INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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Lecture No. – 08
Primary Decomposition
(Contd)

Alright, so now in this lecture after discussion with the associated prime support etcetera, now we will come to what is called primary decomposition. This is a generalization of a prime decomposition that we will see it once we have defined it what it is precisely, so let us first define it what it is, definition, so always let A be a Noetherian ring, and V A-module, and U sub-module of V, alright.

Now finite family, finite family $Q_i, i \in I$ is a finite index insert of sub-modules of V which are primary sub-modules in V, that means the homothesis are either injective or nilpotent and there this Q_i will be primary corresponding to some prime ideal,
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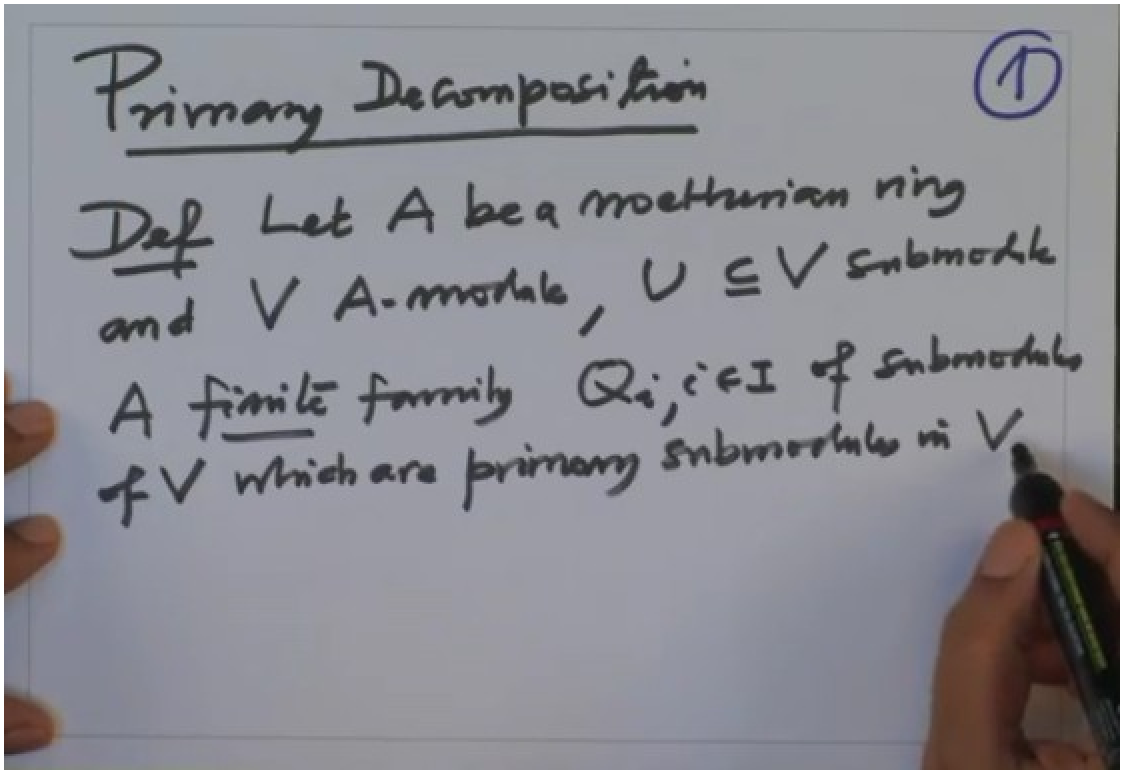

so finite family of sub-modules is called, such that the given sub-module U is intersection of this Q_i 's, this is called, also this equality so this is called a primary decomposition of U in V.

So now there are several questions will come up namely whether does it exists,

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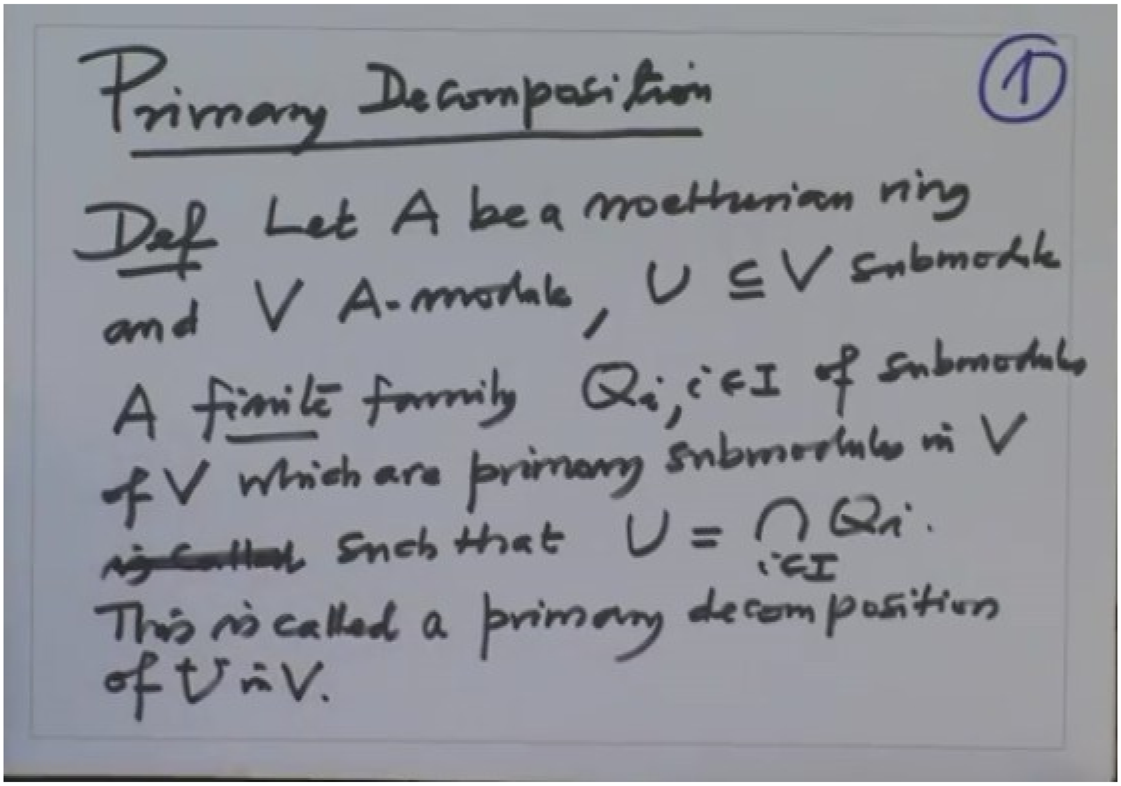

