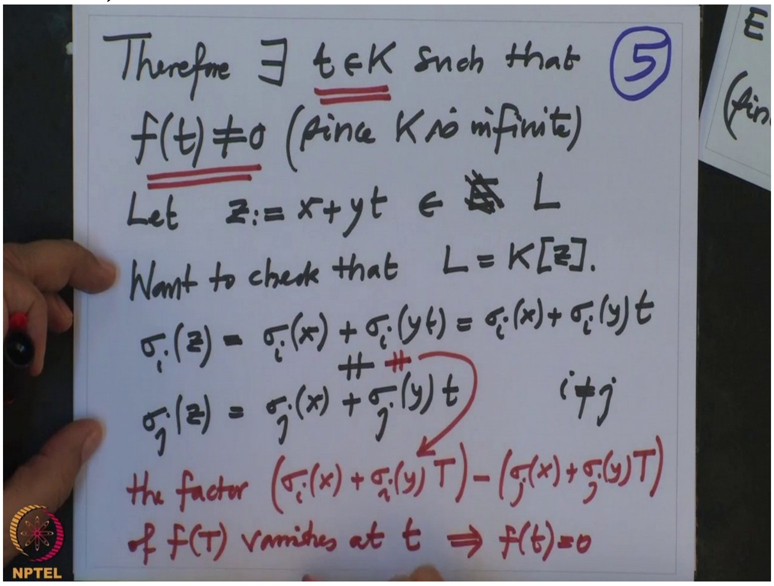

cannot be equal and therefore $\sigma_i(z)$ is not equal to $\sigma_j(z)$.

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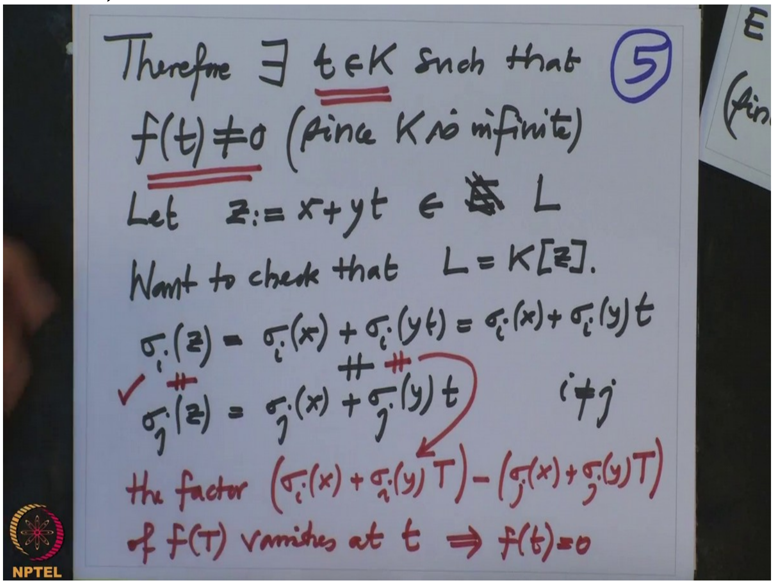

So I have done. That means I have chosen my z; so this means, this means, for this z $\sigma_1(z),\ldots,\sigma_n(z)$ are distinct elements of E and that is what we wanted

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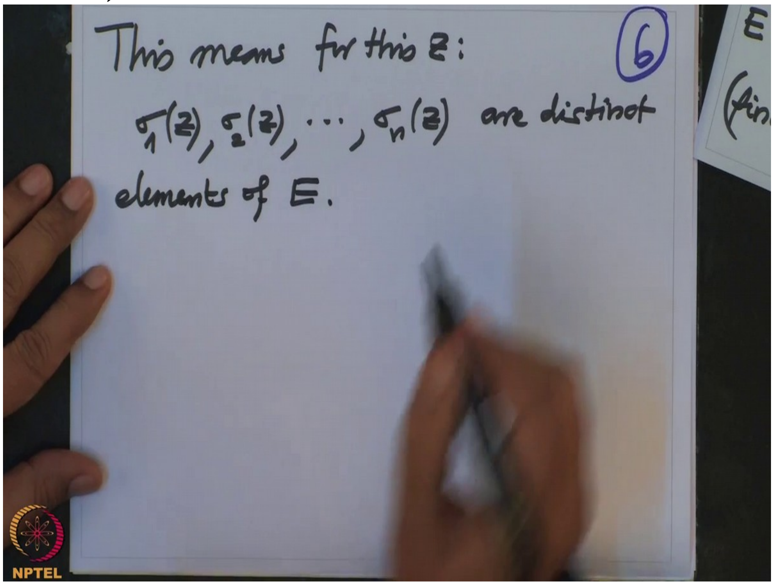

to prove.

And this finish, this completes the proof.

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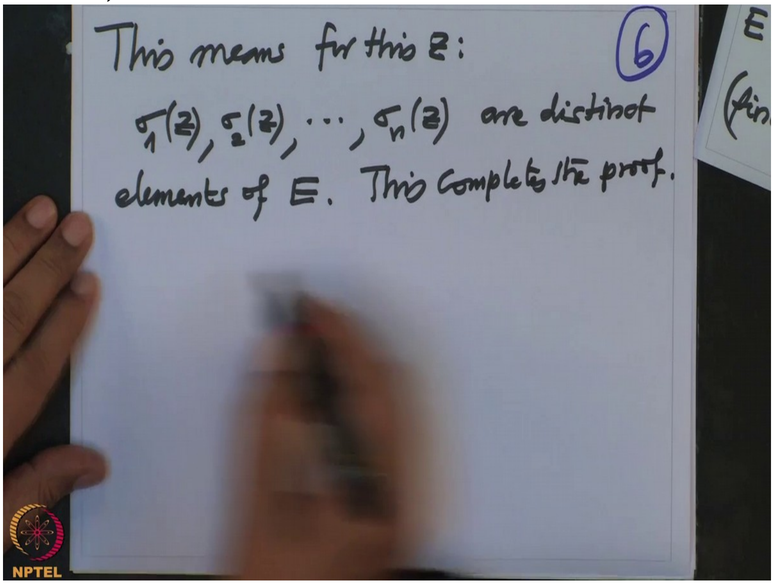

Now I want to indicate also that, this proof we have done it by induction on the number of generators of L over K. L is a finite extension of K and by the induction on the number of generators of L over K we have proved that L over K is simple.

But one can also avoid the induction and directly prove it. So I just want to indicate that direct proof.

So another, so this is remark, this is remark,

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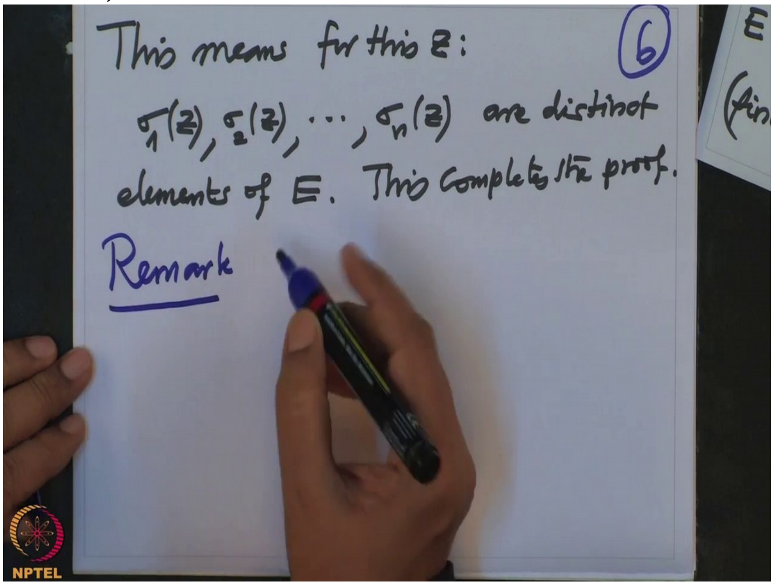

one can do without induction as follows.

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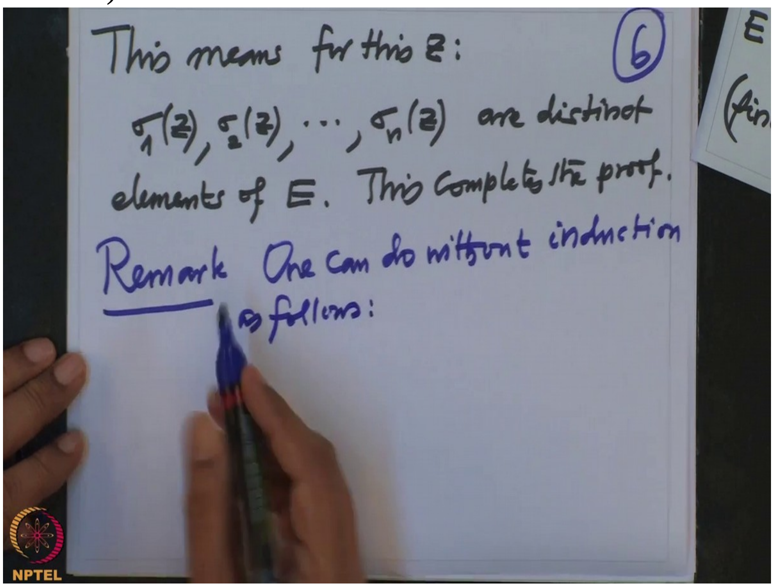

So, so suppose L is a finite extension of K, therefore let us say L is generated by $x_0, \ldots, x_n$.

