Galois' Theory.

Professor Dilip P. Patil.
Department of Mathematics.
Indian Institute of Science, Bangalore.

Lecture-52.

Uniqueness of Algebraic Closure.

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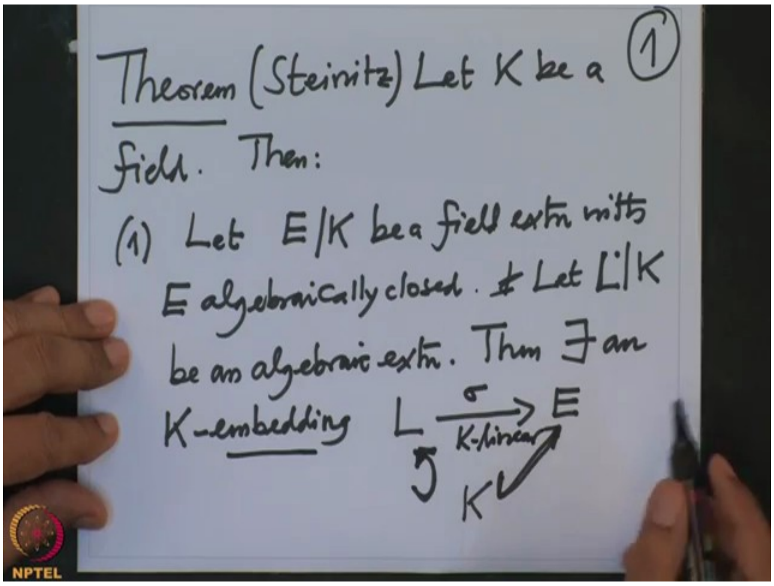

So, in the last lecture we saw that every field has an algebraic closure. Now the question is how many. So now we prove sort of uniqueness, it is not unique but it is unique up to nK isomorphism. So let me state it precisely, then really prove it. So theorem, this will also Steinitz, let K be a field, then, 1 let E over K be a field extension with E algebraically closed. In the last lecture we indicated that how to construct such E. Okay. Then, let L over K be an algebraic extension, may not be finite, then there exists an K embedding from L to E σ .

K embedding means, see, this is an algebraic extension, this is contained here, so we can extend this, this σ is K Linear and embedding means it is a field homeomorphism. So such an embedding exists for every algebraic extension, that is Part-1.

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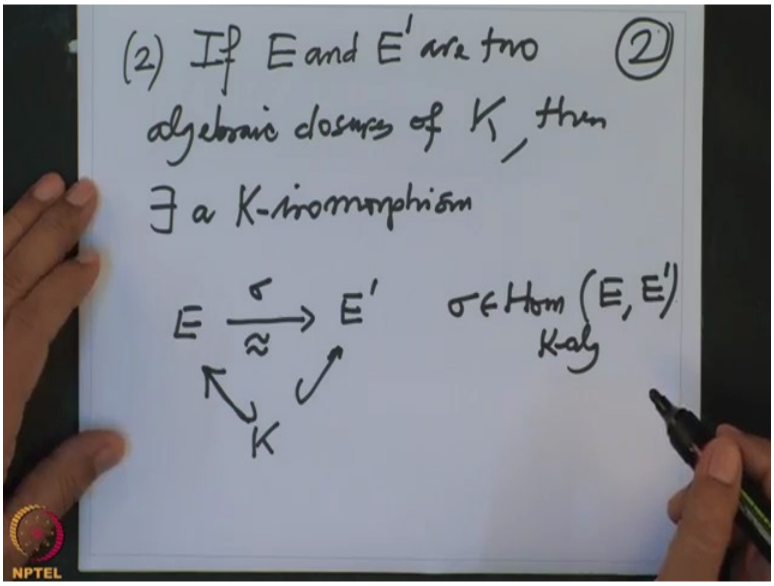

And part 2 we say that, if E and $E^{'}$ are two algebra closures of K, then there exists a K isomorphism σ from E to $E^{'}$, that means this computes with its K Linear and it is a field homeomorphism. So that means this σ is indeed an element in $Hom_{K-alg}(E, E^{'})$. This is K algebra homeomorphism from E to $E^{'}$, which is also, which is K Linear, that is K algebra homeomorphism. So that means that E and $E^{'}$ are unique up to this K algebra your isomorphism. So let us prove this.