is it unique and so on. So but before that as I said I just want to say that this is a generalization of the prime decomposition, so look at this example, example so suppose A is a P ideal, for example you can take either A equal to ring of integers or polynomial ring in one variable over a field K, then we know that if I take any nonzero element $x \in A$, x nonzero, then we know that this x has a prime decomposition that means x is a unique, product of some unique times some $p_1^{v_1} \dots p_r^{v_r}$ where U is a unit in the ring, and this p_1 to p_r are prime elements, that means they generate, each one of them generate it's a prime ideal and this v_1 to v_r are the multiplicities that is bigger equal to 1 and they are distinct prime ideals, because equal ones we have collected together, so such a thing is called a prime decomposition of a nonzero element in a PID, we know that in such,

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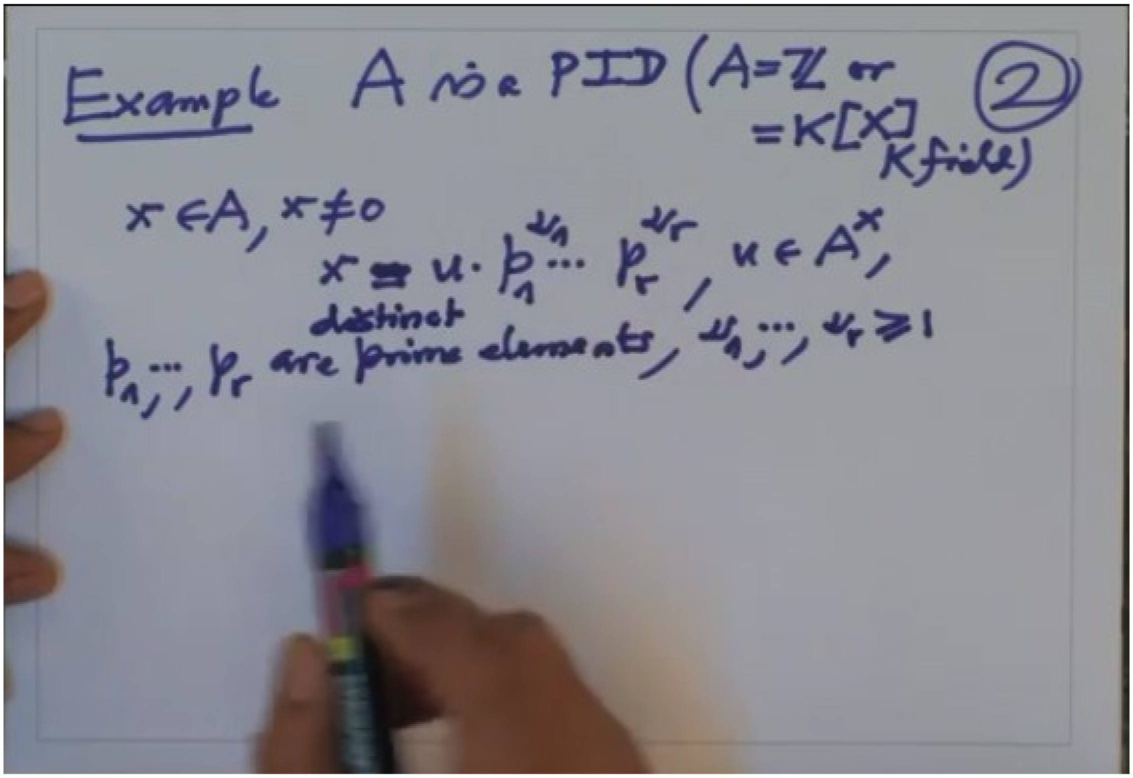

because PID's are UFD so every nonzero element has a factorization like that.

Now when I translate this in terms of the ideals that means ideal generated by x that is this is, this will go away because ideal generated that is A so this is $(Ap_1)^{\nu_1} \cap ... \cap (Ap_r)^{\nu_r}$, this becomes intersection because a product and intersection is same then the ideals are co-maximal but because they are distinct primes, these ideals are co-maximal AP_i and AP_j for distinct i and j, this is the whole ring A for $i \neq j$, therefore this equality holds, and moreover because they are prime elements, this AP_i 's are prime ideals, not only their prime ideals they are actually the maximal ideals because they are nonzero prime ideals in a PID, nonzero prime ideal in a PID maximal therefore this, therefore they are maximal ideals, therefore their powers, therefore $A P_i^{\nu_i}$ this are AP_i primary ideals, therefore what this equality is nothing but it's a primary decomposition of the ideal Ax,
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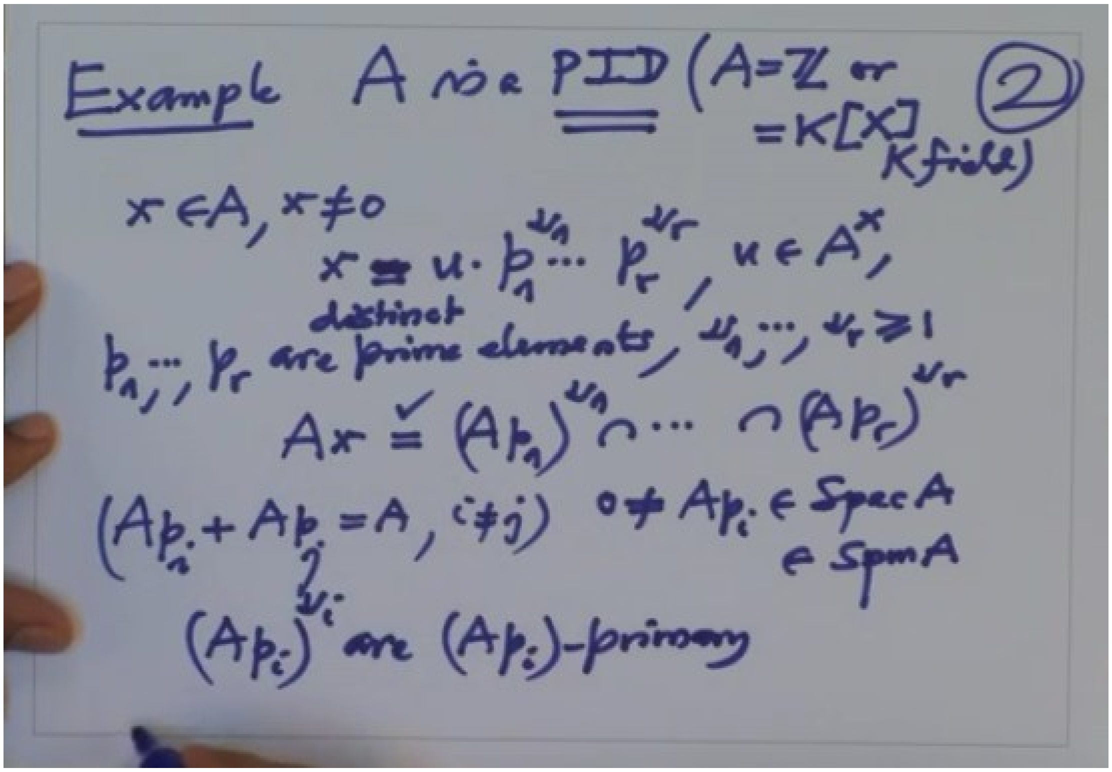