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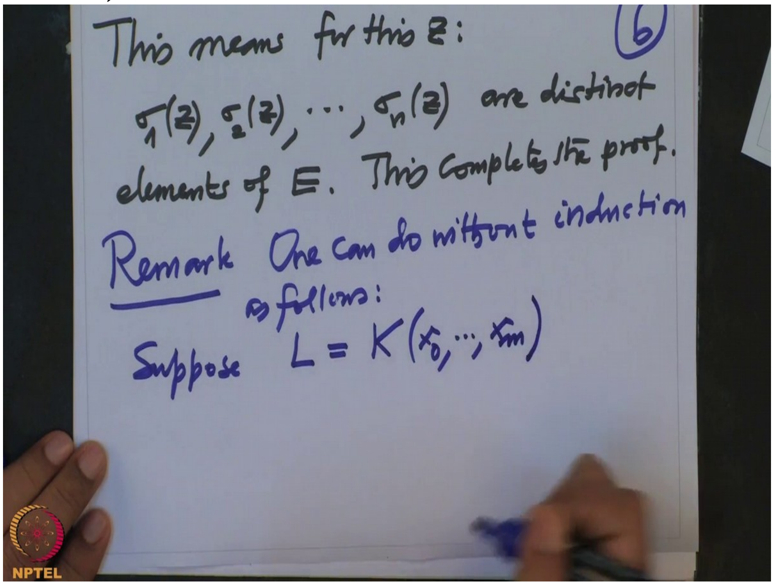

And now instead of, we can consider the following polynomial. Consider the polynomial

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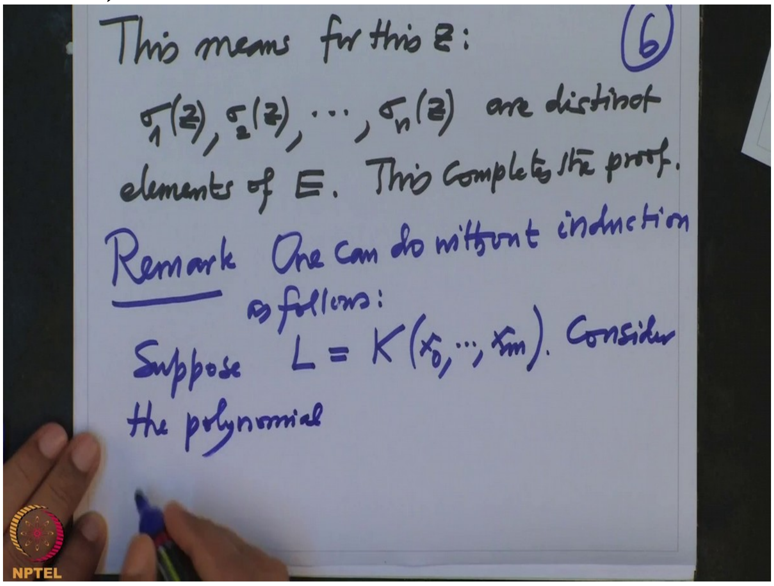

f, f T equal to product, 1 less equal to i less than j less equal to n and then the sum, sum is from k equal to 0 to m $\sigma_i(x_k)t^k$ minus summation k is from 0 to m $\sigma_j(x_k)T^k$.

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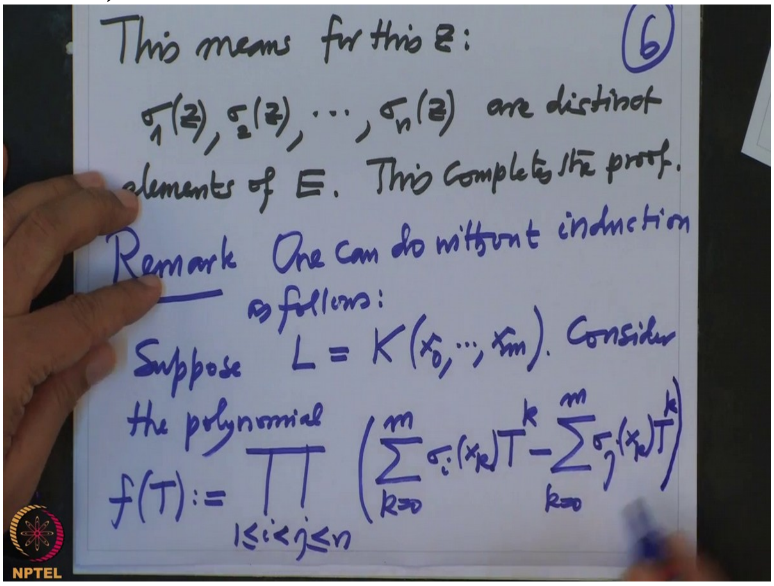

And do the same, do the same trick. So instead of, do the same trick and use the fact that σ_i and σ_j, I will write in the next page, and use the fact that σ_i is uniquely determined by the tuple $\sigma_i(x_0),\ldots,\sigma_i(x_n)$.

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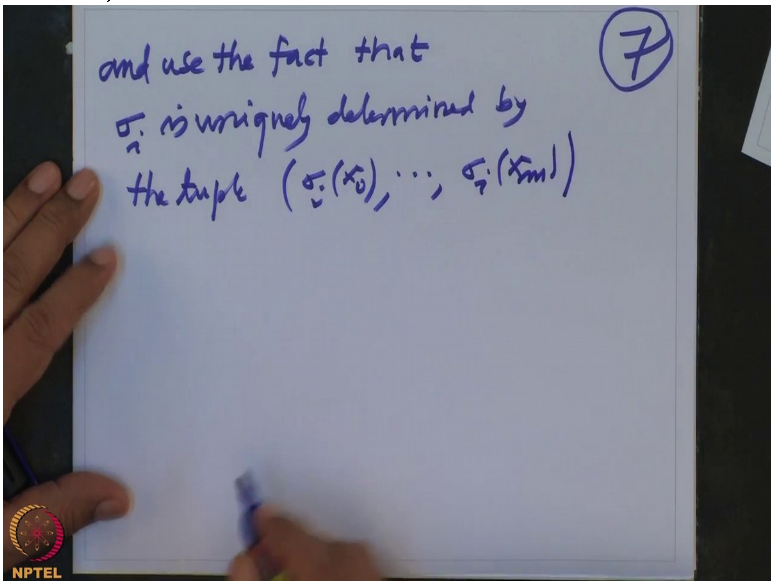

And same trick, so I will, so I would just say here, completed the proof, complete the proof. Is the same trick, no more extra trick is needed

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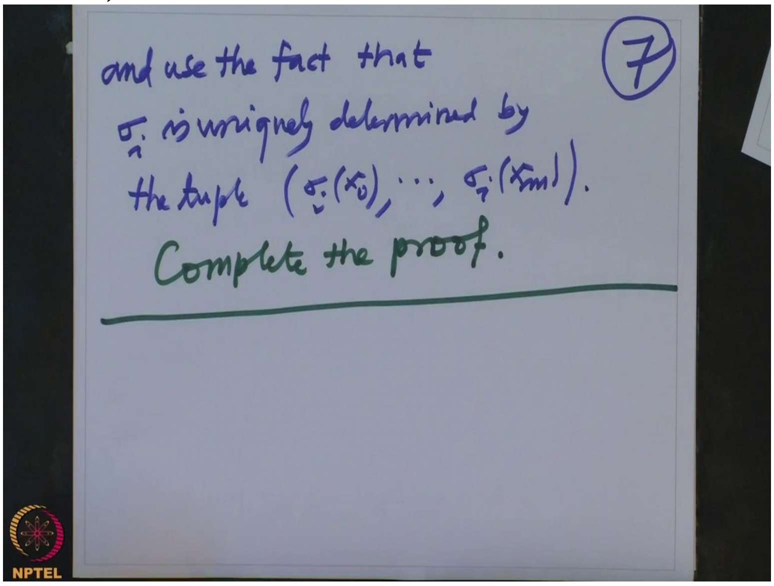

for this, so that was what.

Now let us deduce some consequences from here.