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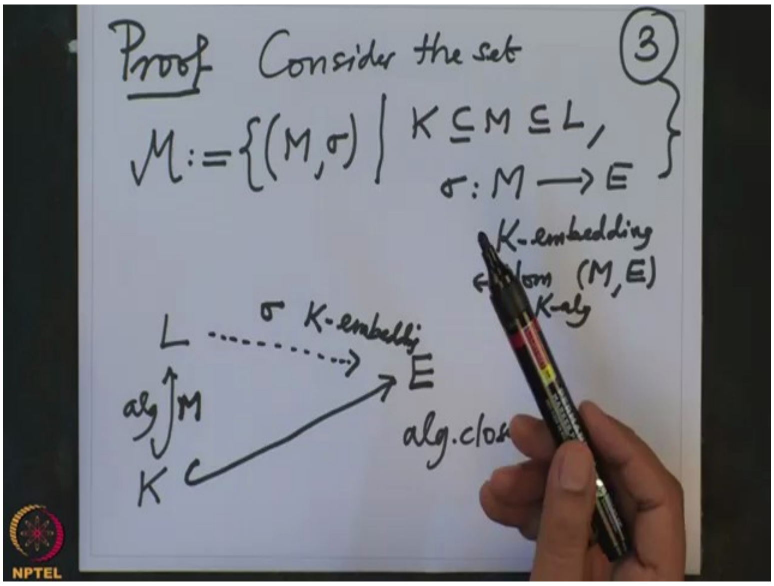

So proof, again proof will involve Zorn's lemma. So, proof. All the theorems, one cannot prove without using the Zorn's lemma. So, consider the set M, this is the set M of all pairs of

M, σ such that M is an intermediary field of L and K and the σ is the K embedding from M to E, K embedding. That is it is an element in $Hom_{K-alg}(M, E)$, this. This is our set M. Remember, what we want to do is a following. Let me draw a diagram. L over K, this is the given algebraic extension and we have given E, E is given here, E is contain this K is a field extension of case and this E is algebraically closed, that is what we have given and we want to extend it here, this σ we are looking for, K algebra homeomorphism or K embedding.

This is what we are looking for. So now what we have done is, we have taken all the intermediary fields M which have the extension and we are going to maximise it and we will prove that this process will reach M. And precisely that means choosing the maximal element. And choosing a maximal element one needs to apply Zorn's lemma and one has to have an order. And we have to check that order set is inductively ordered. That means every chain has an upper bound there.

And one notes that there are maximal elements and those maximal elements will give as the required thing. This also shows that this process is not very natural and this may not, the embedding may not be unique, there may be many embeddings, that is it. Let us now justify that we can apply Zorn's lemma to this set M, that we have defined. So we need an order 1st.

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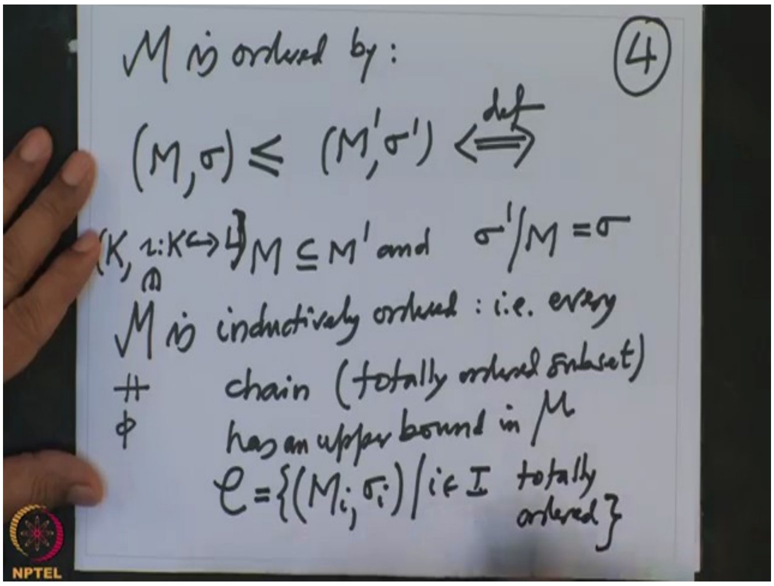