so this equality is a primary decomposition of the ideal generated by x in the module A, so that is the reason why it is called,

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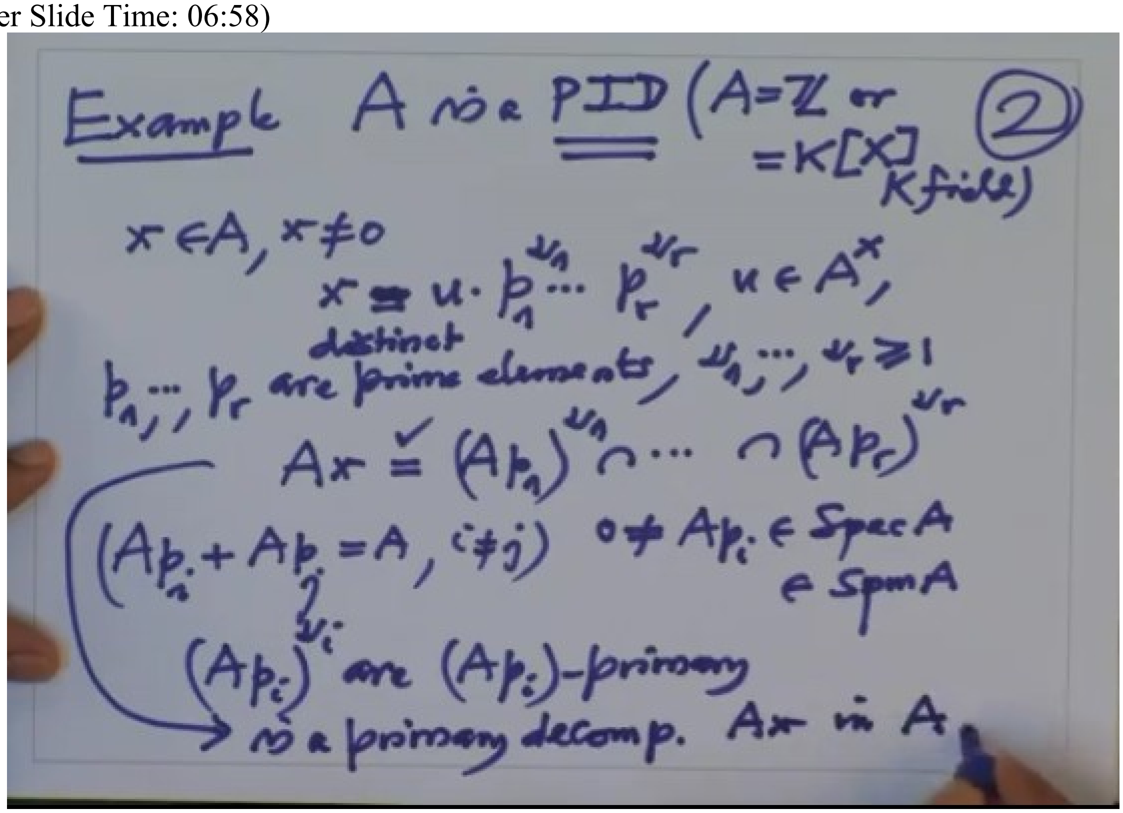

therefore it is generalization of the prime decomposition. So in a ring, ring may not be UFD but whether the question is whether in general rings ideals in the ring have primary decomposition, and the answer is yes your ring is Noetherian, alright.

Now we prove the existence so that is the theorem, this theorem which is called existence of primary decomposition, and for the proof of this we will use the earlier construction of the associated primes and so on, so let A be Noetherian, A be Noetherian ring and V finite A-module and W sub-module or U, U is a sub-module, U is a sub-module of V. Then there exists finitely many sub-modules Q_1 to Q_r sub-modules of V such that the following modes, such that one Q_i 's is P_i prime ideal in V, and this P_i 's are different, P_i not equal to P_j for $I \neq j$, they are prime ideals so P_i 's are prime ideals in the ring A, okay, that is one.

