**INDIAN INSTITUTE OF TECHNOLOGY BOMBAY**

**IIT BOMBAY**

**NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)**

**CDEEP IIT BOMBAY**

# COMMUTATIVE ALGEBRA:

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## Lecture No. – 07
## Primary Decomposition

In the last lecture we saw that when ring is Noetherian, A Noetherian and V is a finite A module then to this V we have attached two subsets of the spectrum of A, one is associated primes of V, and the other is support of V, both these are subsets of in spec of A, and we have checked that this support is actually a closed set, this is V of annihilator of V, and also we have checked that both this have the same, we have also check that this is a finite set
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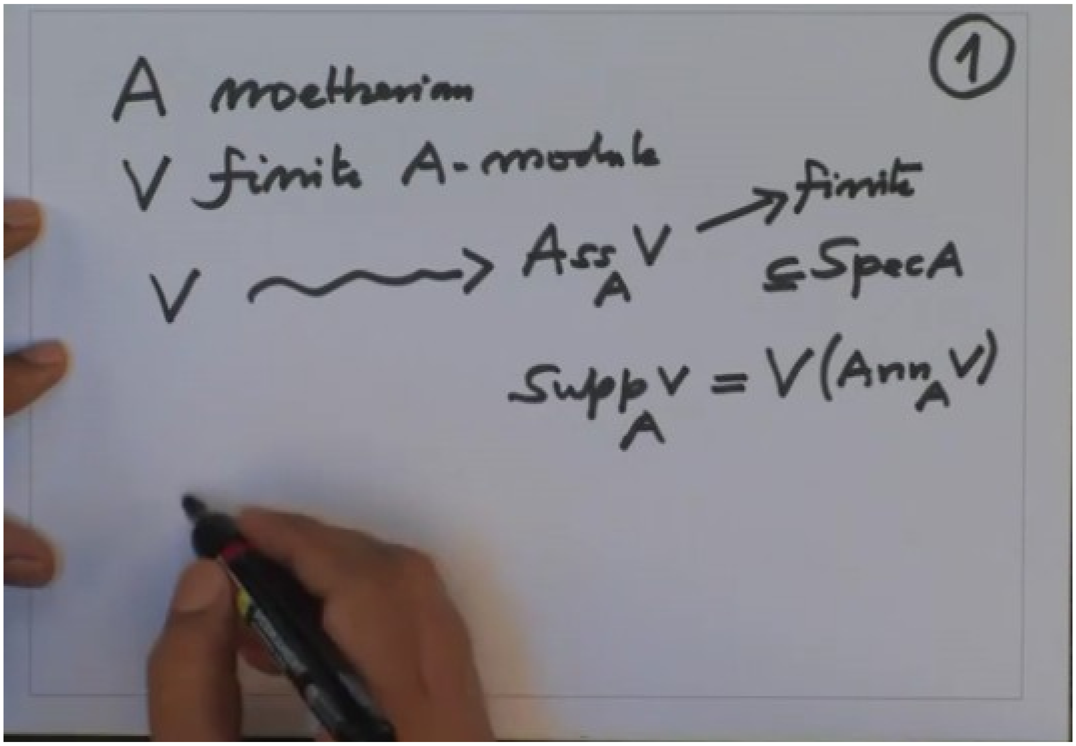


and we have checked that both this sets, subsets of spec A they have the same associated primes of V and support of V, have the same minimal elements with respect to the natural inclusion, alright.