So the first important consequence, so corollary,

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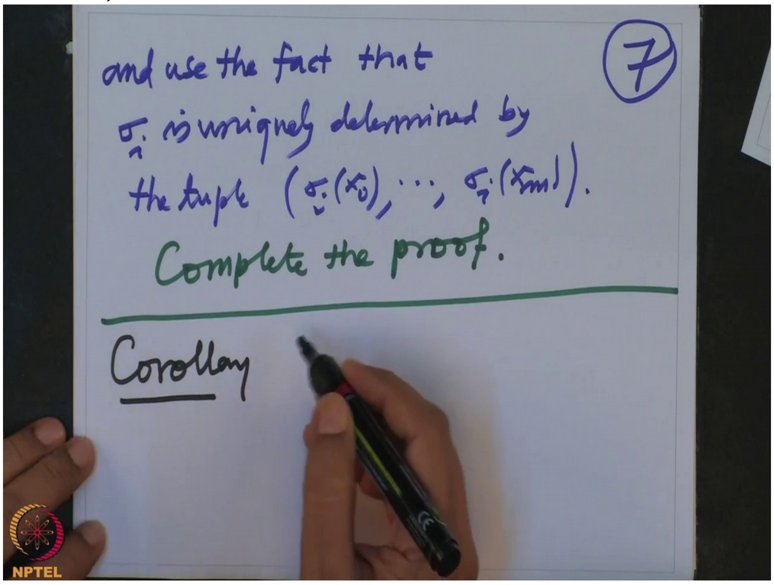

let L over K be a finite separable extension

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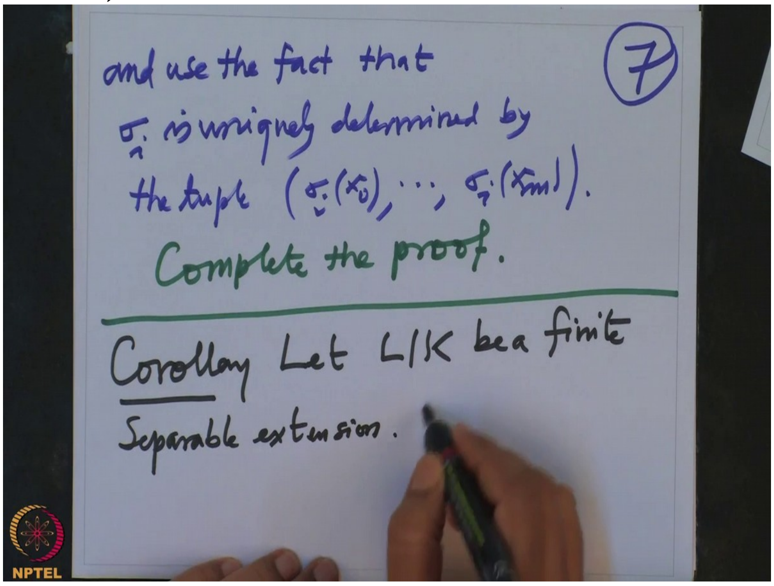

then f L over K, the set, this is the set of intermediary field extensions

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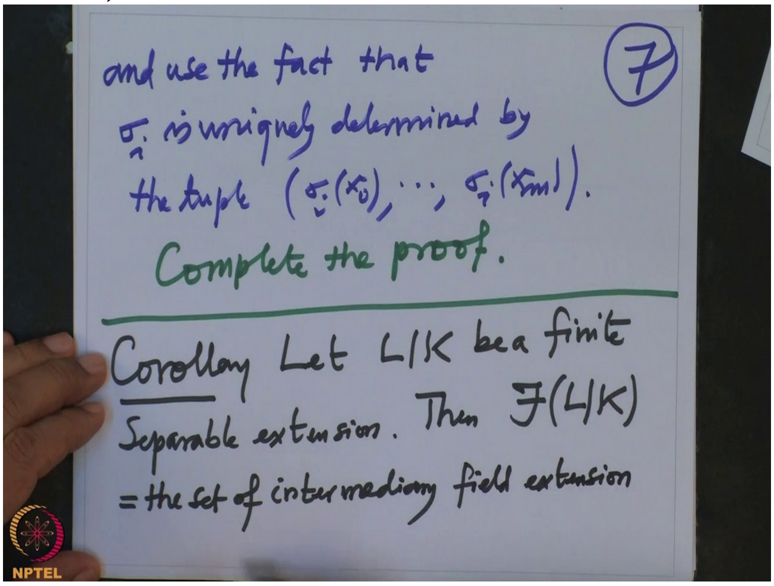

K contained in M contained in L,

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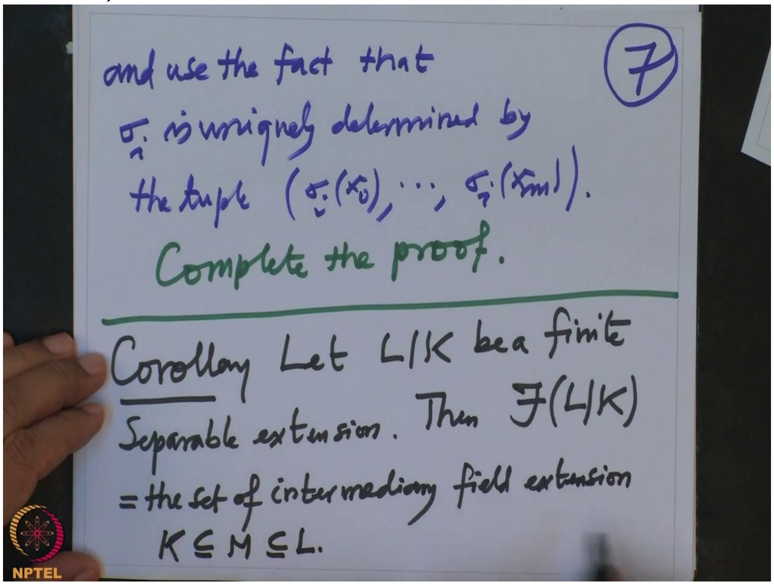

this set, so this is the definition of that set.

Then this is a finite set. That means there are only finitely

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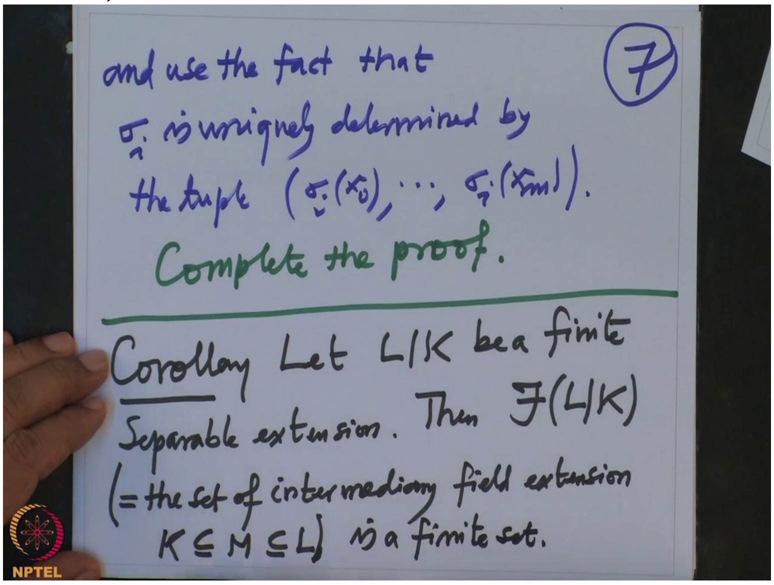

many intermediary extensions.

So proof, proof, since L over K is finite separable, L over K is finite separable, we know it is simple. L is $K[x]$ for some x by theorem.

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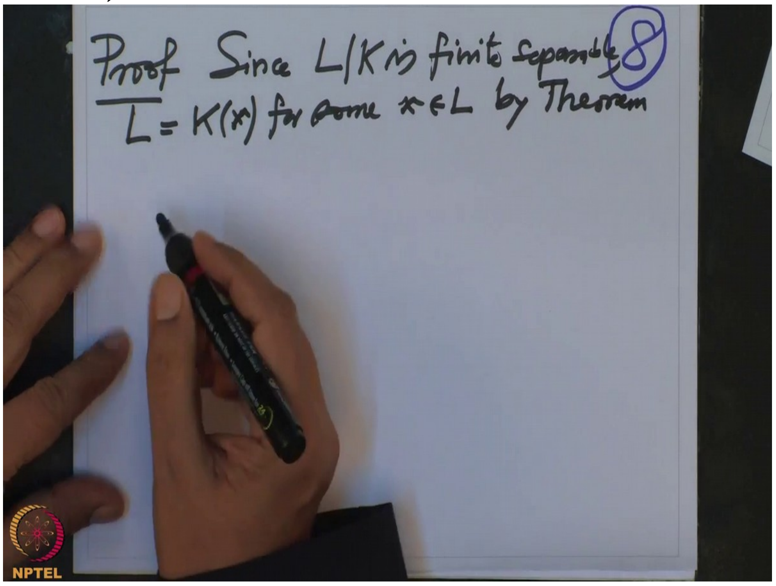

And we want to check that this is a finite set. I want to check this is finite.