Second, U is intersection of Q_1 to Q_r , and third the associated primes of $\frac{V}{U}$ is precisely this set P_1 to P_r , so if you want to find associated primes you will find the primary decomposition and
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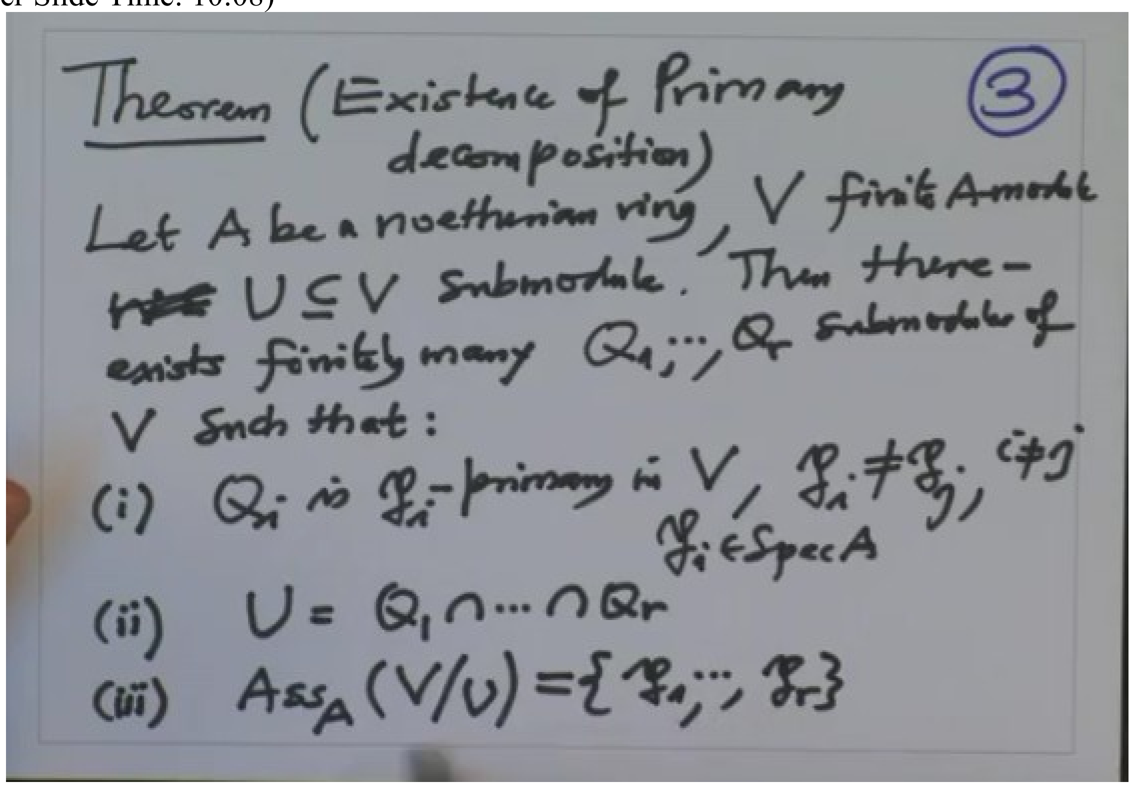

look at their corresponding prime ideals and those will be the associated prime ideals, so this is the theorem we are going to prove, so there will be several steps but all are easy, so for a proof, for a proof note that we can assume U is 0, this is by passing from V to the residue class module $\frac{V}{U}$, then you can assume U is 0.

Then when you go mod that then the primary decomposition will be $\frac{Q_i}{U}$, alright, then okay, so we can assume U is 0, now suppose we want to construct this QI right, so suppose now the third condition is associated primes of U are precisely P_1 to P_r that means so therefore suppose we know already associated primes of V these are finitely many prime ideals, so let us say that P_1 to P_r and then we want to construct Q_i 's, so that Q_i 's are P_i primary and they are different and U equal to the intersection of Q_1 etcetera Q_r that is what we want to prove, because we already 3, alright.

Now suppose this then by the earlier theorem I can construct, so by earlier theorem which says that given any subset we can find, given any subset of the associated prime we can find sub-module whose associated primes are precisely the given set, so by earlier theorem there exists for each i, Q_i , sub-module, this is for each i = 1 to r such that the associated primes of, associated primes of $\frac{V}{Q_i}$ is precisely the singleton P_i and associated primes of Q_i this will be removing P_i , so this is P_1 to P_{i-1} and then P_{i+1} , P_r , this is for i= 1 to r, there exists such a sub-module because that is what we have proved in the earlier theorem that such a sub-module exists.
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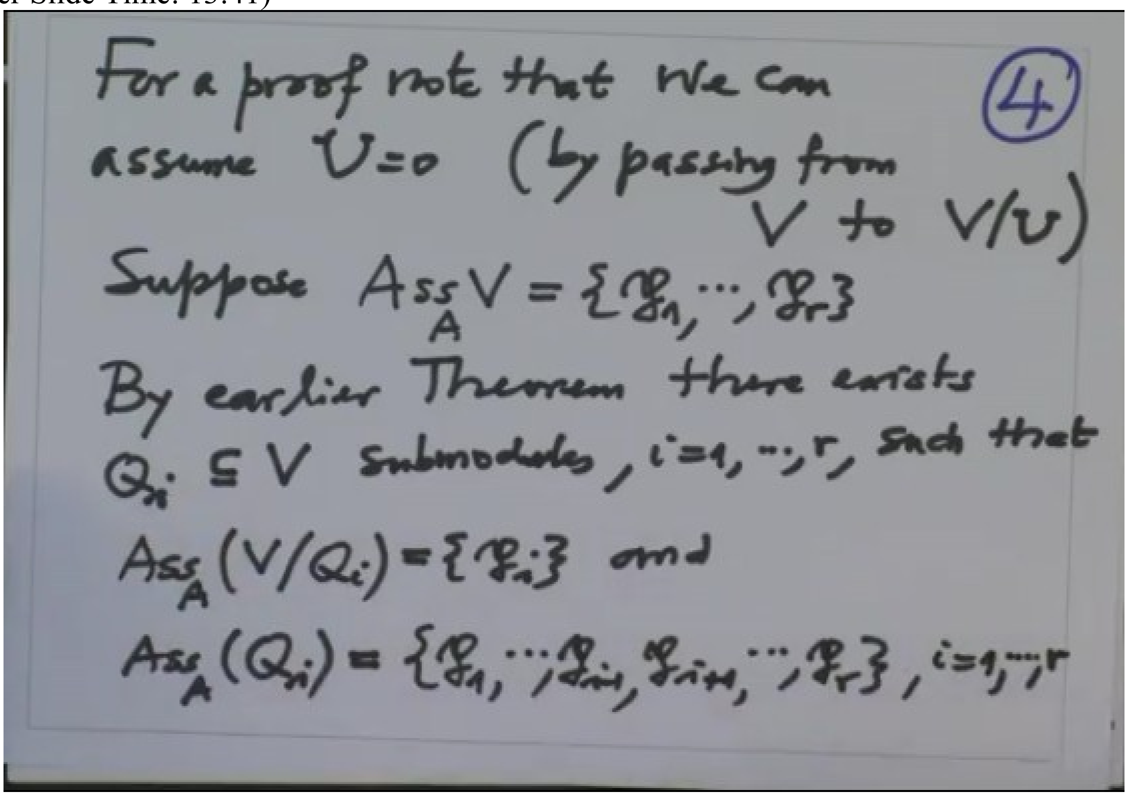

Now first note that we have checked that once the associated primes of this is singleton then we have checked that such a module, such a sub-module has to be P_i primary, so note that then, note that Q_i is P_i primary in V. And now put $U = Q_1 \cap ... \cap Q_r$, and what would we like to prove? Now we want to prove this U has to be 0 that is what we want to prove, alright, so to prove U is 0 this is what we want to prove, alright.