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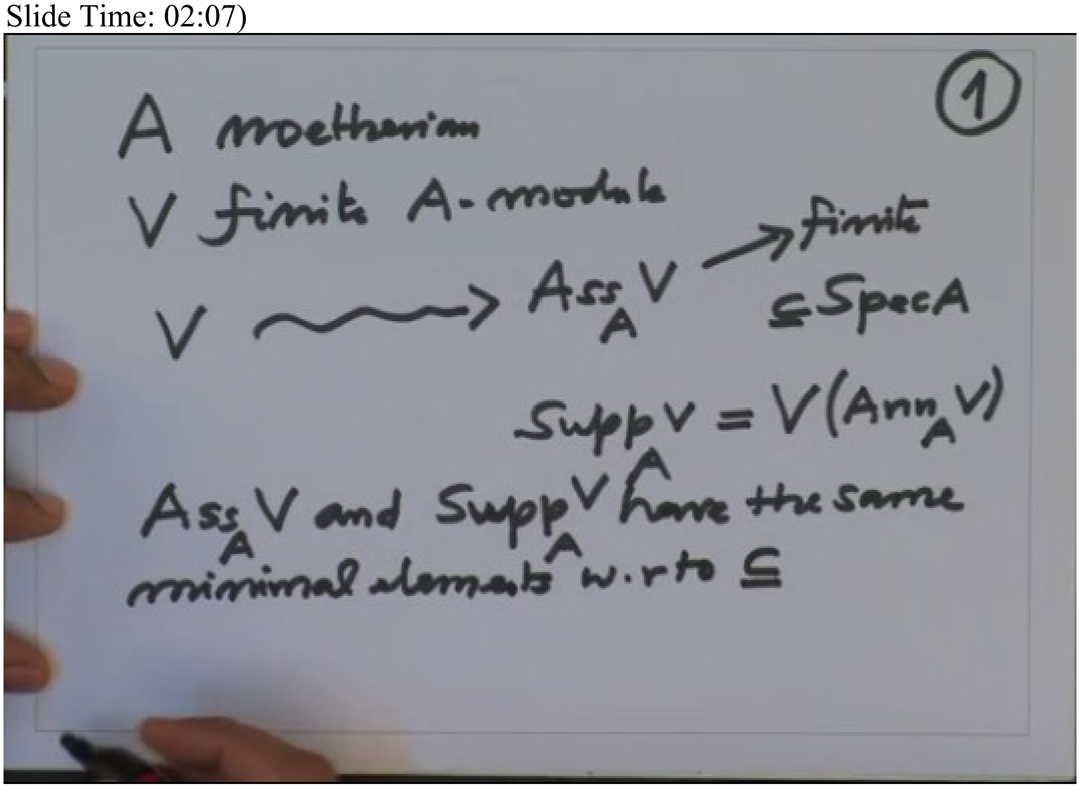

So one of the important consequence is the corollary I have to mention, this is applied to the ring so if A is Noetherian ring, A Noetherian ring then spec of A has finitely many minimal elements with respect to inclusion,

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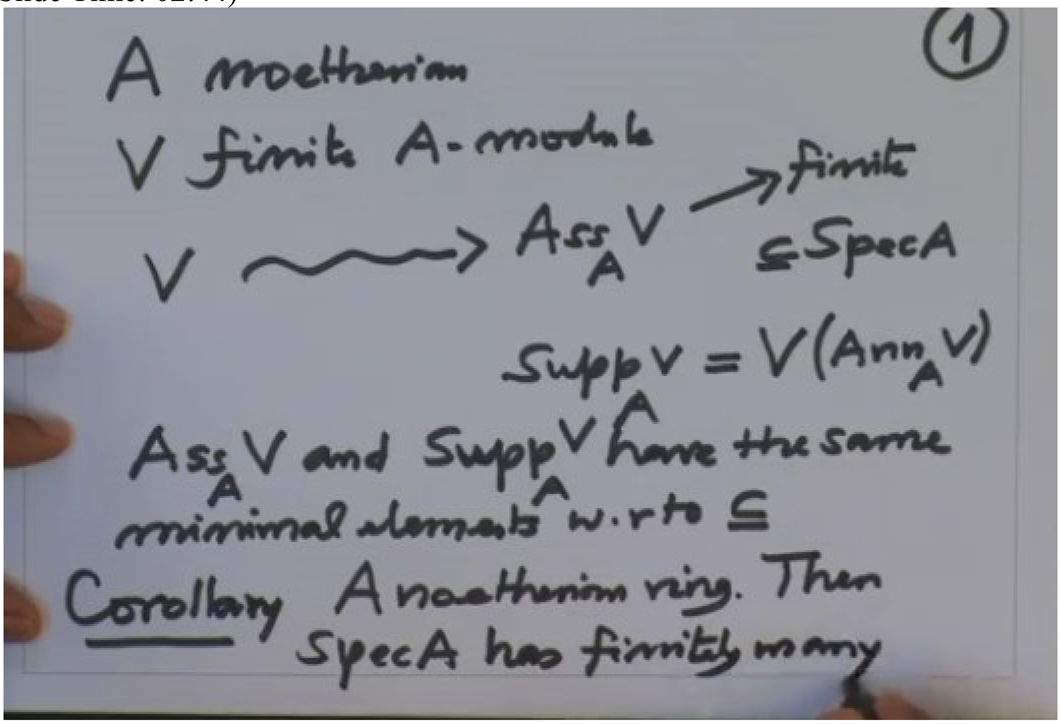


the natural inclusion, this is immediate from the fact that you apply the above observation to the module V = A, then in this case we know spec of A equal to the support of A as A module, and then we know the associated primes, this is a finite set and there minimal elements are same therefore spec A will have finitely many minimal elements, alright.

