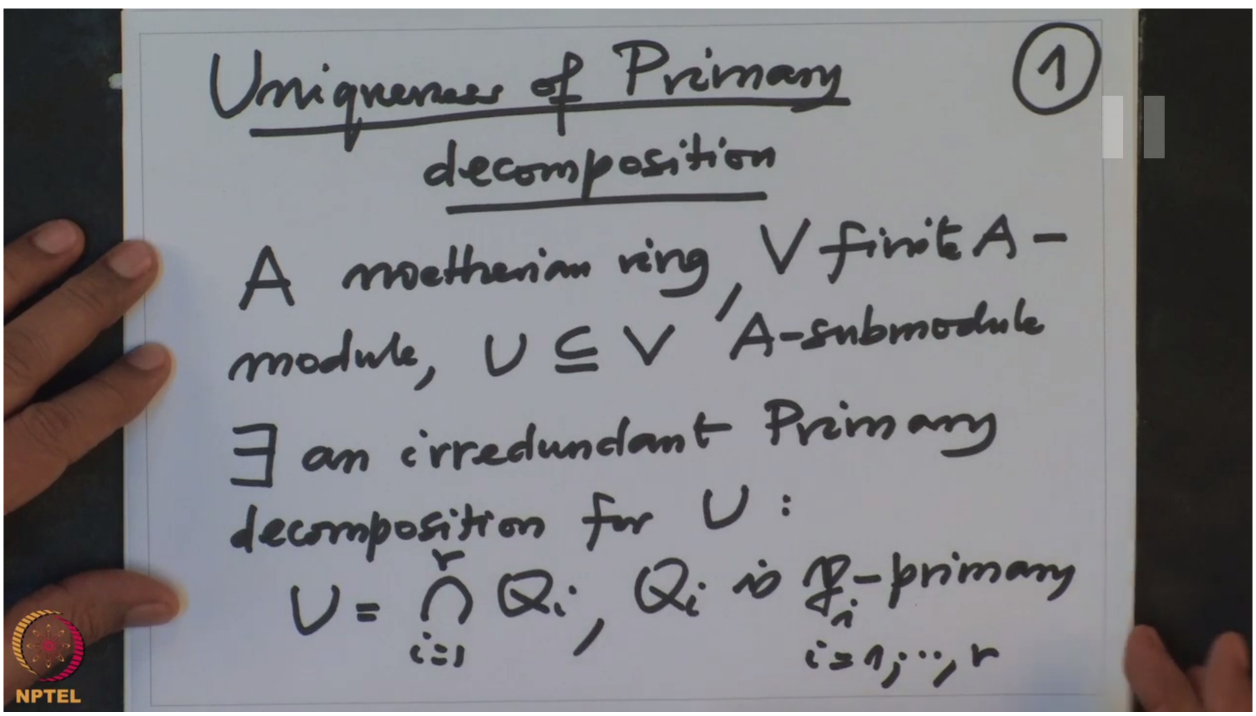
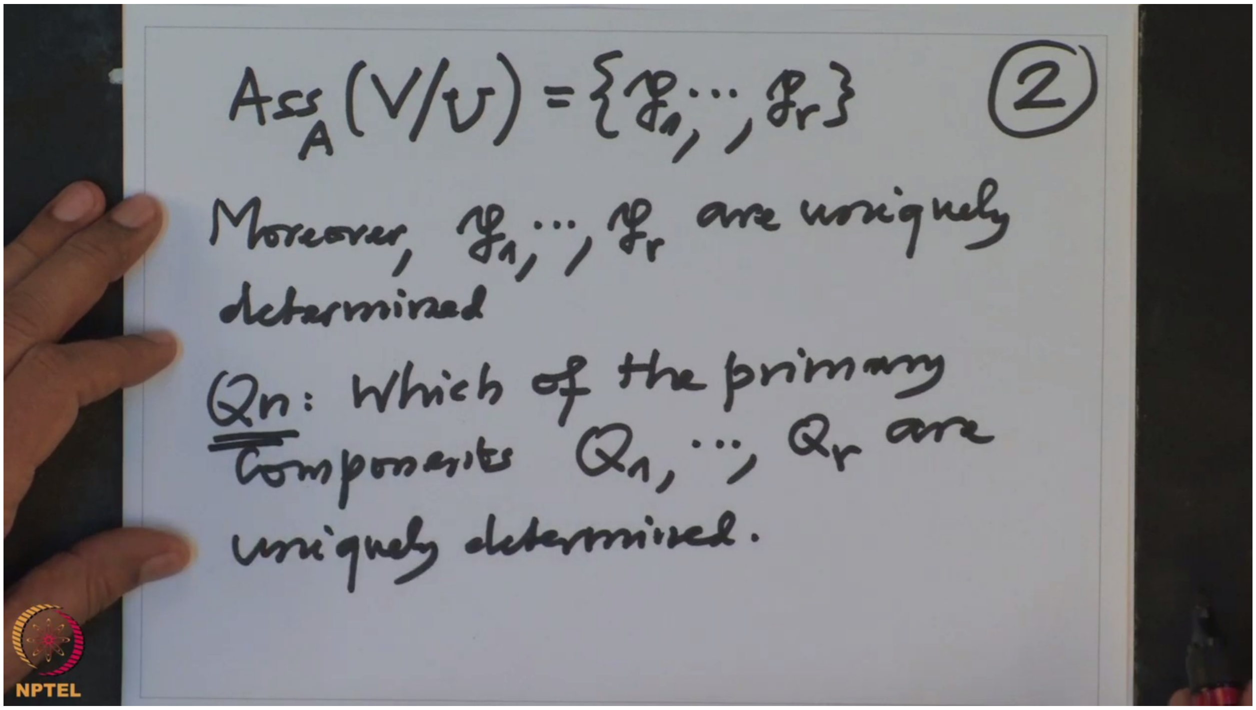
Prof. Dilip Patil: In today’s lecture, we shall discuss about the unique nature of Primary Decomposition. So let us recall what we have done so far.