So, define, so they set M is ordered by the following order. So take 2 elements in M, one is M, σ , the other is M' σ and then we say that this is bigger equal to this, if and only if, this is the definition, this is if and only if, M is contained in M' and σ' is restricted to

M equal to σ. So, with this we need to check now that this M is inductively ordered. So 1st of all this M is nonempty because this K, there is natural inclusion iota from K to L because this L is given to be extension of K, so there is a natural inclusion.

So this pair belongs to this M. Therefore M is nonempty and inductively ordered means that is every chain, what is the chain, chain is a totally ordered subset, has an upper bound in M, this is what we want to check. Alright, so let us that, let us call that chain to be C, C is the pairs (M_i, σ_i), $i \in I$, so this I is totally ordered and we want to check that it has an upper bound.

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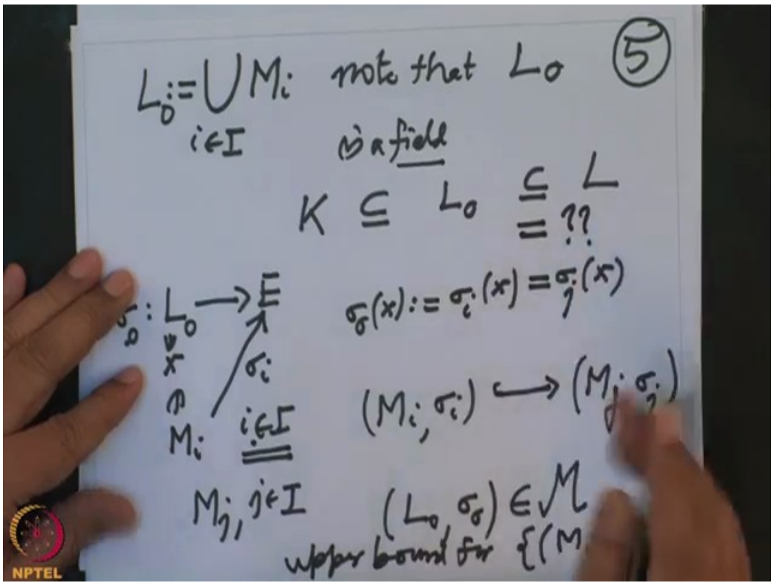

And obviously what we will claim is you take the union of these M_i's, i in I, let us call this as L_0. So 1st of all note that this is a field, this is a subfield of, so note that L_0 is a field and L_0 is in between K contained in L_0 contained in L. So, field is clear because if you take to elements here, both of them will lie in some M_i, one will lie in M_i, the other will lie in M_j but I is totally ordered, therefore I can assume both lie in the same but all these M_i's there who is a field, therefore this is a field, this is clear.

Another thing is we also can extend that, we can extend that σ is, we will define σ_0 from L to E, L_0 to E. How do we define this? Take any $x \in L_0$ and choose i so that X belongs to M_i, $i \in I$ and now define, we will be given this σ_i. So define $\sigma_0(x)$ to be equal to $\sigma_i(x)$. But this will not depend on the choice of this y because this is the chain. That means (M_i, σ_i) and if you were to arrange $M_j, j \in J$ and as you assume this i is totally ordered, so j is bigger, then (M_j, σ_j) , this is contained here and then therefore this is also equal to σ_j .

So it is well-defined, it is no problem. And remember everywhere we are choosing that this is a chain. So therefore anyway, we have therefore got hold of this element (L_0, σ_0) , this is again in M and which is clearly an upper bound, upper bound for the chain (M_i , σ_i). And now we will, obviously we will claim that this L_0 equal to L, we want to claim this equality here, this is what we will prove.