So I am going to give you a map from this set to divisors of $D(\mu_{x,K})$. What is $D(\mu_{x,K})$?

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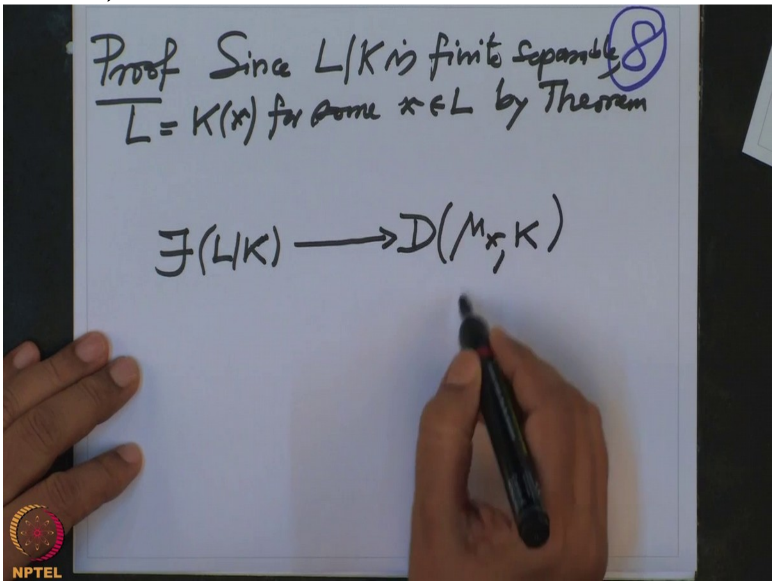

This is, these are, this is the set of monic divisors of $\mu_{x,K}$ in the polynomial ring, in $K[X]$.

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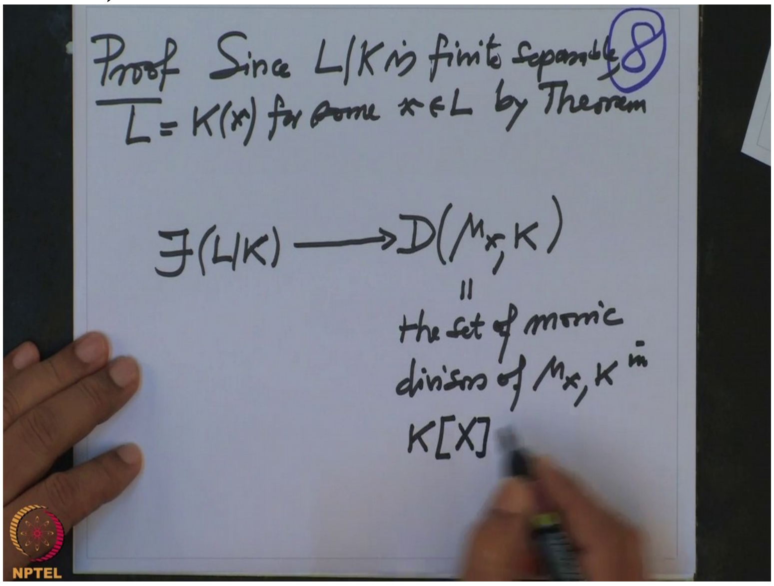

This is a monic polynomial,

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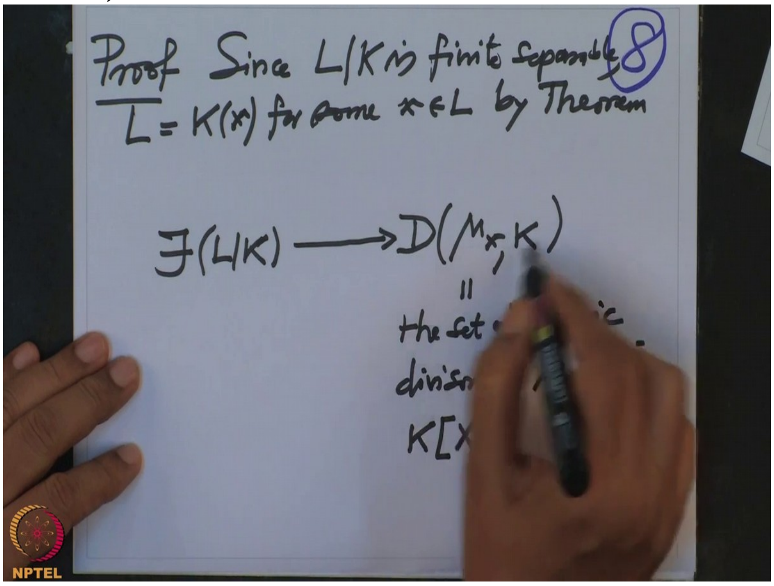

monic irreducible polynomial not in $K[X]$ but in $L[X]$.

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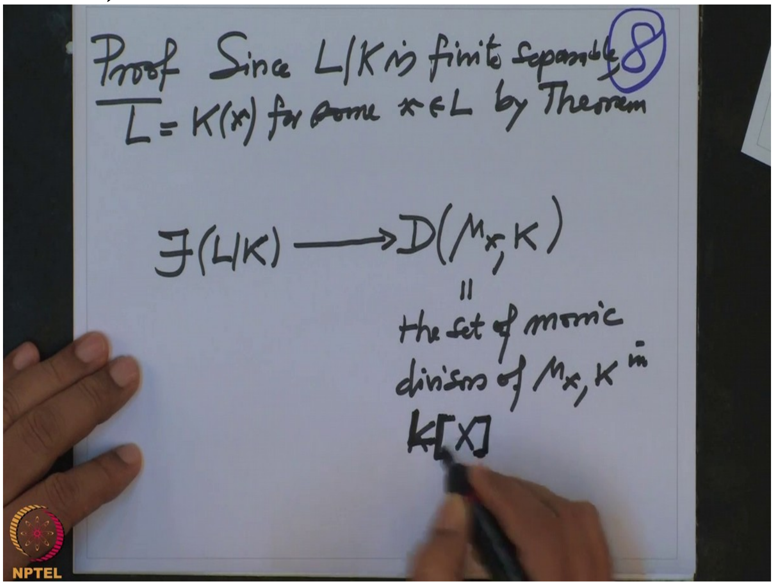

It is a

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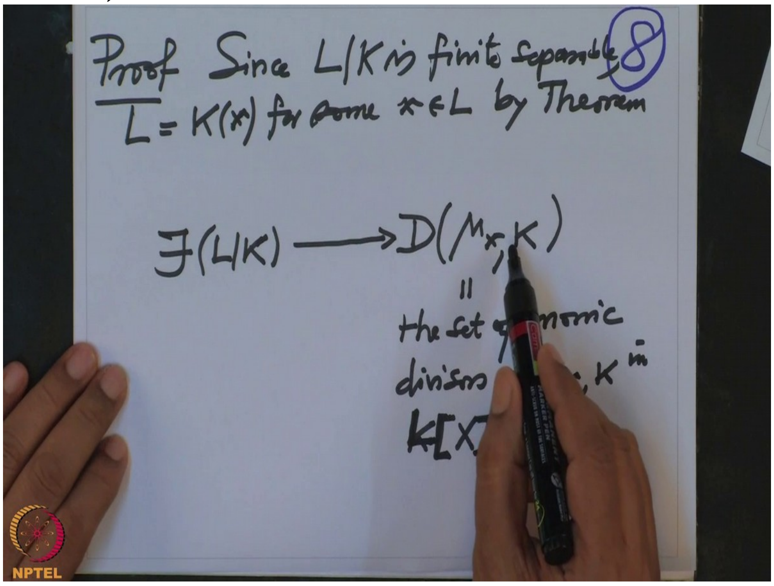

a irreducible polynomial in $K[X]$ but in

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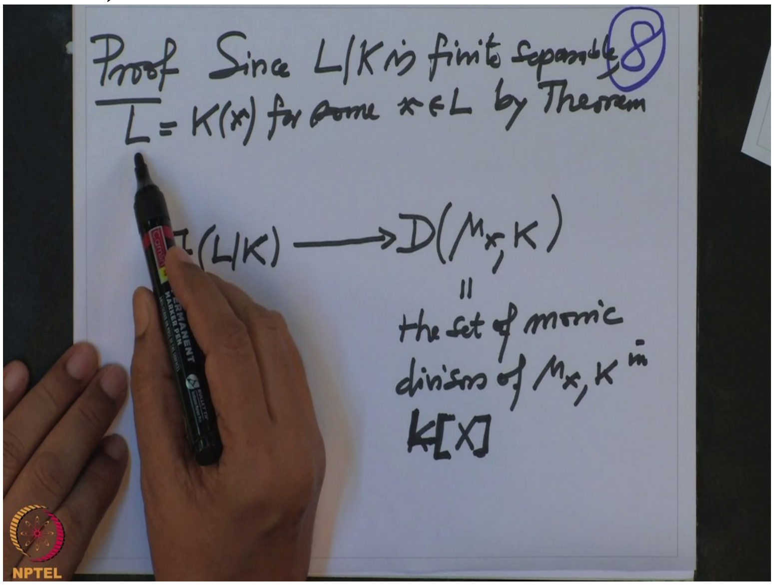