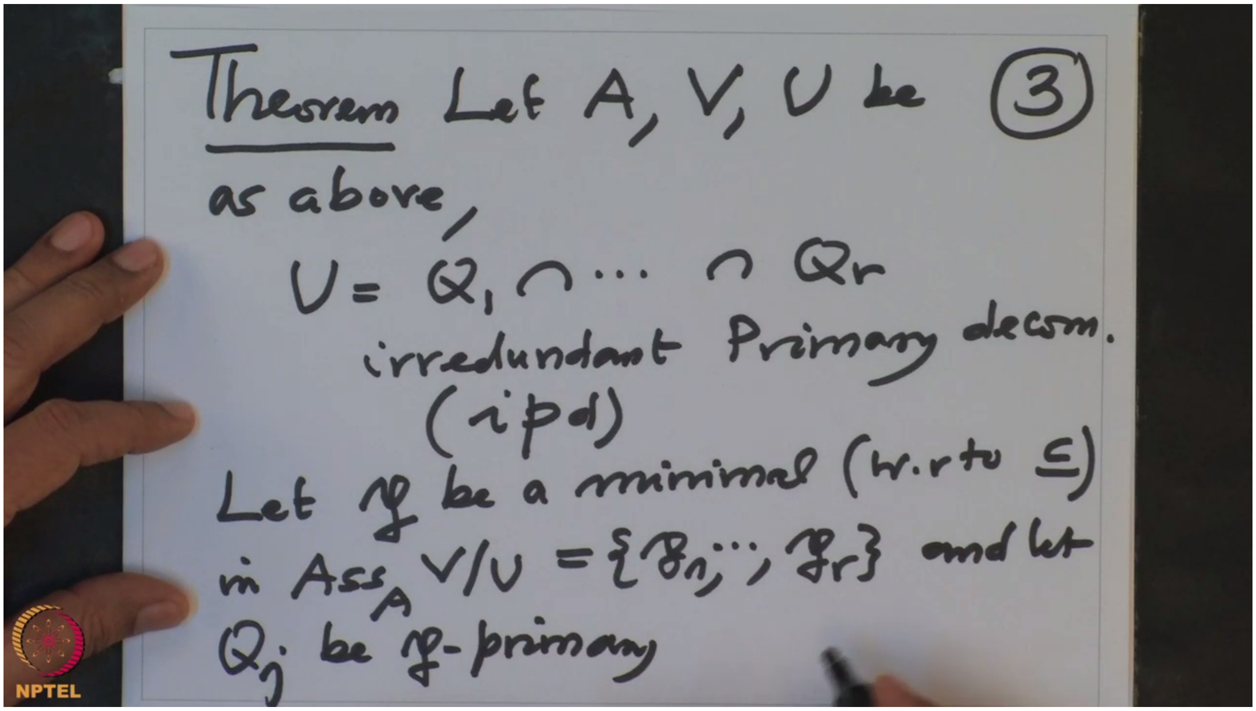


So today, we are discussing Uniqueness of Primary decomposition. So as usual, A is in Artinian ring and V is a finite A - module, and U $\subseteq$ V A-submodule. Then we have proved earlier that there exists an irredundant Primary decomposition for U. That means U is intersection of Q_i

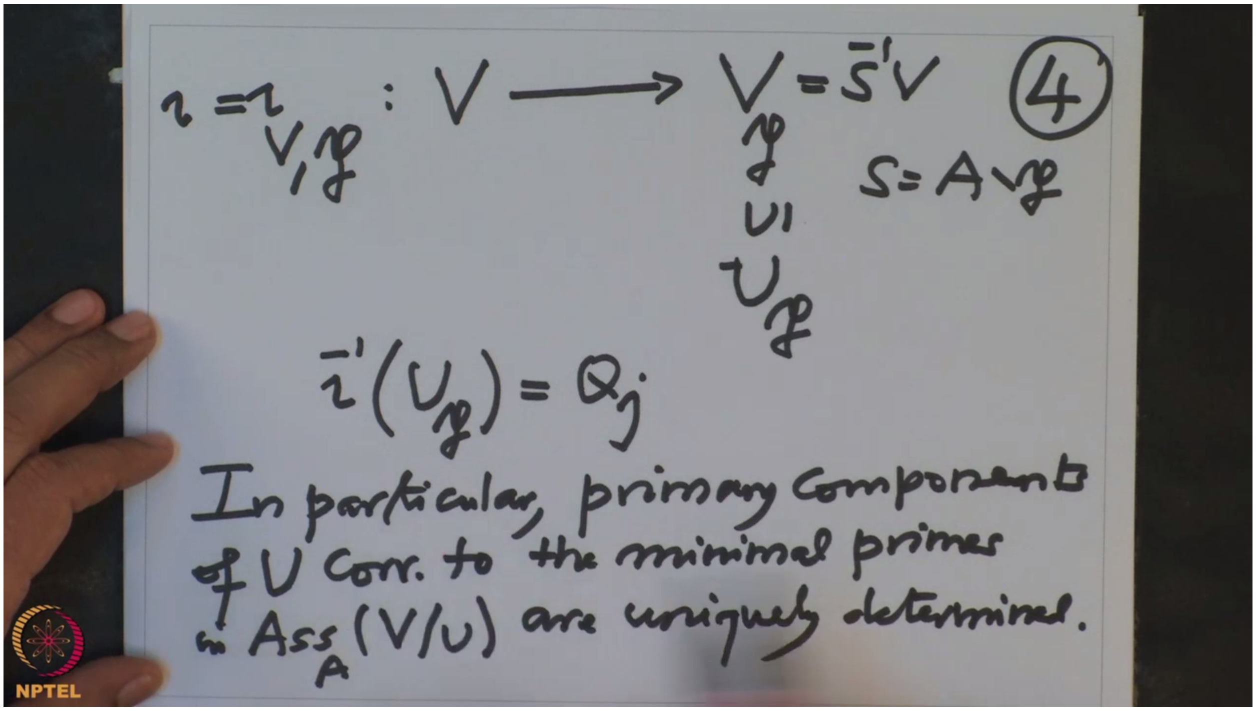
finite limiting and this Q_i is P_i-primary for $i=1,\ldots,r$, and irredundant means you cannot drop anyone of them.