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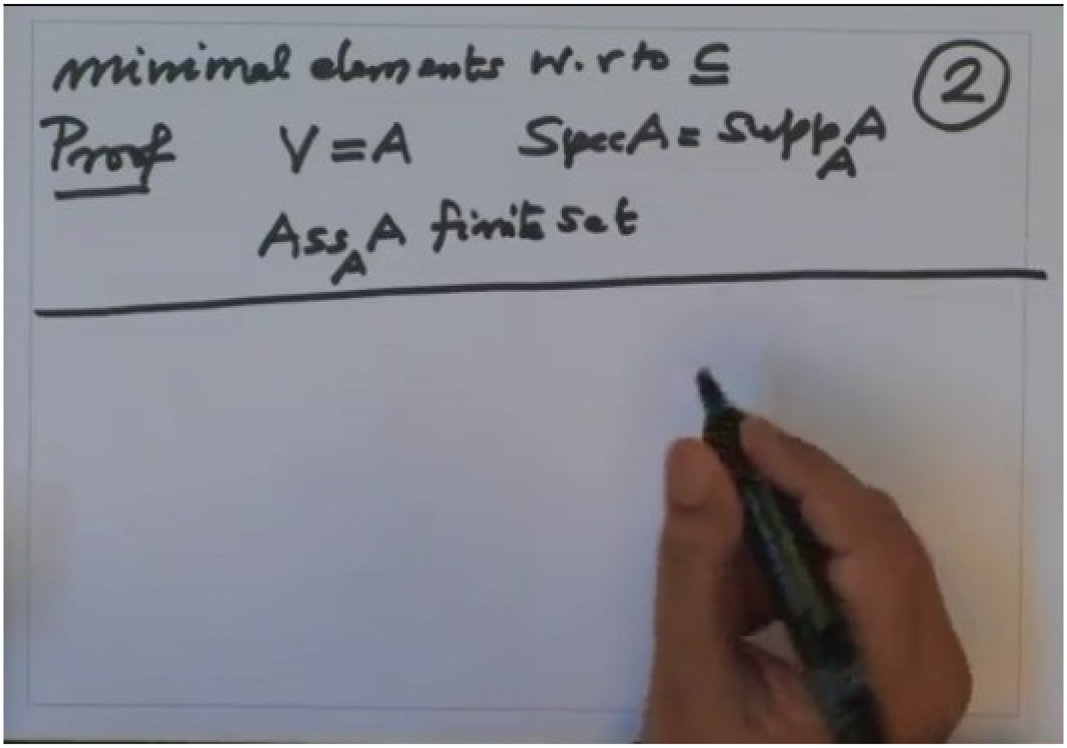


Now we want to do little bit further more detail study about associated primes and what is called primary decomposition. So now I want to study what is called primary decomposition, alright, so in this we will mostly assume the ring is Noetherian, so let us assume the A is always, I'll not keep saying but this is our blanket assumption now that A is Noetherian ring, and V is a finite A module, this generated by finitely many elements, alright. So observe that we have this, because V is A module, we have this ring homomorphism theta from the ring A to the endomorphisms of (V,+), that is a going to the homothecy, these are also, these $\theta_a$ is also called homothecy by a, so this is any element x sending it to ax.

So given ring homomorphism like that we can define a module structure on the Abelian group (V,+) conversely if you have a module that means we have this ring homomorphism, and this element this $\theta_a$ is called homothecy by a, alright so that is the notation,