So this is equivalent to proving if I want to prove some sub-module is 0. I want to prove that the associated primes of this U is empty, is empty set there is no associated prime that's what we are noted when we define associated primes, so we want to show this, okay, alright.

But in any case we know associated primes of U, this is contained in associated primes of Q_i, because this is a sub-module of each one of them so this is sub-module, this is associated primes is a subset here, and this Q_i is a sub module of V therefore this is a subset of associated prime ideals of V and this is true for every i, i from 1 to r.

So if this has some element, so therefore I know there and these we know it is P_1 to P_r, so therefore if at all this contains, we want to prove it is empty set, if at all it contains some element it must be one of the P_i, so note that if P_i belong to associated prime ideals of A in U, then P_i will also belong here, then P_i will also belong to associated prime ideals of Q_i, but I know I have chosen Q_i so that the associated prime ideals of Q_i, but I know I have chosen Q_i so that the associated prime ideals of Q_i are precisely omitting P_i 's, so this is we know we have chosen Q_i so that this is P_1 to P_{i-1}, and then P_{i+1} to P_r, so therefore P_r definitely doesn't belong here, so this is a contradiction, so this means so this proves that associated primes of U should be empty set and that is equivalent to saying U is 0 and once U is 0 we have done, we have proved everything,
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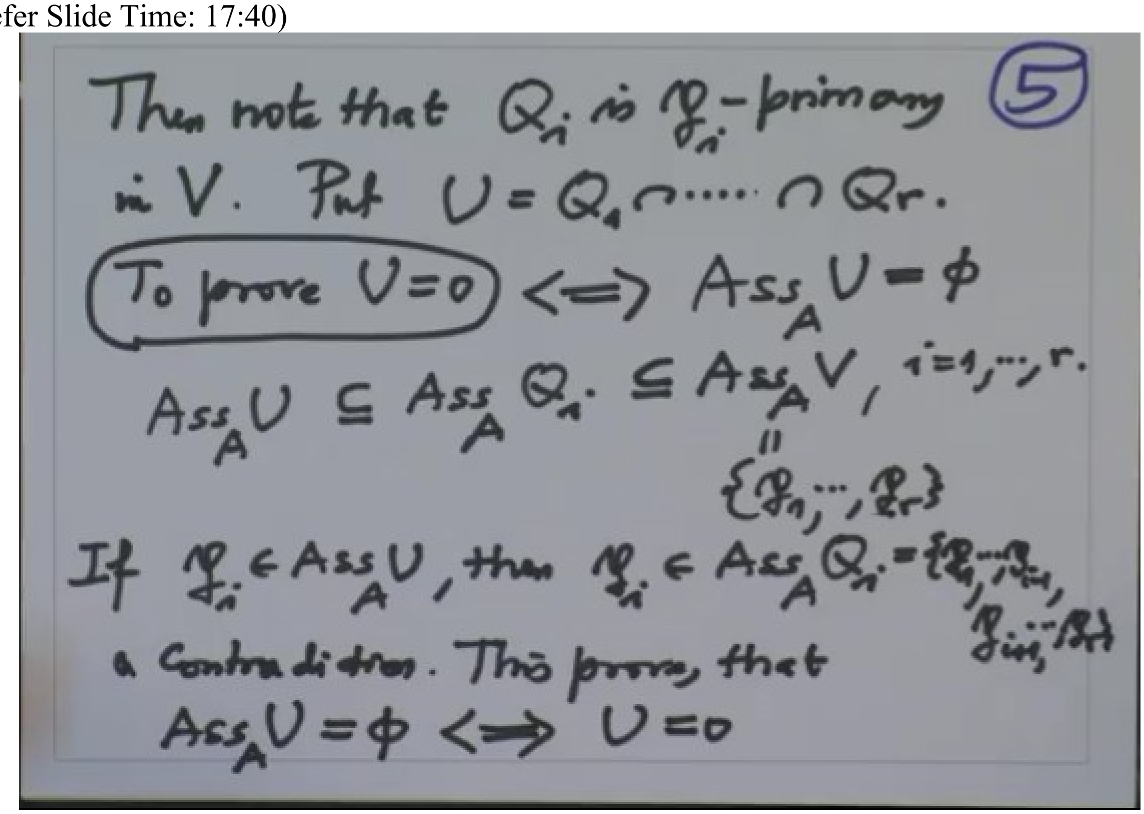

so that proves existence, alright, so these two like distance.

Now the next job is to find the uniqueness, alright. So now we are precisely defining what does one mean by the uniqueness, so let us define, so this is a definition so this will tell more precisely what does one should mean by uniqueness of the decomposition, so again A Noetherian ring, remember that our assumption is V is finite module, so V A-module finite and U is a sub-module of V, sub-module and we are talking about primary decomposition of U in V, and its uniqueness, so the primary decomposition $U = Q_1 \cap \ldots \cap Q_r$ of U in V is called irredundant, irredundant if all necessary that means what, so if the two conditions one there is, there exist so intersection of Q_i , Q_j , $j \neq i$ this is not contained in Q_i for every i, for every I this thing happens, that means this Q_i is really necessary in this intersection.