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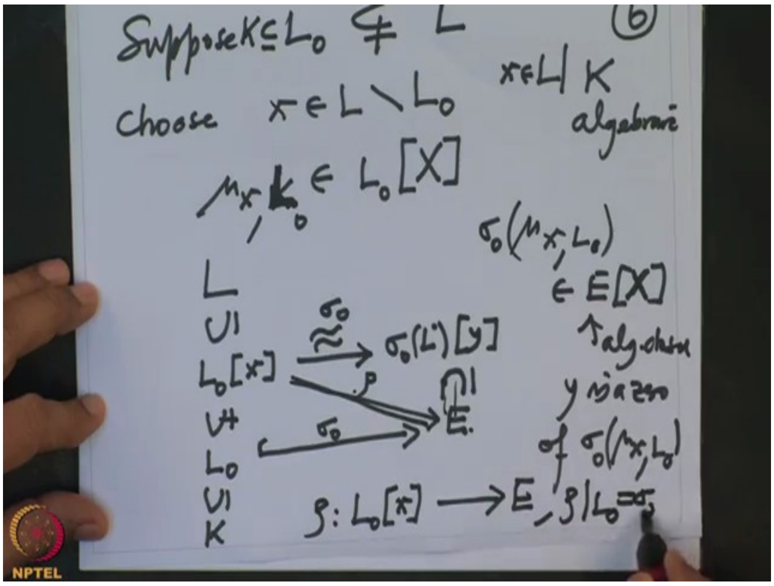

Alright, so suppose on the contrary L_0 is not full I, full L. Suppose L_0 is properly contained in L. Then what can we do, we can choose an element which is not in L_0 and which is in L. So choose $x \in L$ which is not in L_0 and remember this L is L over K is algebraic. That is given to us and this x is given there in L. So minimal polynomial of x over K makes sense, this is minimal polynomial of x over K, this is not over K, over L_0 . This is a polynomial in $L_0[X]$, x over K is algebraic, therefore L over L_0 is also algebraic because L_0 contains K and therefore minimal polynomial makes sense and it is a polynomial in L_0 , it is a polynomial in $L_0[X]$.

So look here, this is L, this is $L_0[X]$, this simple extension of L_0 which is contained in L, this is L_0 here, and this is K here and we have an embedding here to E. And we are looking for a contradiction now, this is proper. Anyways, so therefore this polynomial is in

L_0 over K and we have L_0 to E there is an extension, this is σ_0. So therefore this L_0, this is, this is, E is algebraically closed and this L_0 is contained in this, so therefore I can always find, if I take the image of this, if I look at μ_{x,L_0} and take, + σ to this, σ_0 to this, so you get a polynomial in E, this is a polynomial in $E[X]$.

And E is ultimately closed, therefore this polynomial will have 0, so therefore I can always extend. So this polynomial has a zero, so y is a zero of $\sigma_0(\mu_{x,L_0})$. So I have these, the image of this. So, let me identify, you do not have to keep saying this one, or you have this σ_0 of L, this is a subfield here which is the image of this and to this I adjoint y, so that these 2 films are isomorphic, that is just given by this σ_0, this is a subfield here. So therefore if I take this, this composition, this, that will give me an extension, that will give me an embedding, let us call this embedding as, this is an embedding, so let us call this as rho.

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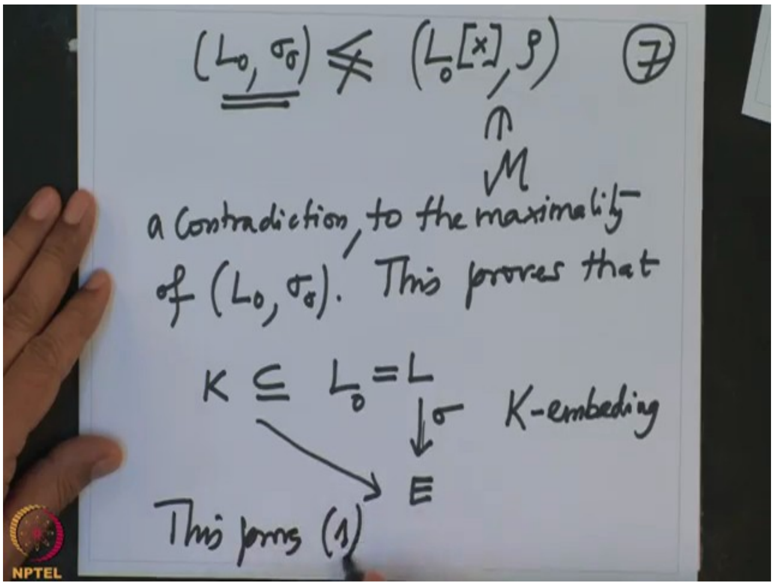