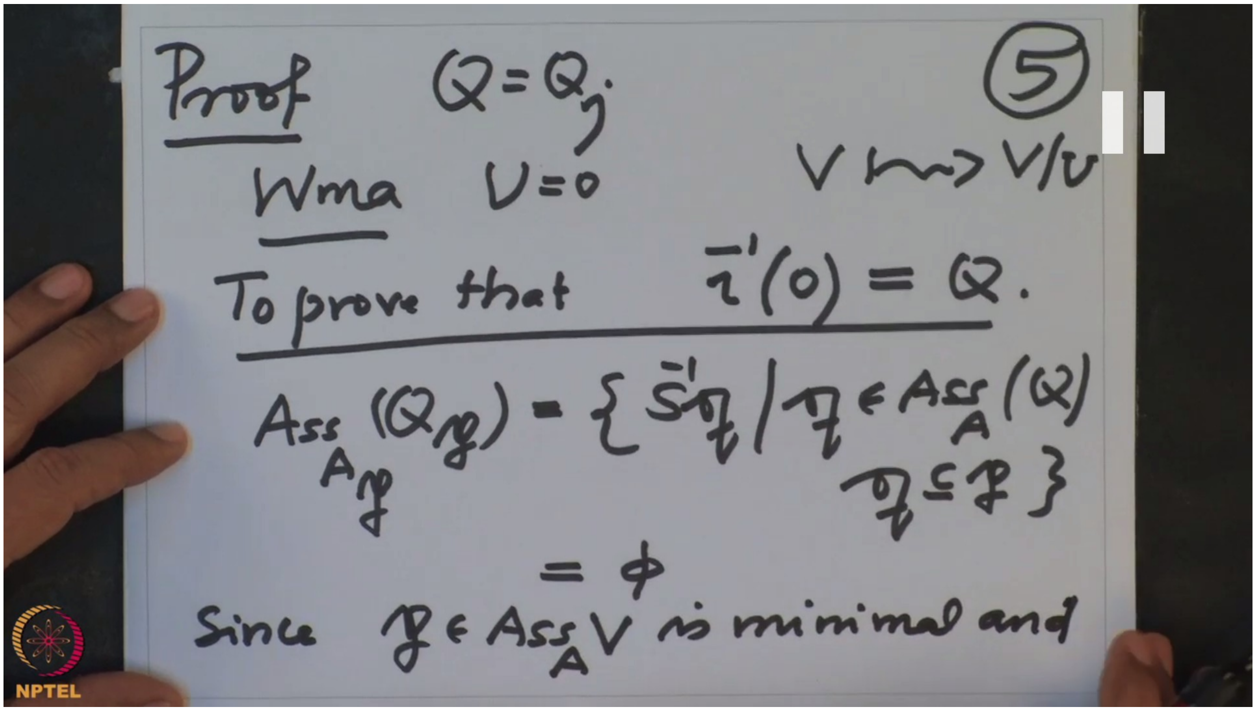
Also, we have seen that the associated prime ideals of $\frac{V}{U}$ is precisely these $\{p_1,\ldots,p_r\}$. These are the prime ideals corresponding to the primary components of U. Moreover, we have also seen that this $p_1,\ldots,p_r$ are uniquely determined. And today, we are going to discuss whether the primary components are uniquely determined or not. So question is, which of the primary components is $Q_1,\ldots,Q_r$ are uniquely determined. As you will see the answer to this question, not all of them are uniquely determined, but some of them are uniquely determined.



So we are going to find out which one. So for this, we want to first prove the preparation, we want to prove that the following theorem, which will help us, which is the preparation for answering this question, so theorem. Let A, V, U be as above and $U = Q_1 \cap \ldots \cap Q_r$, irredundant primary decomposition. These also I will abbreviate by (ipd) and let P be a minimal with respect to the inclusion in the Associated primes, in the set -- this is the finite set of associated prime ideals, and I've chosen P to be the one minimal among them, and because it is a finite set, minimal exists, and let Q_j, which is appearing here, be P-primary. Only one of them corresponds to this p, therefore, I call it Q_j.

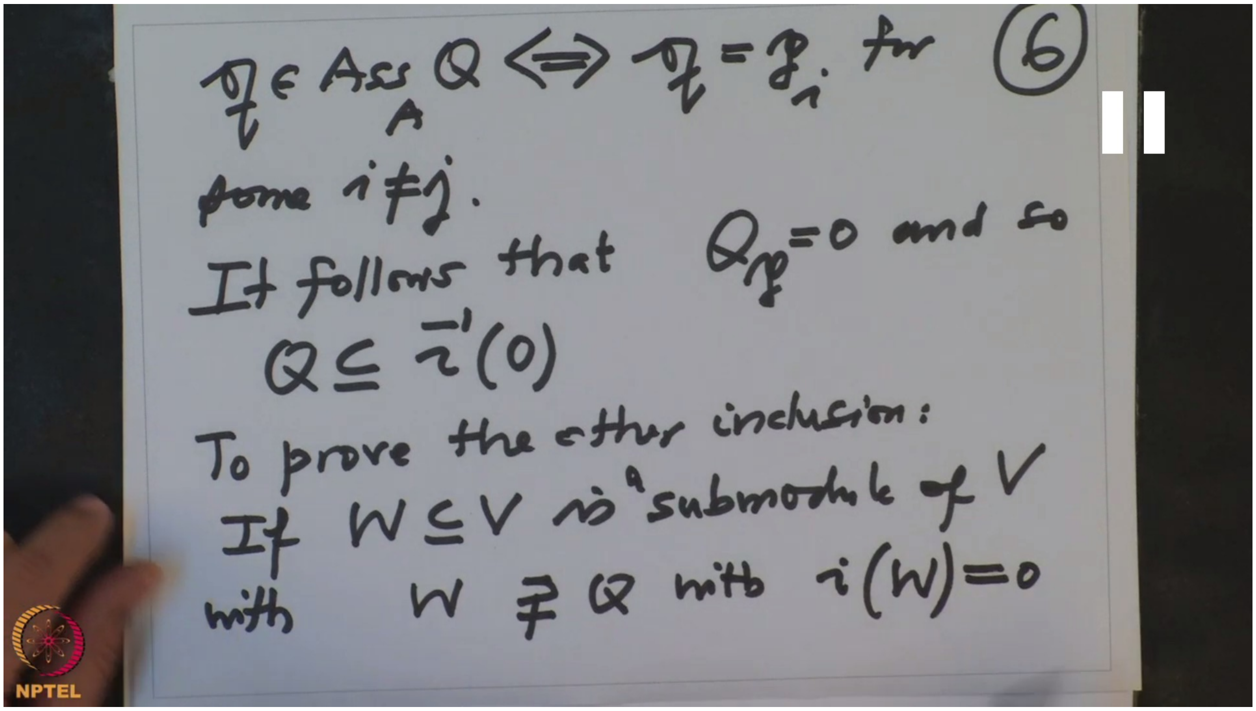


And then, I want to look at the map, see we have a map, $i_{V,P}$, this is just a notation, this is a map, a localization map from V→V localized at p. V_P is by definition $S^{-1}V$ where $S = A \setminus P$, and now you have a submodule U of V and I localize it here. And I want to pull it back under this natural map, so $i_{V,P}$ let us abbreviate this by i only, $i^{-1}(U_P) = Q_j$, that is an exception. Once I approve this, because this side is uniquely determined by U and p, therefore, this side is uniquely determined. So that will prove that Q_j is uniquely determined. So in particular, let us record the statement, in particular, primary components of U corresponding to the minimal primes in the Associated $(\frac{V}{U})$ are uniquely determined. And the prime ideals -- the primary components corresponding to the non-minimal prime ideals, they need not be uniquely determined. I will leave this to find an example of such kind to the participants, but let us do this theorem first and then we will digress a little bit for the ideals and about the Zariski topology.