L, when you go to L, it has a 0 in x so definitely this polynomial

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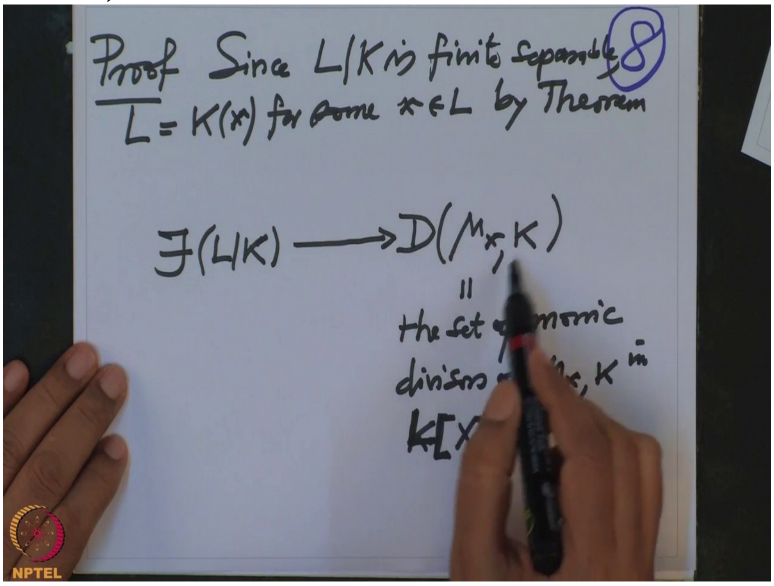

is not irreducible over $L[X]$ so I look at the monic divisors of that polynomial in $L[X]$.

And what is the map? Map is very simple. Take any M, intermediary field and look at the minimal polynomial of X over M.

This minimal polynomial of X over M and what is the relation between minimal polynomial of X over K, this divide this where in $M[X]$ but if

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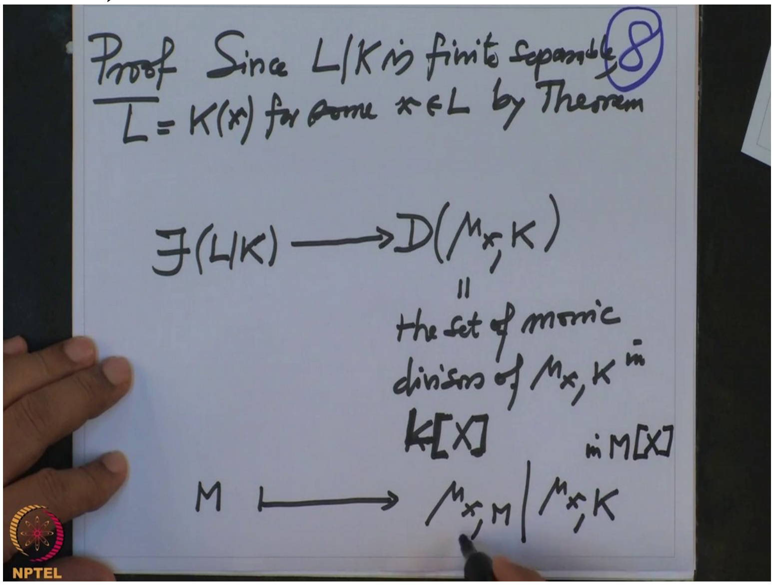

it divides in $M[X]$ then it will also divide in $L[X]$ because L is bigger field.

So our map is very simple, M going to the minimal polynomial of X over M. And I claim that this map is injective.

So the map we will check, we will prove, prove that this map, what is the map, M going to $\mu_{x,M}$ is injective.

What does that mean?

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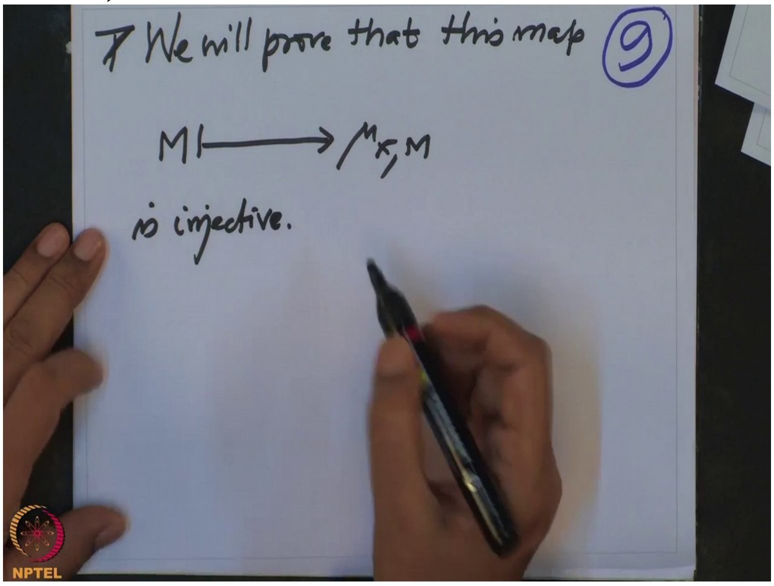

How does one prove that map is injective? That means if I know this

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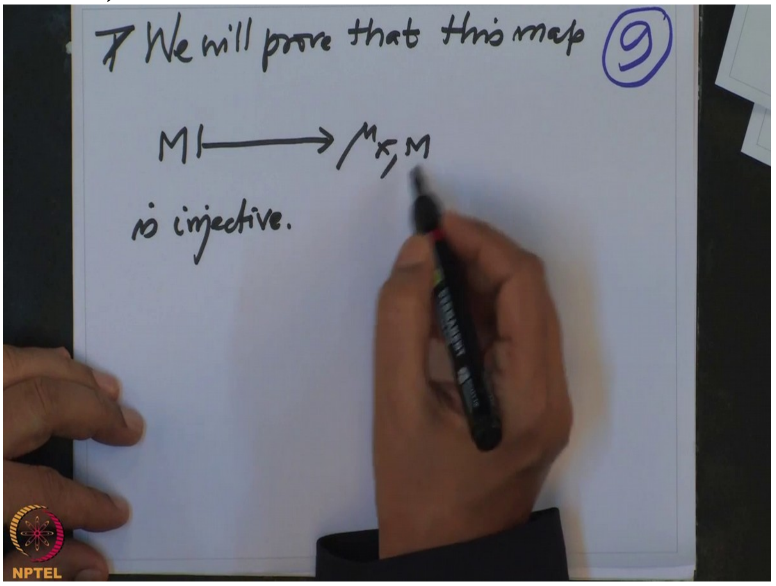

polynomial then I should get back this M.

So let us write this polynomial. Suppose this polynomial where, where, this is a monic polynomial of some degree. So suppose it is $X^m + a_{m-1}X^{m-1} + ... + a_1 X + a_0$,

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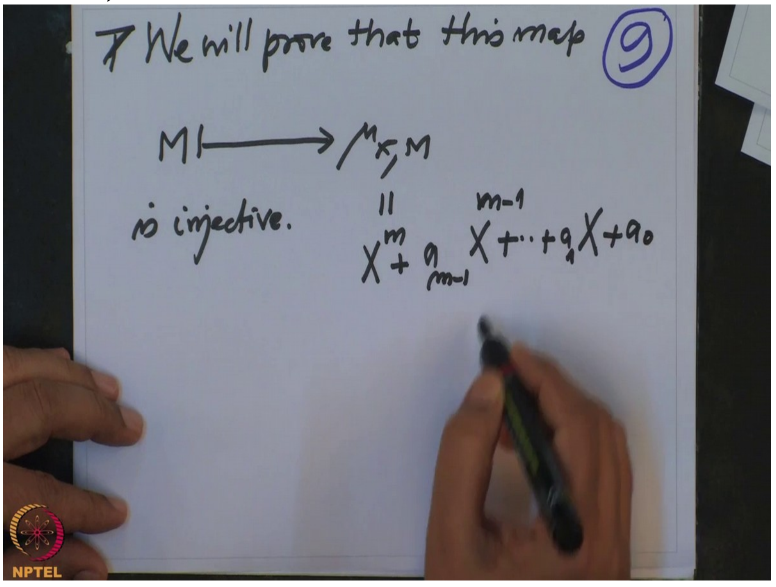

this is the polynomial in $M[X]$.