And second condition the associated primes of $\frac{V}{Q_i}$ this is singleton P_i , and this P_1 to P_r are pair wide distinct, that means so that is $P_i \neq P_j$ for $i \neq j$, then we only say that it is irredundant, that means all the P's are different
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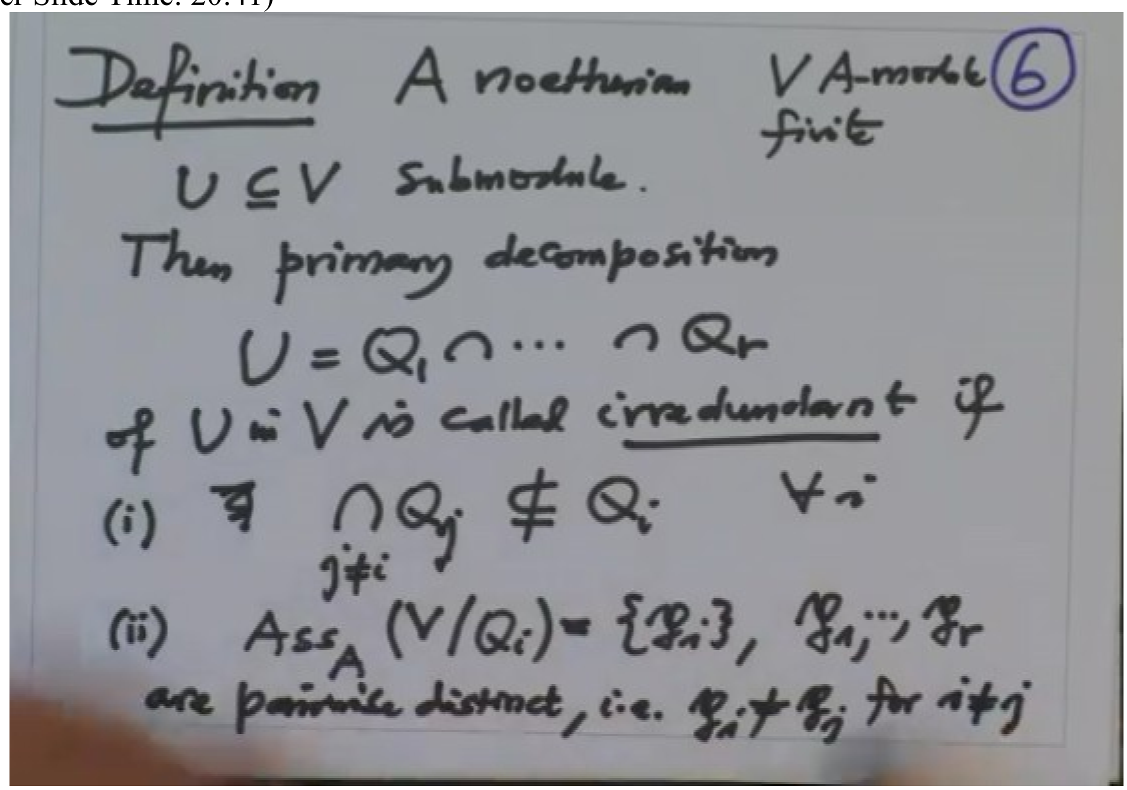

and all the capital Q's they are needed to in the intersection.

Now let us prove a proposition, so proposition, so let U=¿Q_i , i is from 1 to r be an irredundant primary decomposition of U in V, V is a finite module over Noetherian ring, and this is a primary decomposition, then one the associated prime ideals of $\frac{Q_i}{U}$, this is the union of the associated prime ideals of Q_j mod U, where j is running not equal to i, and j is from 1 to r, alright then B, 2, the associated prime ideals of $\frac{V}{U}$ is precisely the union of associated prime ideals of $\frac{V}{Q_i}$, if it is irredundant then this must be, this two equalities, so proof so each are equality, so one I would like to prove one inclusion at a time and then the other inclusion another time, okay, so I'm proving one and that two this inclusion, this inclusion in prime, that means every element of this should be an element in this union that is what I want to prove.

So alright, so let us take the given U is, U take intersection over j, $j \neq i$ and then Q_j, and then intersection with Q_i, so I have separated of the time, because I want to prove that if somebody an element here, then it is here, so this j is from 1 to r, so this I just rearranged it,
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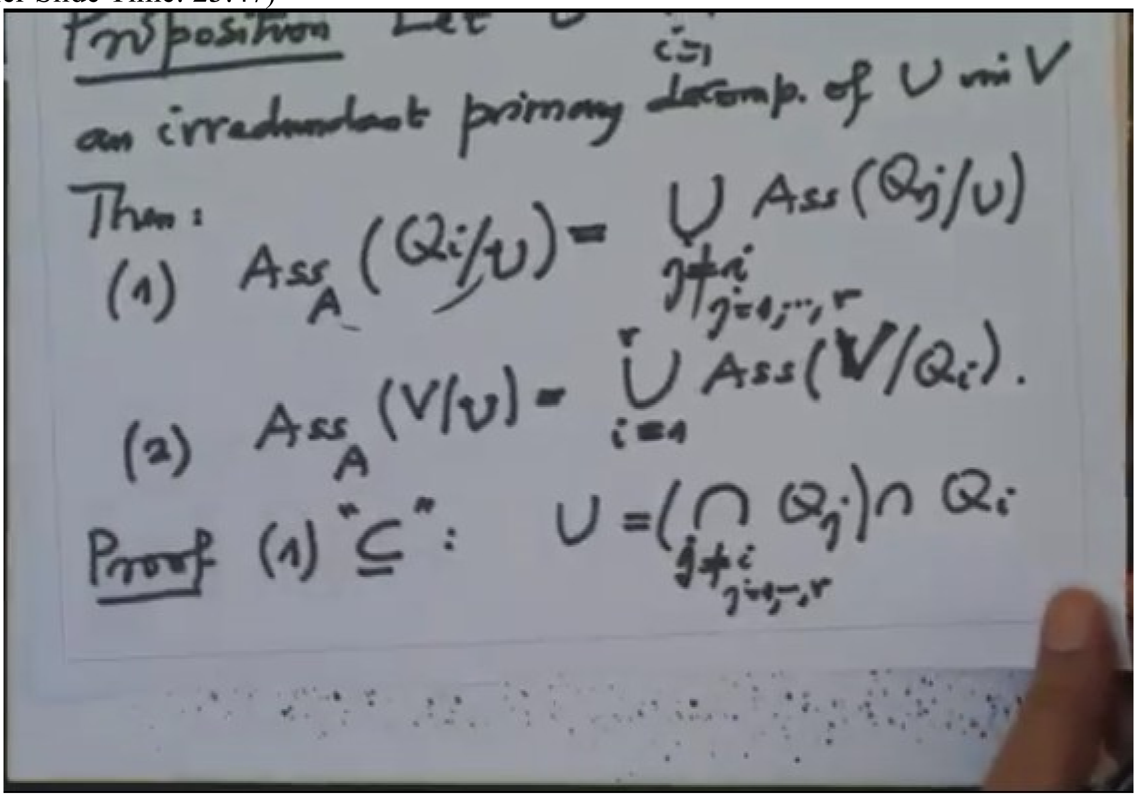