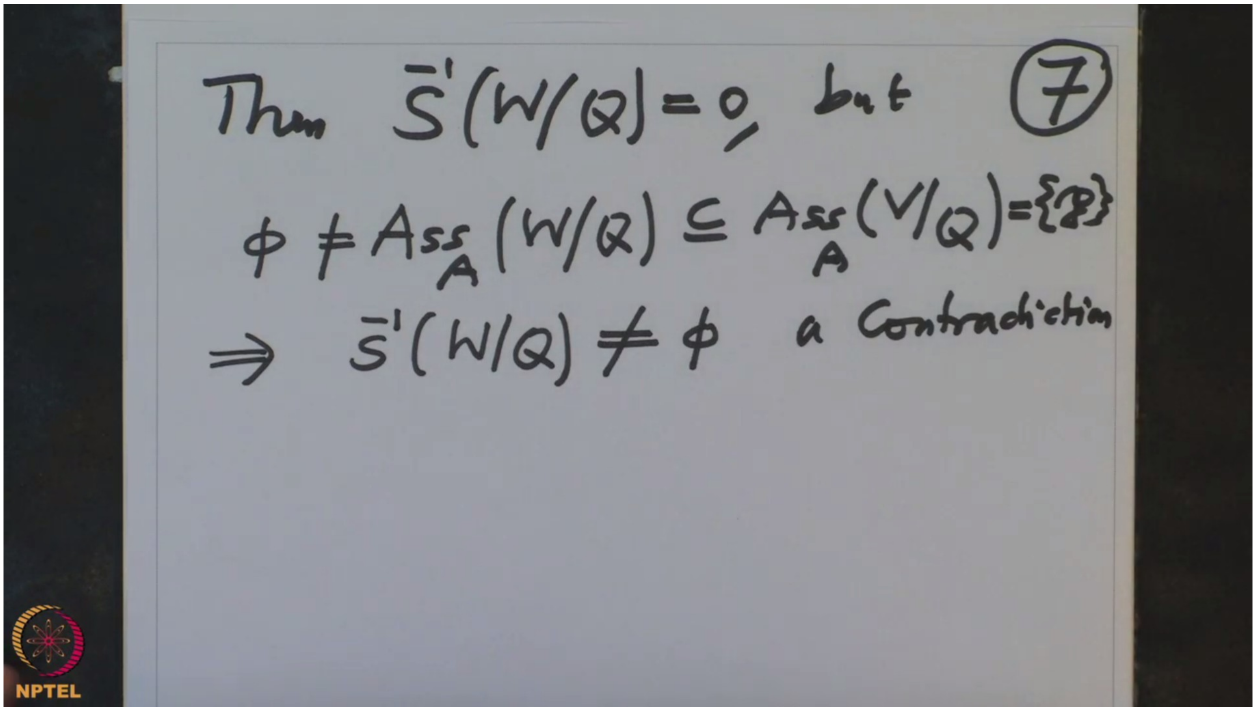


So proof is very easy, so Proof. Let us put $Q = Q_{jj}$ and we may assume that U = 0 bypassing to the replaced V → $\frac{V}{U}$. And let us put -- so we want to prove therefore, so with this notation, we want to prove that $i^{-1}(0) = Q$. This is what we want to prove, all right.

So we have earlier proved that if I localize p and look at the associated prime ideals of (Q_P), these are precisely the localizations. So {$S^{-1}Q$ where Q belonging to the associated prime ideals of (Q), and obviously it should be proper ideal to Q should be containing P. This we have proved earlier, but remember this set, because Q_P is minimal, P is minimal among them. It is clear that this is empty set since P belonging to associated prime ideals of V is minimal --

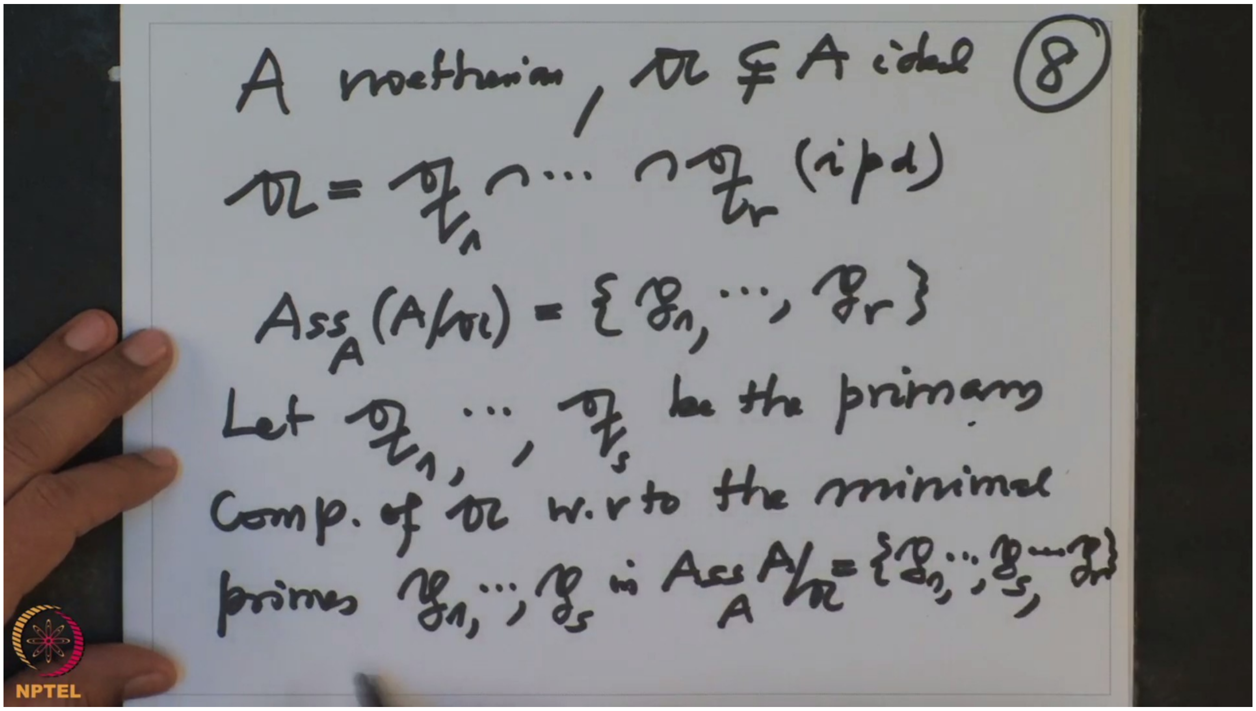


And Q belonging to he associated prime ideals of Q if and only if Q is one of the P_i for some $i \neq j$. So therefore, from this it follows, so it follows that because the set of associated prime ideals of the localization Q_P is empty set, therefore, the model Q_P has to be 0 model only, and hence this means, and so Q is containing i^{-1} of 0. So we wanted to prove equaled here, we proved one inclusion. So now to prove the other inclusion, suppose this is not equal, then I can find a bigger submodule. So if W containing B is a submodule of V with W properly contained in Q with $i(W) = 0$. That means when you have pushed this W to the localization, it becomes 0, then we should get a contradiction.