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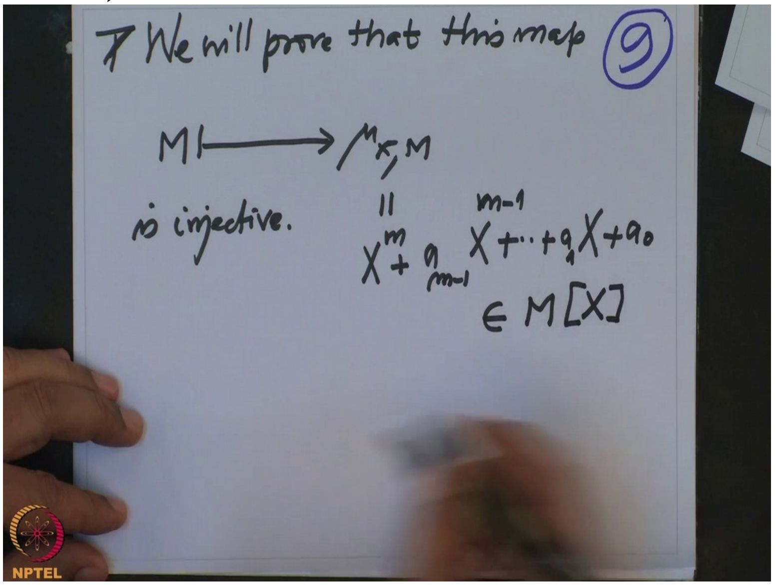

Then I say, then I want to prove M has to be equal to generated over K by these coefficients $a_0, \dots, a_{m-1}$.

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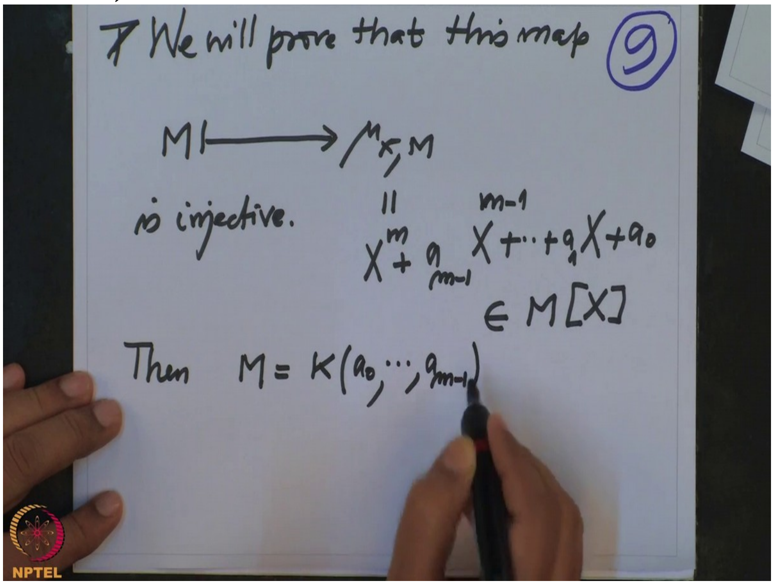

If I prove this then injectivity

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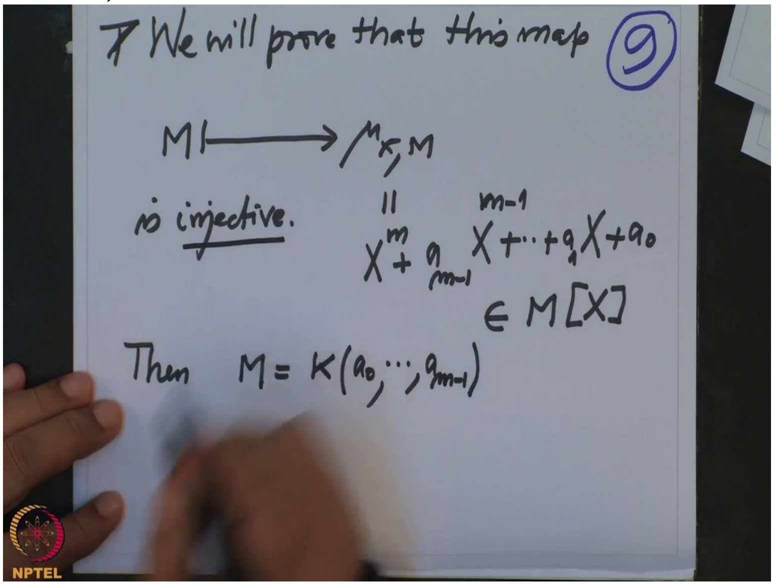

will follow because if I know the polynomial, I can get that my field M.

So I have to prove this equality.

But to prove this equality, let us call this as M_0

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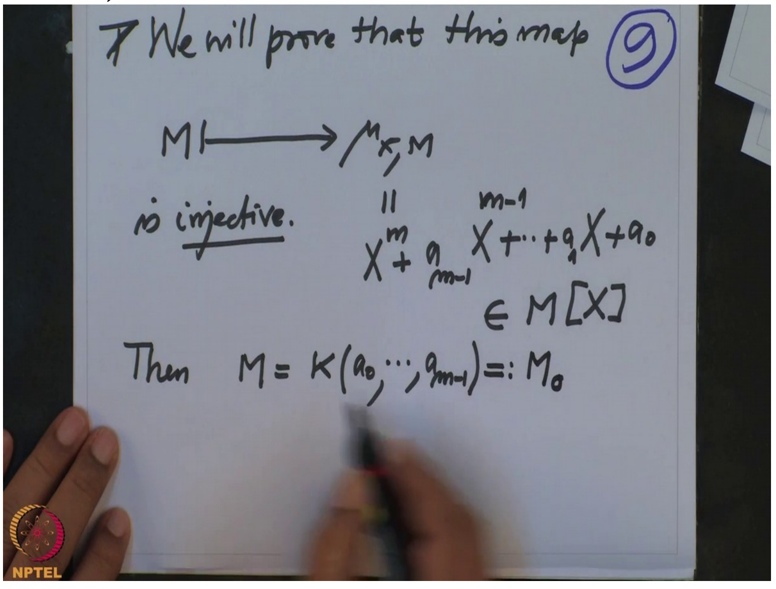

and obviously this is contained here because this polynomial

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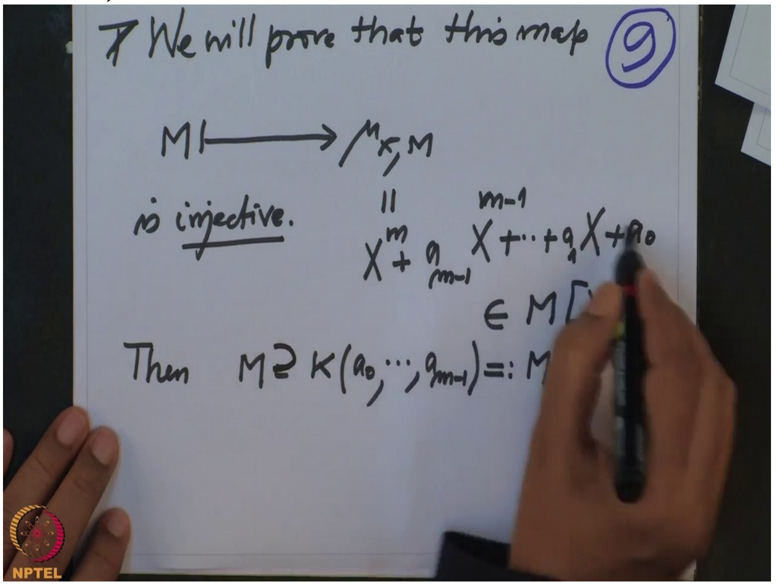

has coefficients in M.

So this is ob/obvious, so this is clear. So I have to prove the other way.

So to prove the other way I just have to compute this, this, this, everything is a subfield of L.

So I want to compute L over M and also I want to compute L over M_0.

And I want to show they are equal.

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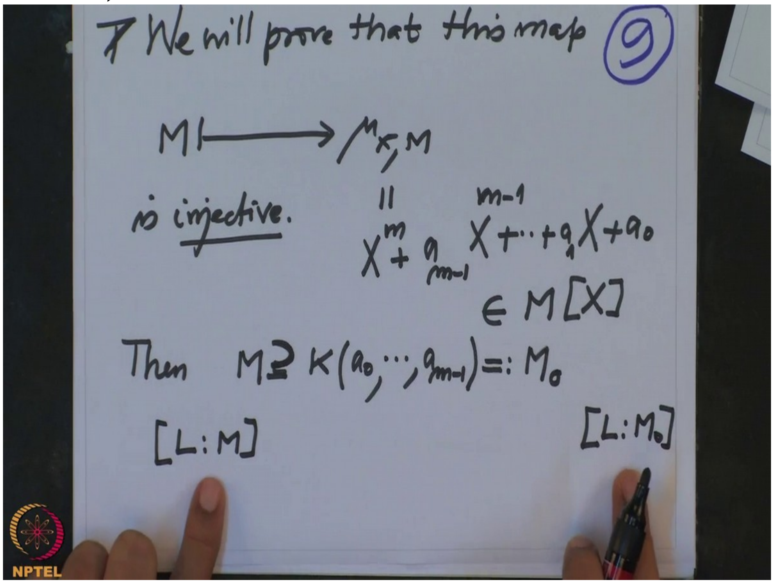

If I show they are equal then M will be equal to M_0 because this is a subfield, therefore this, this equal to L over M over, now times M over M_0, this I know.