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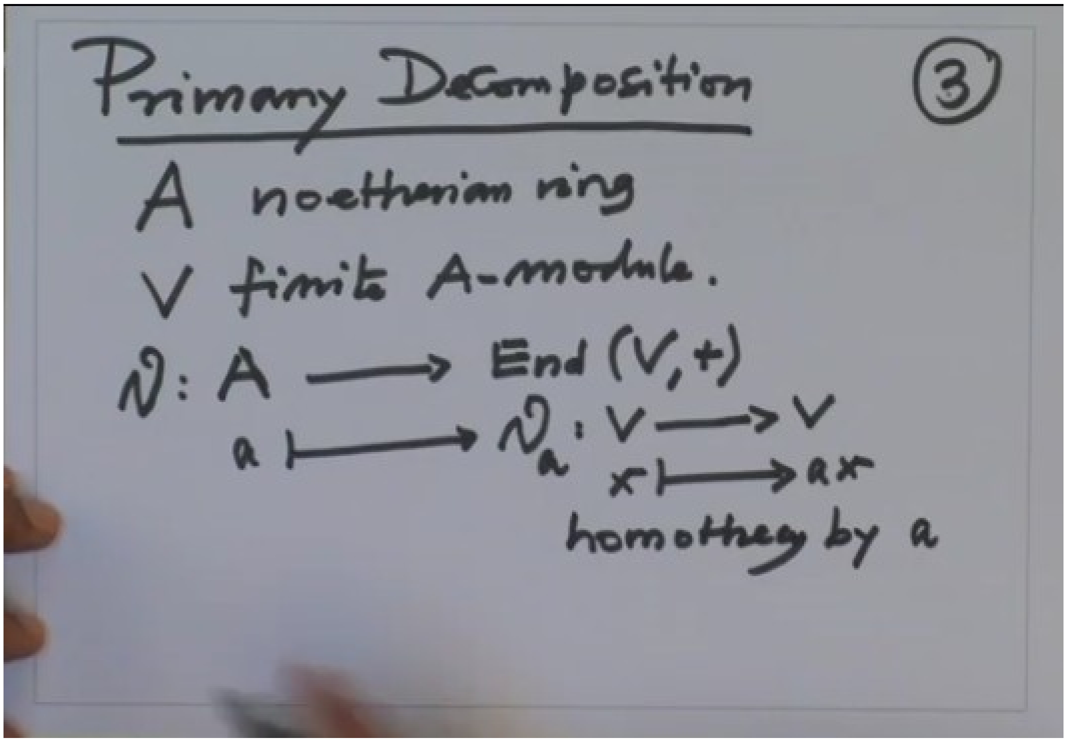


so note that homothecy is nilpotent, this $\theta_a$ is nilpotent that means composition of $\theta_a$ finitely many times it becomes 0 that is if and only if a belong to intersection of all P where P belonging to the associated primes of V,

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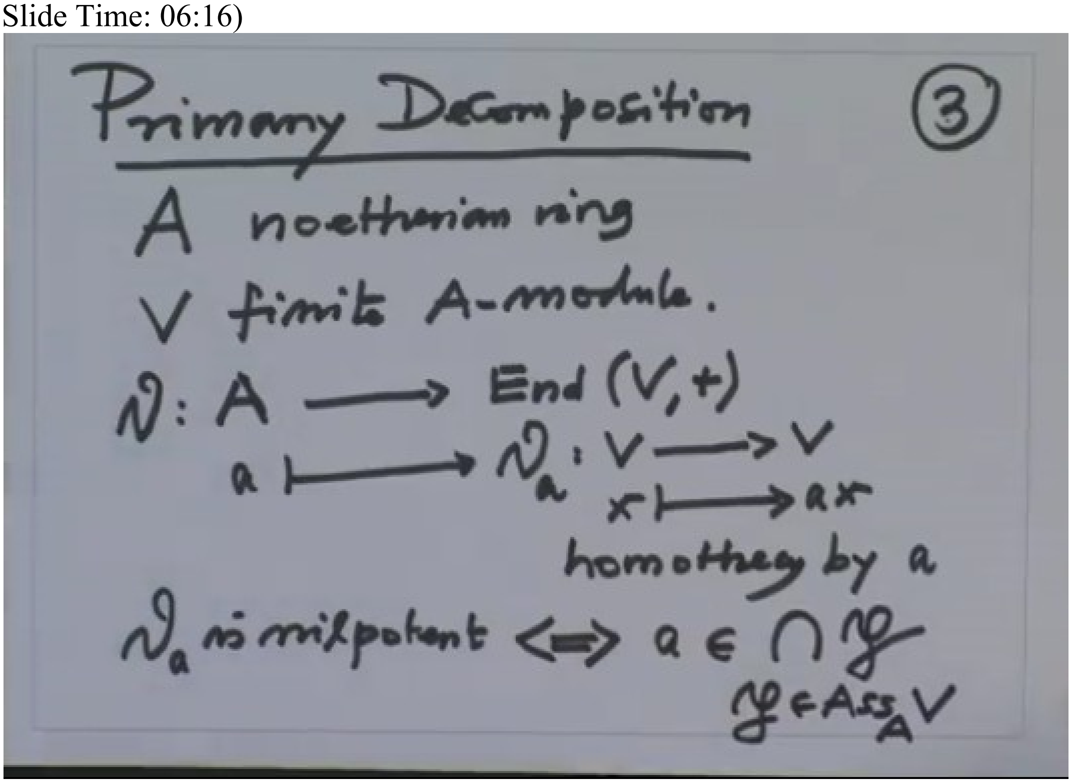


this is very easy because I’m going to use a fact that the associated primes of V and a support of V they have the same minimal elements, this is what I’m going to use, alright.