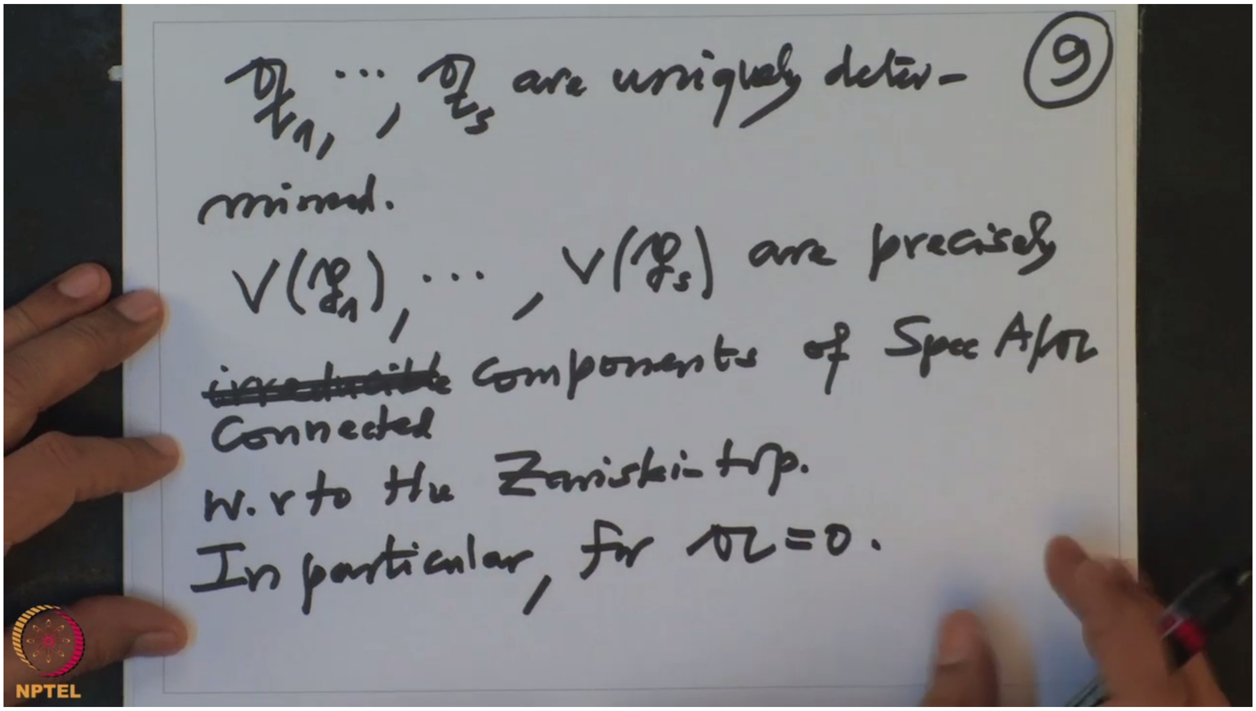


And how do we get a contradiction? That is because -- okay, so then first note that in $S^{-1}\left(\frac{W}{Q}\right)=0$, but -- this is 0, therefore, the associated primes of V should be empty set, but what happens. If you look at the associated prime ideals of ($\frac{W}{Q}$) which is containing associated prime ideals of ($\frac{V}{Q}$) but Q is pre-primary, therefore, associated prime ideals of ($\frac{V}{Q}$) = {P} and this is because Q is not W, we are assuming, this is non-empty. Because of this when I localize this, this P will still survive. So that implies $S^{-1}\left(\frac{W}{Q}\right)\neq\varnothing$, but we have just shown earlier that it is empty, therefore, a contradiction.

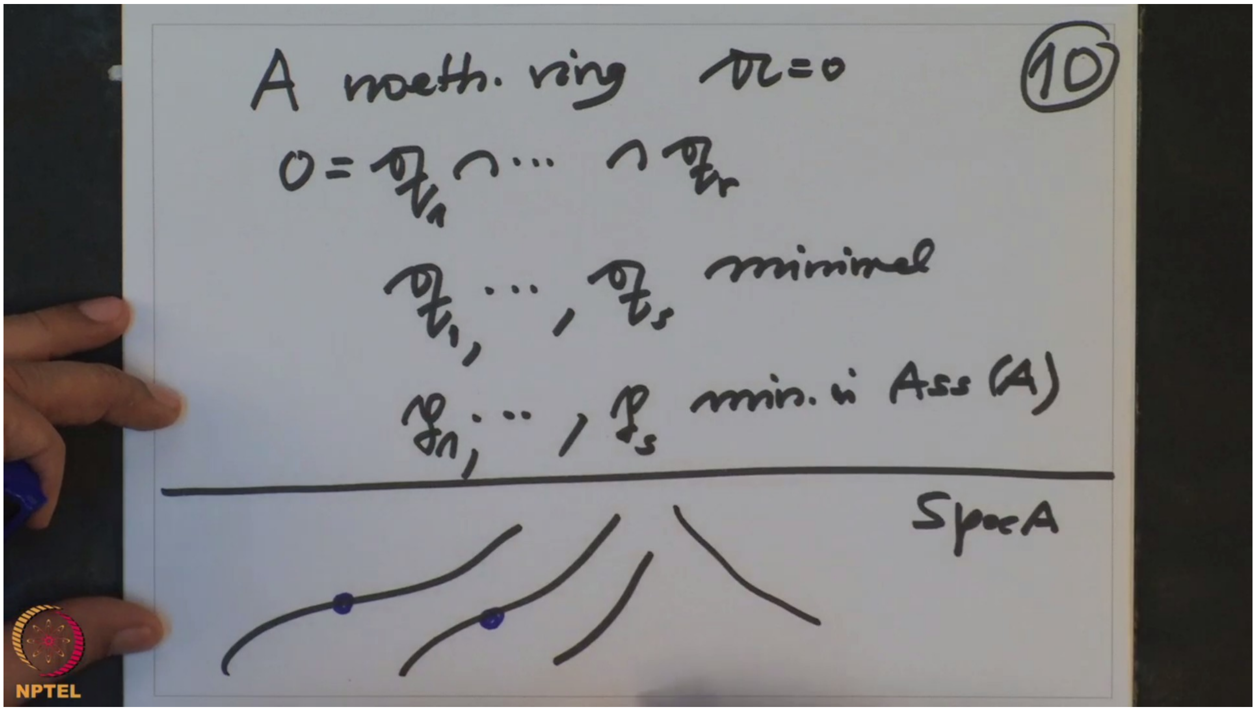
So this proves the proposition and now I just want to remark. So what did we prove? We proved hat the associated prime ideals corresponding to the minimal primes, they are uniquely determined.