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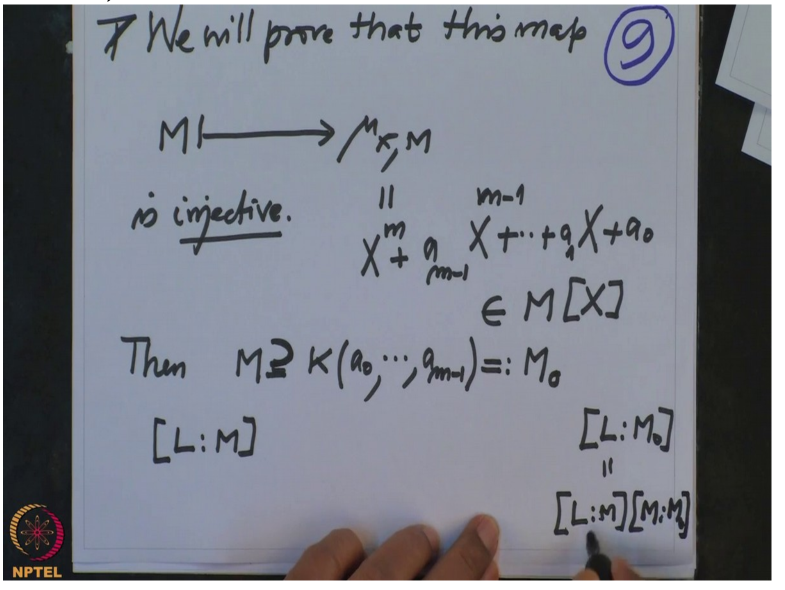

So if I prove that this equal to this, then M over M_0

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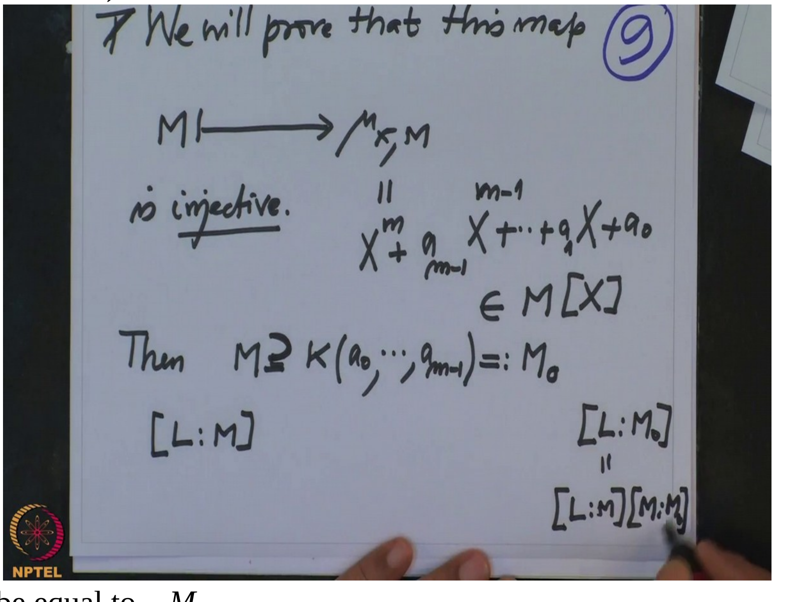

is 1, and M will be equal to M_0 .

So I am aiming to prove these are equal.

But, Ok what is this? This is L is generated by x over K therefore M is also generated by x over M.

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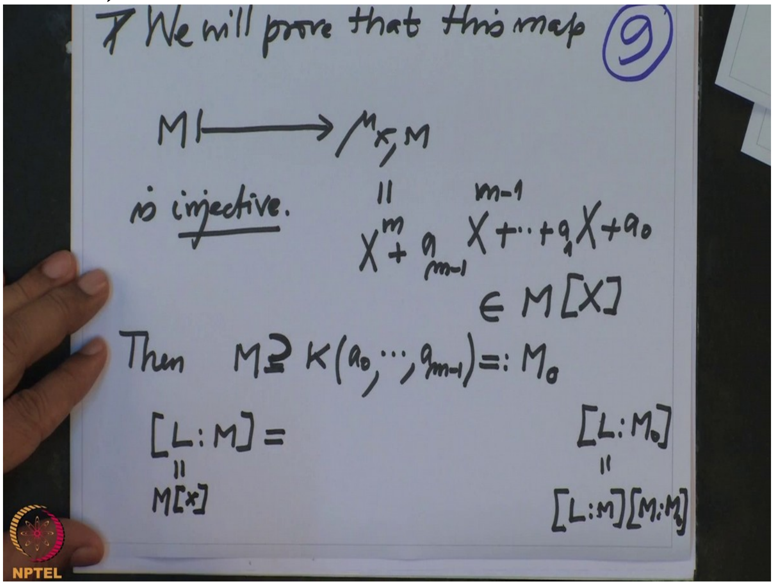

Therefore this equal, degree of this, this is a simple extension over M. So this is the degree of $\mu_{x,M}$.

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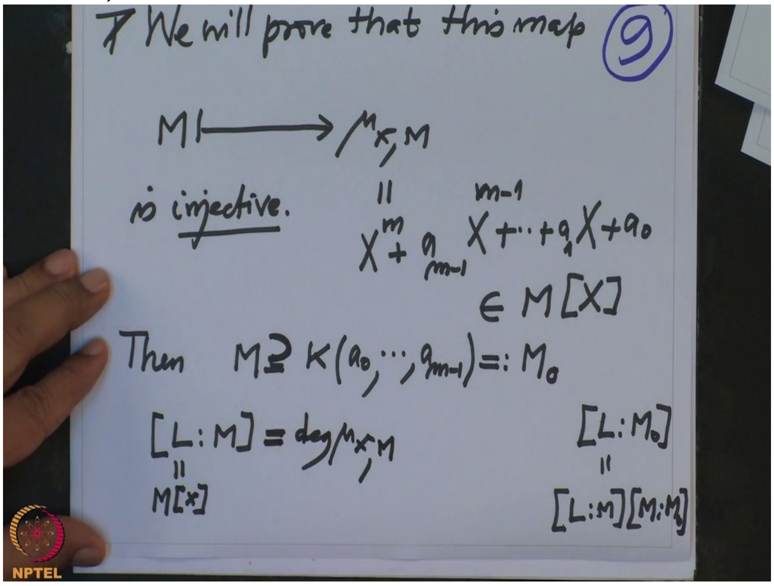

But $\mu_{x,M}$, this is same thing as degree of μ_{x,M_0} because

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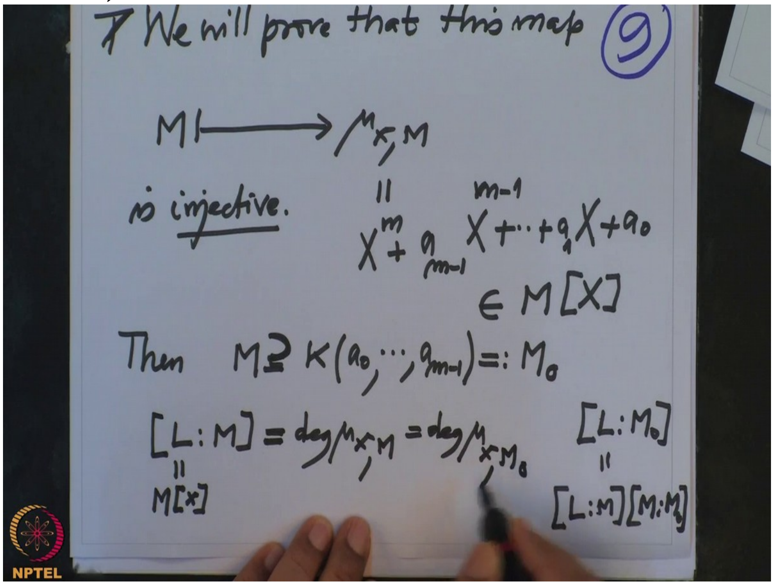

these two polynomials are same, because the coefficients are same.

Therefore these two polynomials are same. These polynomials individually, they are same.

This polynomial and this polynomial are equal therefore their degrees are equal.