so because of this what do I know? This U is equal to the intersection of this two sub-modules therefore if I want to look at the associated primes, so therefore associated primes of $\frac{Q_i}{U}$ this is contained in associated primes of, I have to take union, union over $j \neq i$ j is from 1 to r, associated prime ideals of $\frac{Q_j}{Q_j \cap Q_i}$, since because this inclusion follows because I have the inclusion map from $\frac{Q_i}{U}$ to the direct sum $\frac{Q_i}{Q_i \cap Q_j}$, and this is running over $j \neq i$, and here also this should be the i.

So this is clearly an inclusion because if somebody goes to 0 here, that's in the intersection of each one of them therefore it is, so that is easy to see, therefore if I look at Q_i, this one

$\frac{Q_i}{Q_i \cap Q_j}$, this is isomorphic to $\frac{Q_i + Q_j}{Q_j}$ which is a sub of $\frac{V}{Q_j}$ therefore this one is clear, therefore associated prime ideals of $\frac{Q_i}{U}$, this is contained in the union, union $j \neq i$ associated prime ideals of $\frac{P}{Q_j}$, so that proves one inclusion,

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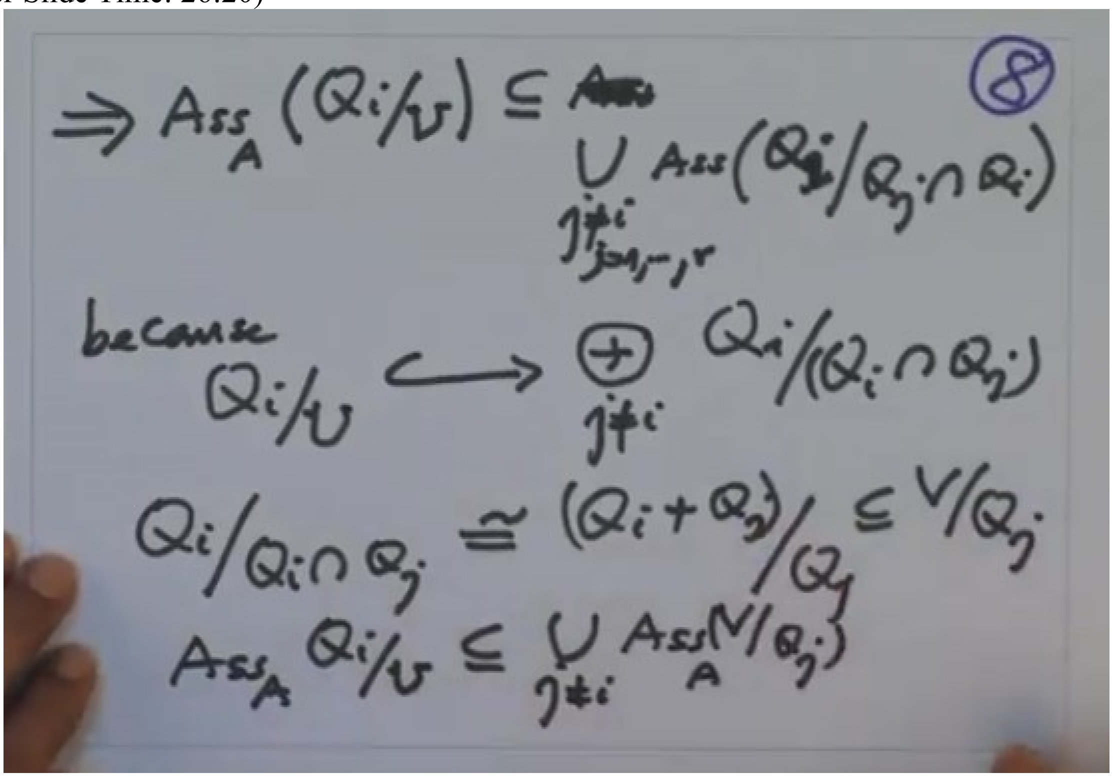

and the other inclusion is will be clear by 2, so I just want to prove 2, so the other inclusion in one will be clear from 2, so proof of 2, alright, proof of 2, alright.

So what is that we want to prove? We want to prove that the associated prime ideals of $\frac{V}{U}$ is contained in the union, so look at the associated prime ideals of $\frac{V}{U}$, we want to prove that this is contained in union, union is from i = 1 to r and associated prime ideals of $\frac{V}{Q}$ as singletons, they are singletons P_i 's because Q_i 's are P_i primary, so this on the other side so this is clearly contained there, now other inclusion, so this is clear, this is clear, on the other conversely if you put P_i = intersection Q_i , now this is running over Q_j , j is running, $j \neq i$ j is from 1 to r, then $P_i \cap Q_i$ is given U, and $P_i \neq U$ because it was irredundant primary decomposition and again therefore PI/U this is nonzero, and this is contained in $\frac{P_i + Q_i}{Q_i}$ which is contained in $\frac{V}{Q_i}$, and therefore we get associated prime ideals of $\frac{P_i}{U}$ is singleton P_i , and because $\frac{P_i}{U}$ is contained in $\frac{V}{U}$ therefore all this P_i 's, P_i 's belong to the associated prime ideals of $\frac{V}{U}$,