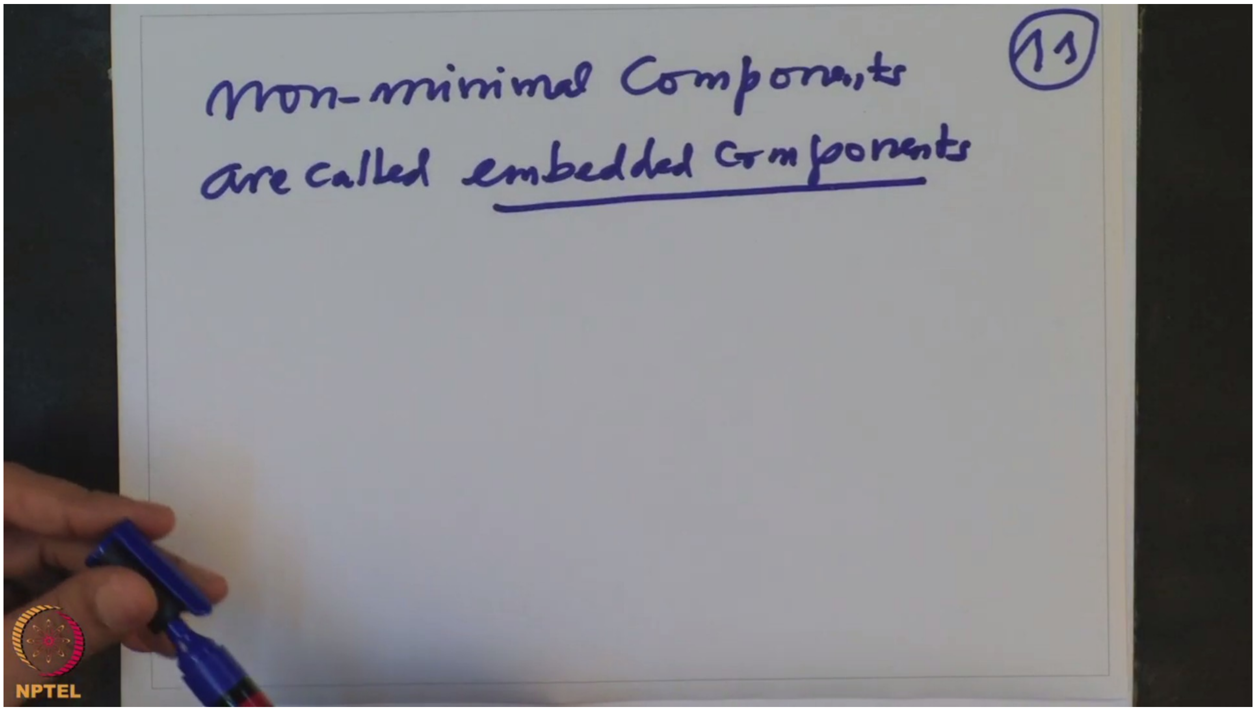
So when you, for example -- now let us take a particular case, which is very important. So suppose I have a Noetherian ring, A Noetherian and A is an ideal there, a proper ideal, then we know that A has a primary decomposition irredundant. So primary components are like this, $q_1 \cap \ldots \cap q_r$, and among them I choose -- so we have this associated prime ideals. This is $P_1, \ldots, P_r$. These are uniquely determined we know but qs may not be uniquely determined. So I choose the minimal one among them. So that means let $q_1, \ldots, q_s$, maybe minimal among them, so $q_1, \ldots, q_s$, definitely there is at leas a few more minimals, few minimals because this is a finite set. So let this be the primary components a with respect to the minimal primes $P_1, \ldots, P_s$ in the associated prime ideals of this $\frac{A}{a}$, which is a the set $P_1, \ldots P_s$ and a few more, P_r. This is (ipd). We know that these q_s are minimal, q_s are uniquely determined.



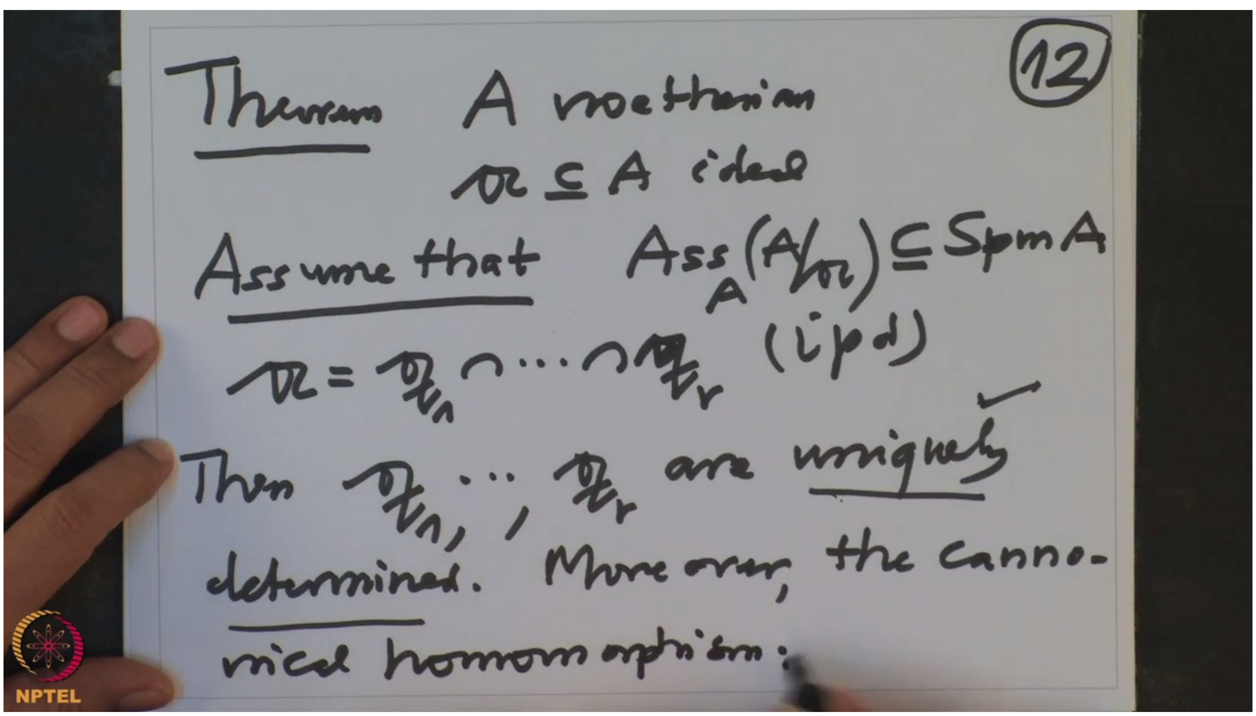
So therefore, we have proved that $q_1, \dots, q_s$ are uniquely determined and now if you look at the corresponding V, recall that V is -- this is by definition all those prime ideals, which contain P_1. So this is given to $V\left(P_s\right)$, these are precisely irreducible components of spec $\frac{A}{a}$. So they are actually not irreducible, but they are connected components of this with respect to the Zariski topology. So I have recalled Zariski topology is given by where Vs are declared to be the closed sets. We have seen Vs are here. So pictorially what we proved that this -- so in particular, for ideal a = 0, then I would like to draw the picture of the spectrum.