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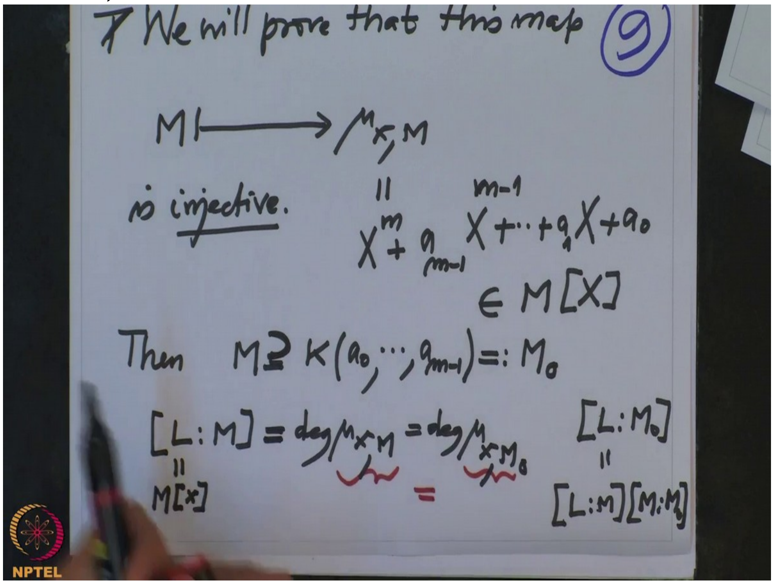

And by the same argument this equality,

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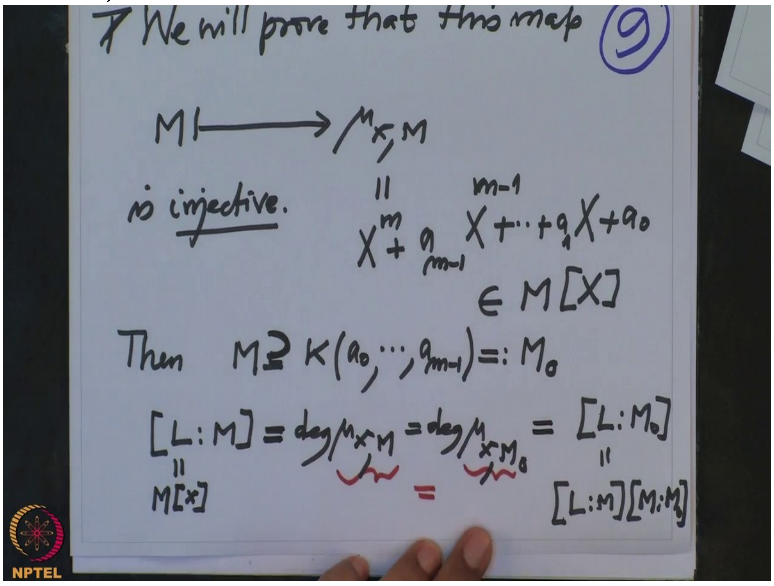

therefore these are equal. Therefore from here you conclude M equal to M_0 and therefore

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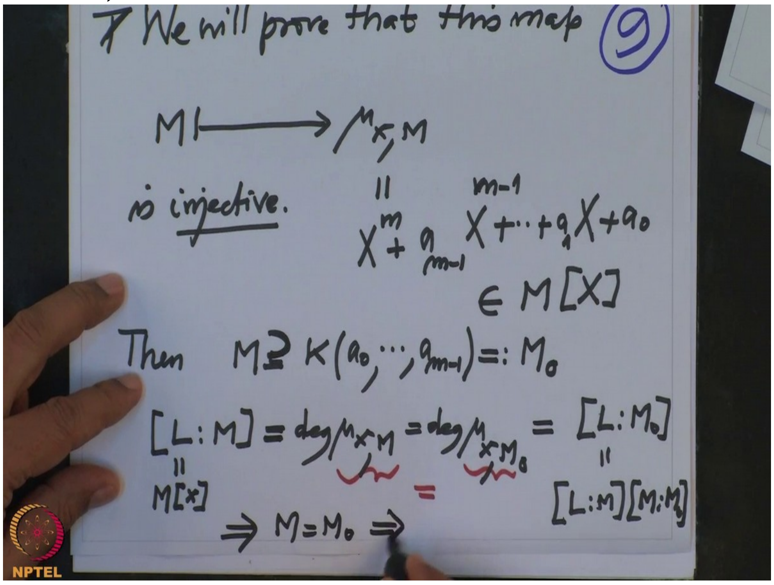

we conclude the map is injective.

The map is injective.

So if you want to give the name δ, the map δ is injective.

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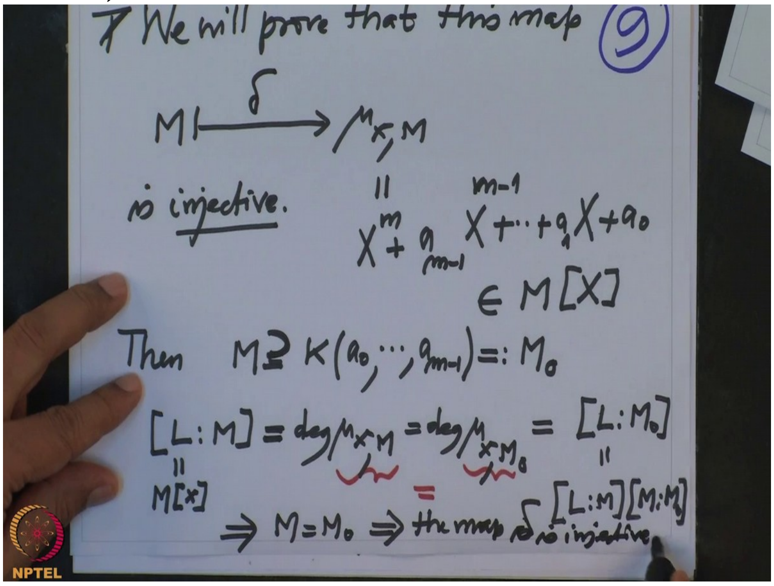

So that proves that, so if you have a injective map from some set to a set where the bigger set is finite, this is a finite set

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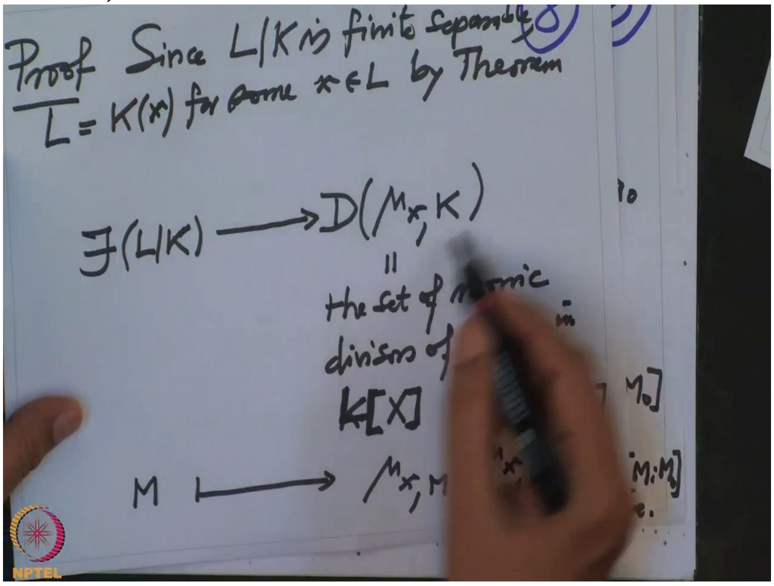

because this is a monic polynomial of some degree and therefore the number of divisors, monic divisors will be also finitely many.

Therefore

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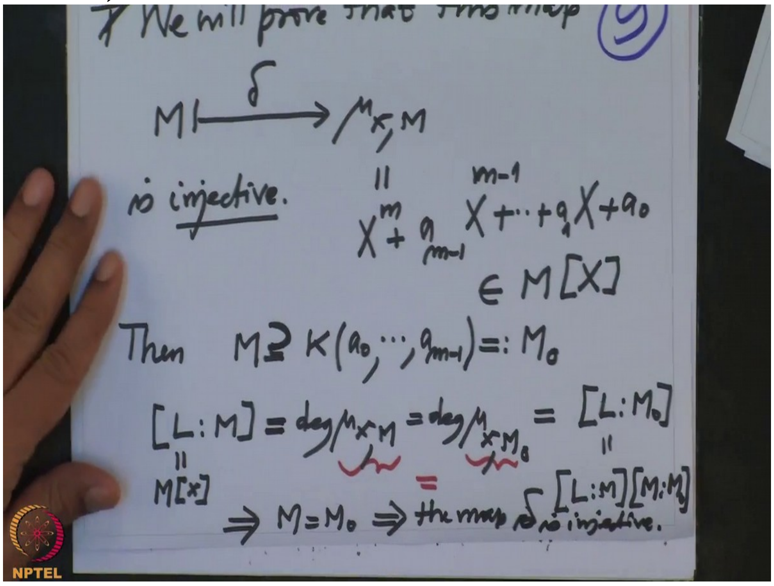

we finish the proof and next time we will continue therefore the consequences of this very important theorem,

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and we will also use this to compute some examples of some Galois groups. So thank you and we will continue next time.