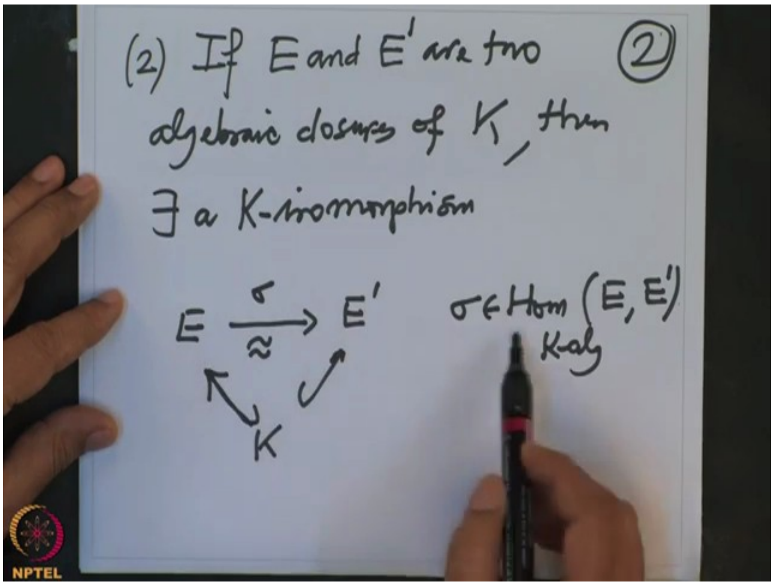

So therefore I got an embedding ρ from $L_0[x]$, this field to E. And obviously rho restricted to L_0 is given σ_0. Therefore, what we check, what we check that that this pair (L_0, σ_0), this pair and $(L_0[x], \rho)$, this pair obviously, this is strictly bigger, this is strictly bigger, that this is also in M but this was the maximal element in M. So a contradiction to the maximality (L_0, σ_0). So this proves that that L_0 has to be L, this is K, this, and then that σ_0 is an embedding in E.

So therefore we have an embedding of, this is K embedding, so there is a K embedding, so that this diagram is commutative. So, that proves the 1st part, so this proves 1. Call it, so now we have to prove 2, I just wanted to show you 2 is, if E and $E^{'}$ are two algebraic closures of K, then there exists a K isomorphism such that E to $E^{'}$, this is the K algebra, isomorphism from E to $E^{'}$. All right.

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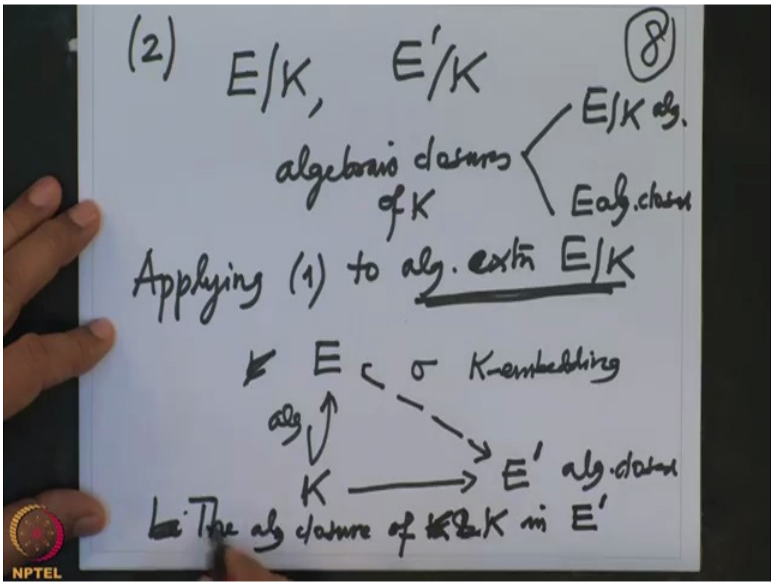