So if you have ring A, A is a Noetherian ring and we are applying the above result to the ideal 0, a is a 0 ideal, then we have 0 is the intersection of these primary components and $q_1,\ldots,q_s$ are the minimal one. These are minimal and they are corresponding to the prime ideals $P_1,\ldots,P_s$, they are minimally in the associated prime ideals of (A), then if you want to see the picture of spectrum of A, these are the Vs, they are the connected components, so they don't intersect because the components, so like this, but there could be an embedded component. That means non-minimal one will be lying somewhere here or here. So these will be called embedded components.

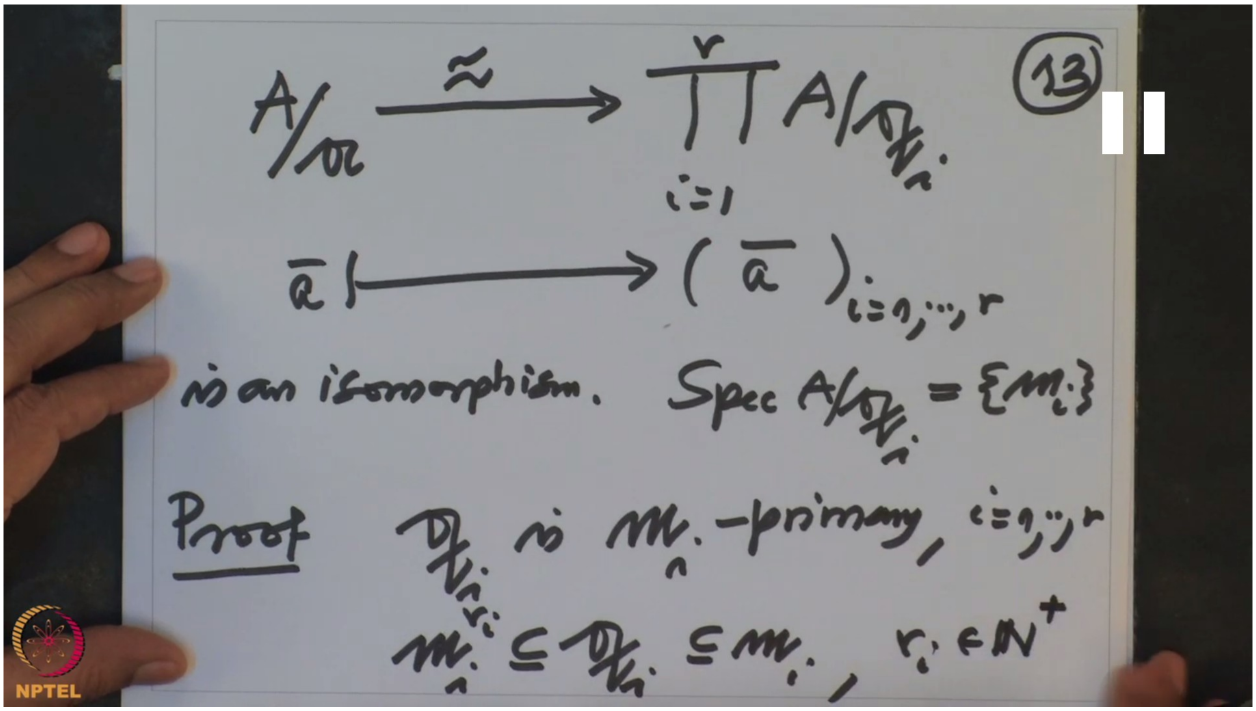


So the components, which are not minimal, they are called -- so non-minimal components are called embedded components. That means they are embedded ins some other connected components, all right.



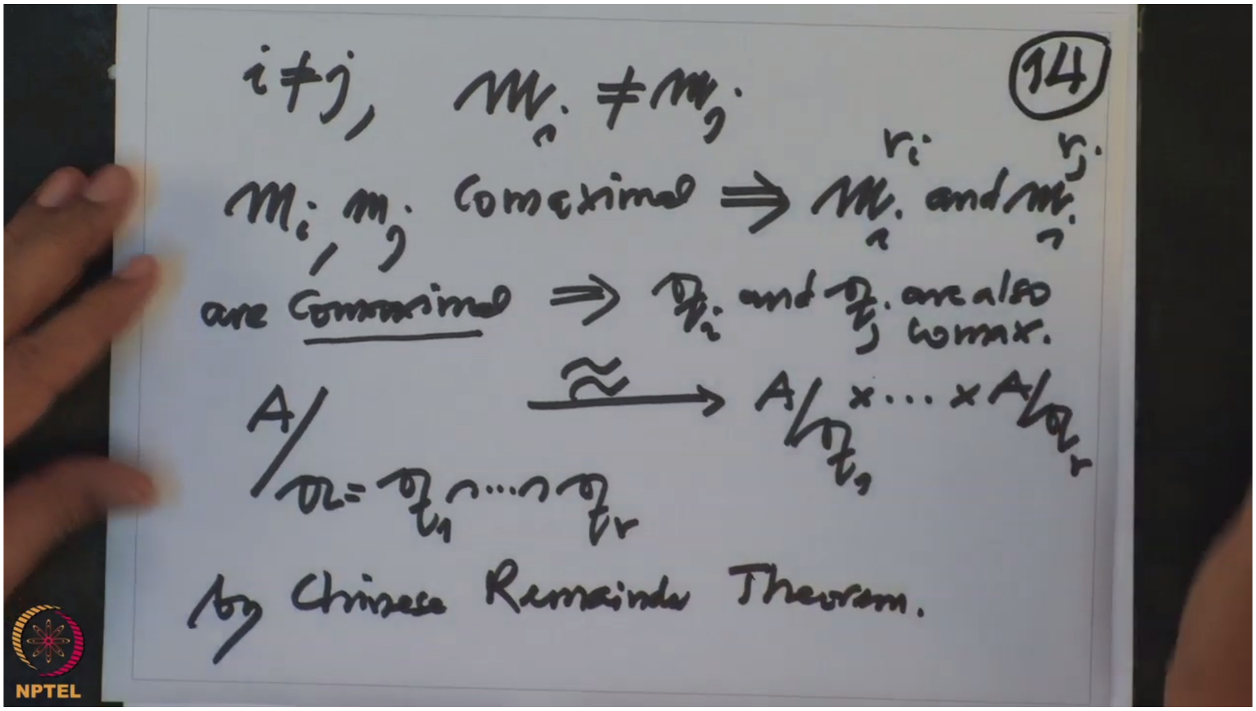
So now I just want to give one more observation, which will be also very useful in some later lectures, namely this theorem, this is for the rings. So as usual A Noetherian, a is an ideal in A, and assume that the associated prime ideals of $\frac{A}{a}$, they are full of maximal ideals. They are not only prime -- by definition they are prime ideals, but we are assuming, they are maximal ideals. This is the set of all maximal ideals in it.