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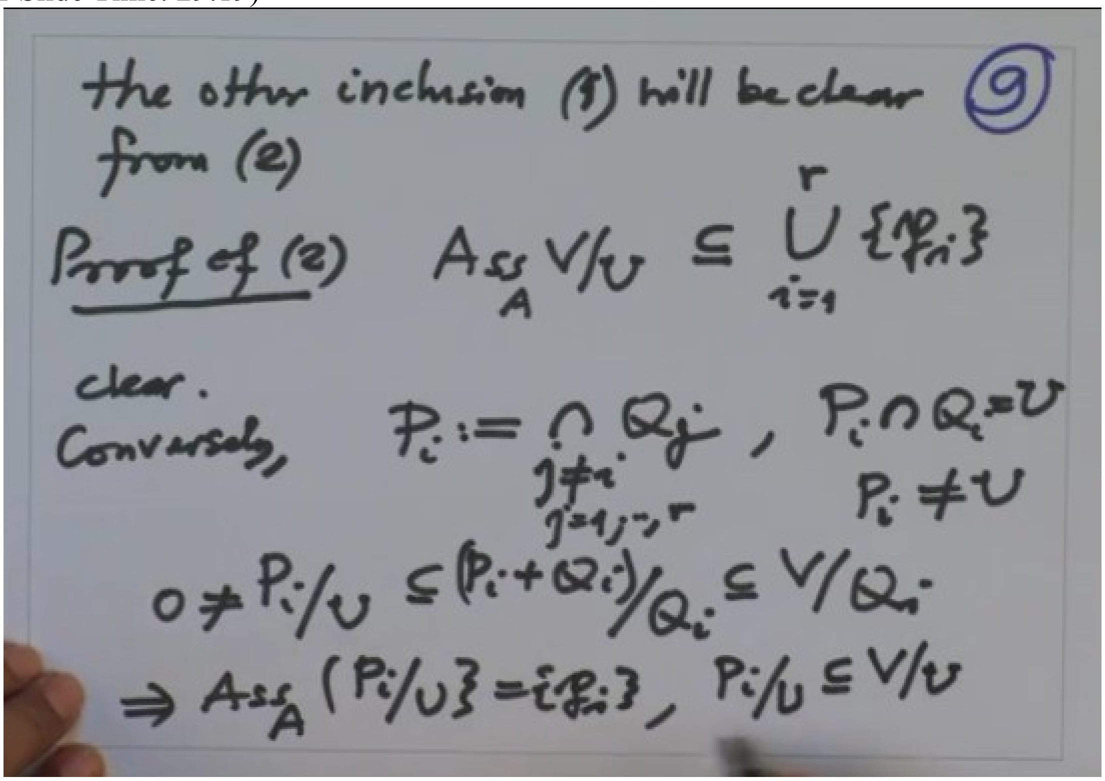

so we have proved that equality here, so that proves equality here and that proves the proposition and therefore we prove that irredundant primary decomposition exist, and we will still reduce one corollary from this, that means write that corollary because corollary, so A Noetherian ring, A Noetherian, we finite A module and U is a sub module, then there exists a primary decomposition of U in V, which is irredundant.

So in fact what do we do is, so if U is intersection Q_i this is a primary decomposition, primary decomposition, Q_i is P_i primary, then P_i 's are uniquely determined, and they are, they belong to, and they are precisely elements P in the associated prime ideals of $\frac{V}{U}$,

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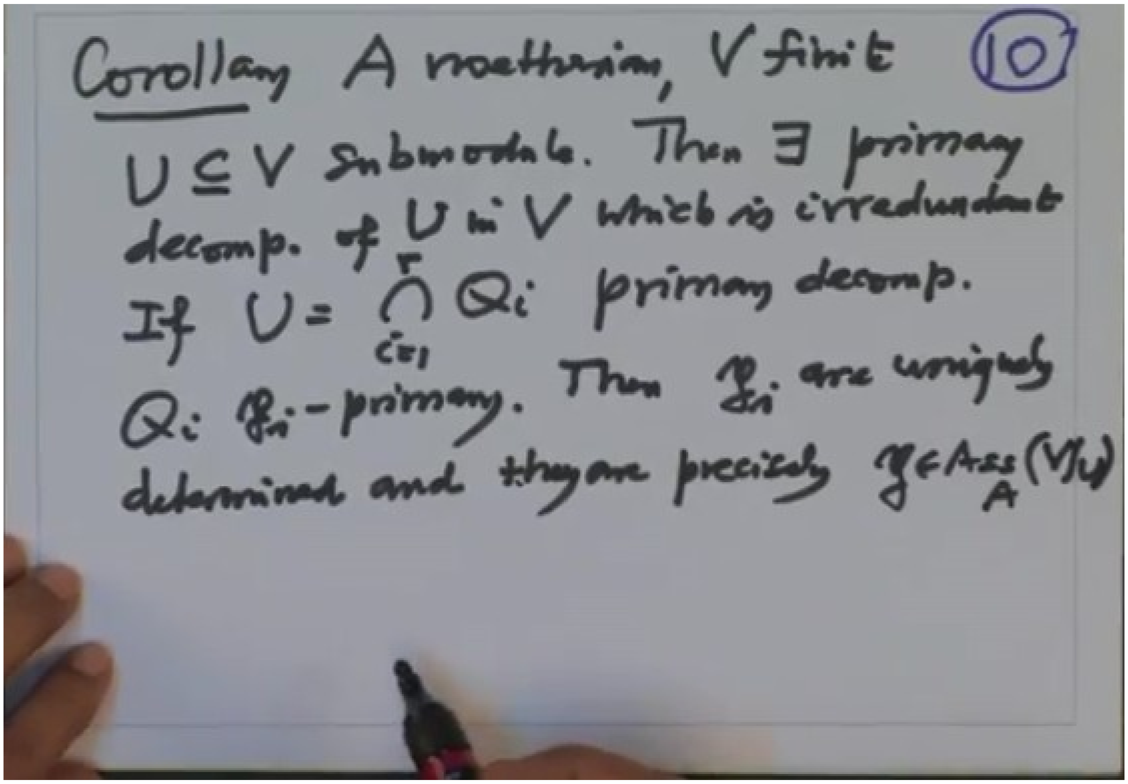

so that proves the irredundant primary decomposition of a sub-module exists in a finite module and maybe in the next time I'll deduce the uniqueness, what no one mean by the uniqueness of the primary components, so we will do that in the next time. Thank you very much.

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