So we have given 2 algebra closures E over K and $E^{'}$ over K, they are algebraic closures of K. So that means 2 things, that is E over K is algebraic and E is algebraically closed, this means algebraic closure. So similarly for $E^{'}$. So, the Part-1 shows that I am going to apply to E. So applying 1 to algebraic extension E over K. Now I am looking at E over K has algebraic extension and I am looking at $E^{'}$ as algebraically closed field and I want to apply 1.

So, that will tell us E here, this is algebraic extension of K and this is algebraic closure, so this is algebraically closed field, algebraically closed, so I can extend this to embedding σ, K embedding. This is by Part-1, now I want to show that this σ is indeed an isomorphism. This is an embedding and now therefore E, if I take the algebraic closure of, so take, so this, take let, so the algebraic closure of K, this way of E, of K in $E^{'}$, I have this. So let me write on the next page.

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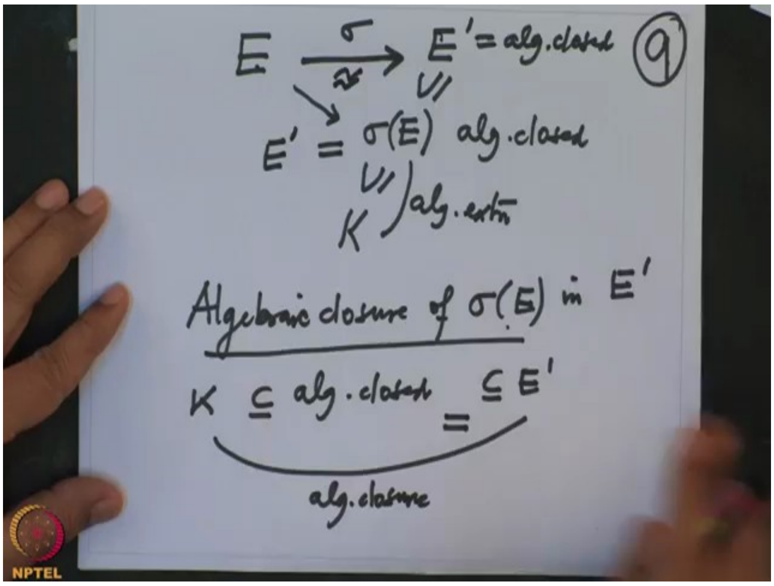

So we have this diagram, so E, E to σ , E to $E^{'}$, we have the σ , I want to show the σ is an isomorphism. So, look at σ of E. See, it is a field, therefore I only have to check it is surjective. So this is the image of σ and this one, now $\sigma(E)$ is algebraically closed, E is algebraically closed, therefore Sigma E is also algebraically closed. This is also algebraically closed and all this contains K, this is algebraic extension. So if I take this field and take its algebraic closure in $E^{'}$, so algebraic closure of $\sigma(E)$ in $E^{'}$.

Algebraic closure of $\sigma(E)$ is also algebraically closed and this is contained in a prime, it contains K, so but this is algebraic closure, so that will show that this is equality here. So therefore we note that Sigma E is $E^{'}$. That means this σ is surjective and that means it is a K isomorphism, that is what we wanted to prove.