Then suppose $a = q_1 \cap \ldots \cap q_r$, this is the primary decomposition (ipd), an irredundant primary decomposition. Then $q_{1,} \ldots, q_r$ are uniquely determined. That is because these associated prime ideals are full of maximal ideals, so therefore, every element there is a minimal element. There can't be any associated prime ideals which contain a maximal ideal, because maximal ideals are maximal with respect to the inclusion. So therefore, by earlier theorem, they are uniquely determined. Moreover, the canonical homomorphism:

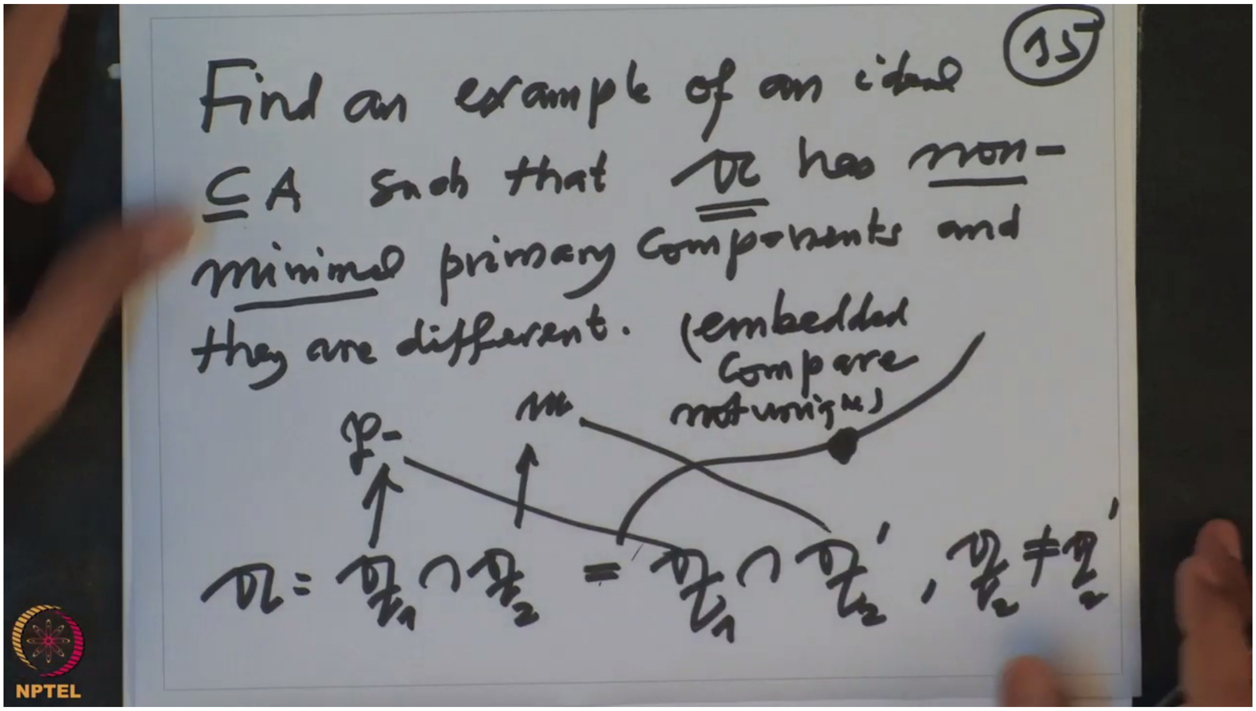


That is $\frac{A}{a}$ to this is a diagonal $i=1,\ldots,r, \frac{A}{q_i}$ just map $\bar{a}$ to the corresponding images, these are the corresponding images in mod q_i and this is from $i=1,\ldots,r$. This canonical map is an isomorphism, and spectrum of $\frac{A}{q_i} = m_i$.

So proof is as I have said, uniquely determined follows immediately from the earlier theorem, because all of them are minimal ones. Now I only have to justify these isomorphism, but for that, I will use Chinese Remainder Theorem, because this q_1 is mi -primary and mis are maximal ideals, therefore, this q_i will contain a power m_i, some power $m_i^{r_i}$ for some r_i belonging to some natural, non-zero natural number.



And then because m_i -- if I take different i and j, if $i \neq j$, then $m_i \neq m_j$, so therefore, m_i, m_j are comaximal, therefore, the powers are co-maximal, $m_i^{r_i}$ and $m_j^{r_j}$ are comaximal. Therefore, if I look at this $q_1 \cap \ldots \cap q_r$ from here to I have $\frac{A}{q_1} \ldots \frac{A}{q_r}$, and because they are comaximal -- because these are comaximal, therefore, q_i and q_j are also comaximal. Therefore, Chinese Remainder Theorem will tell you this is an isomorphism by Chinese Remainder Theorem. So that proved the assertion we wanted and spectrum is single, that is clear. So that proves minimal isolated -- sorry. That proves that the primary components corresponding to the minimal primes are uniquely determined, and that is the best we can do.



And I would just write for record that -- so find an example of an ideal ⊆ A such that A has non-minimal primary components and they are not unique, and they are different. This is very easy to see, but I want you to construct. So one possibility is if you take the suitable prime ideal and maximal ideal containing that, the maximal ideal which contains that. Now use this configuration to find an ideal a so that these -- it has two primary components, one of them is with respect to p, the other is with respect to m, and the m one is not uniquely determined. So you have to find two different. So a you can't write it as $q_1 \cap q_2$, this will be p primary, this will be m primary, and also it is equal to -- obviously q_1 is going to be determined, so $q_1 \cap q_2^{'}$, this is m primary, this is P primary, and equality holds, but $q_2 \neq q_2^{'}$. So that will show that primary components, corresponding to the non-minimal are not uniquely determined. These non-minimals, they are also called embedded, because it's embedded in this, the primary component corresponding to the minimal prime. So I'll just write the name embedded components are not unique in general, all right.

So with this, I will stop and w will continue later. Thank you.