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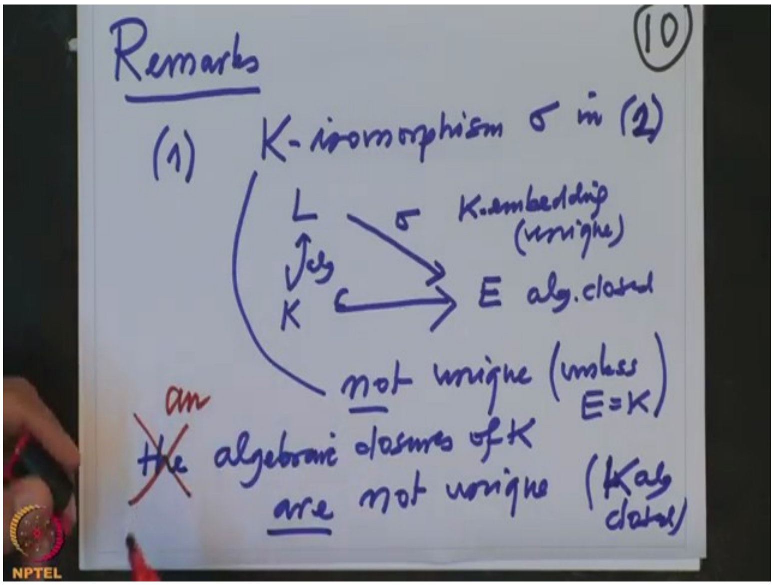

Alright, now I want to make 2 remarks, 2 very important remarks, one this K isomorphism, K isomorphism σ in 1, remember that is L over K algebraic, is algebraic and this is algebraically closed field, E is algebraically closed. Then we have extended this here, this σ which is a K embedding, this is not unique in general, as I remarked earlier also. So therefore in 2 also, the K isomorphism in 2, that is also not unique. Because all this we got by using Zorn's lemma and by choosing a maximal element and partially ordered set may have more than one maximal element.

So therefore, it is not unique in general, unless, unless E equal to K. Anyway, but if you do not have this, that means if you feel is not algebraically closed, then the algebraic closures, algebraic closures of K are not unique in general, unless E, unless K is algebraically closed. So, therefore we cannot say the, we cannot set the, we are not allowed to say the. So we will keep saying an algebraic closure, that is one remark. This is very very important because one may get a feeling that these are unique.

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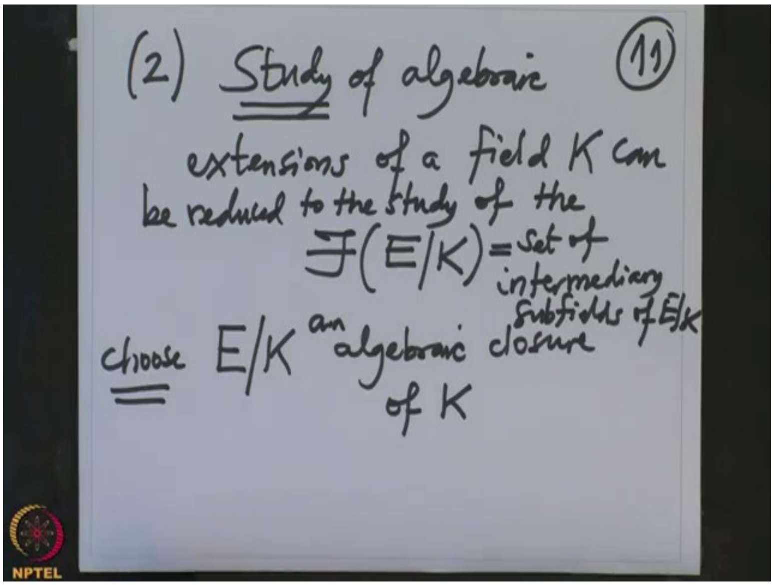

Okay, the next remark, remark 2, if I want to study algebraic, so study of algebraic extensions of a field we study, this study, we will reduce it to, if I choose, E over K algebraic closure of K, an algebraic closure, so choose an algebraic closure of K, that we know it exists now and then this, therefore F of intermediary field of E over K makes sense. So these are, this is a set of intermediary fields, subfields in between E and K. So if you want to study algebraic extension of this, it is enough to study this.

Study of algebraic extension of a field K can be reduced to the study of the set E over K intermediary subfields of E over K where we have chosen algebraic closure of K. So, this I have not used earlier but this is very important in the study of this. Okay, now the next one is, so I would rather stop here and prepare for proving that the field of complex numbers is algebraically closed. And for that proof I am going to use the fundamental theorem on symmetric polynomials, this I will do it in the next lecture. Thank you.