So and we know that this support of V is the V of the annihilator, and therefore we know the intersection of P, P varies in the associated primes also same as intersection P, P varies in the support, this is because they have the same minimal elements and this is nothing but the radical of this annihilator, so that means some element A belonging to the intersection, P belonging to the associated primes if and only if some power of A belonging to the radical of this ideal means, some power of that A belonging to the annihilator of V for some n, $n \in \mathbb{N}$ , but this is equivalent to saying the homothecy by A power N is 0, but that is equivalent to saying $\theta_{a^n}$ that is composition of $(\theta_a)^n$ times that is 0 that means it is nilpotent,

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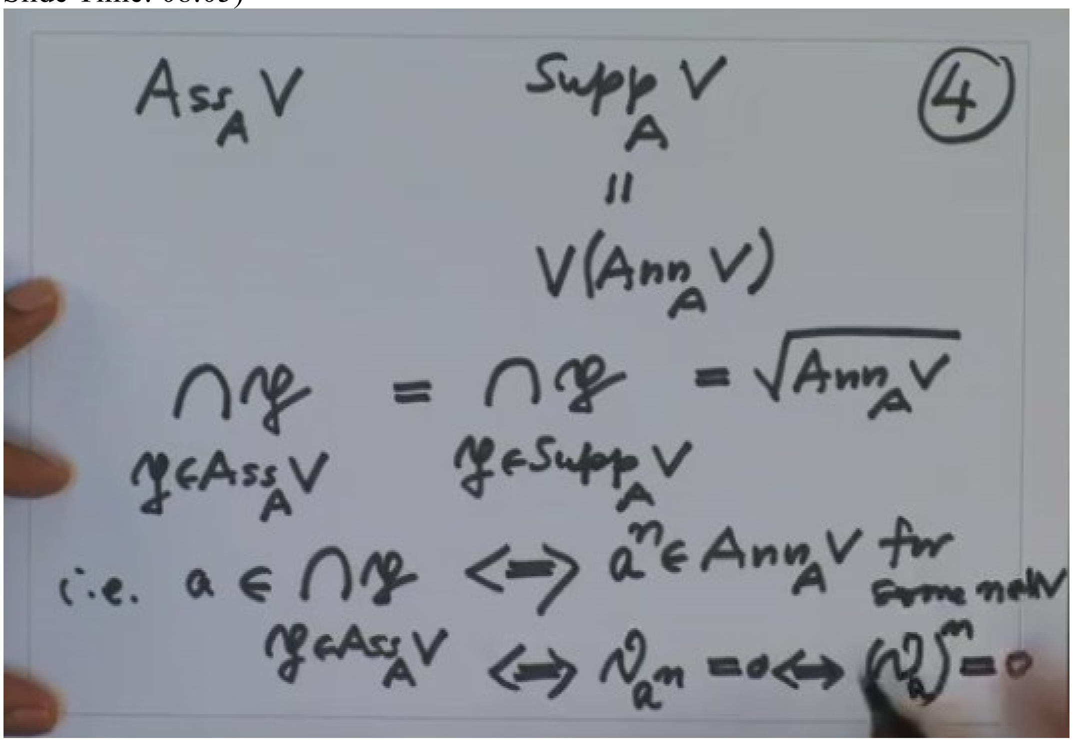


so we proved our assertion that homothecy is nilpotent if and only if a belongs to the intersection of all associated primes, alright.

When that the next lemma I want to analyze saying that, when will the associated primes be singleton? So as the same notation as above V is a finite A module, and then I want to know when so associated primes of V is singleton set, only one prime ideal belong to that, that is equivalent to saying for all a in A, homothecy is either injective or nilpotent. How do we prove the equivalence of this? So proof, okay this condition homothesis as a injective or nilpotent that is equivalent to saying, so let me write it RHS if and only if, what is that? Every element of A, so a we know when it is nilpotent by the earlier observation that is precisely the intersection of P, P running in associated primes of V that is nilpotent or it is injective, an injective means that it is not in the annihilator, so that is not in the, so this is injective means union, let me write it below here union A – union P, P varies in associated primes of V, that is precisely the definition of the associated primes.

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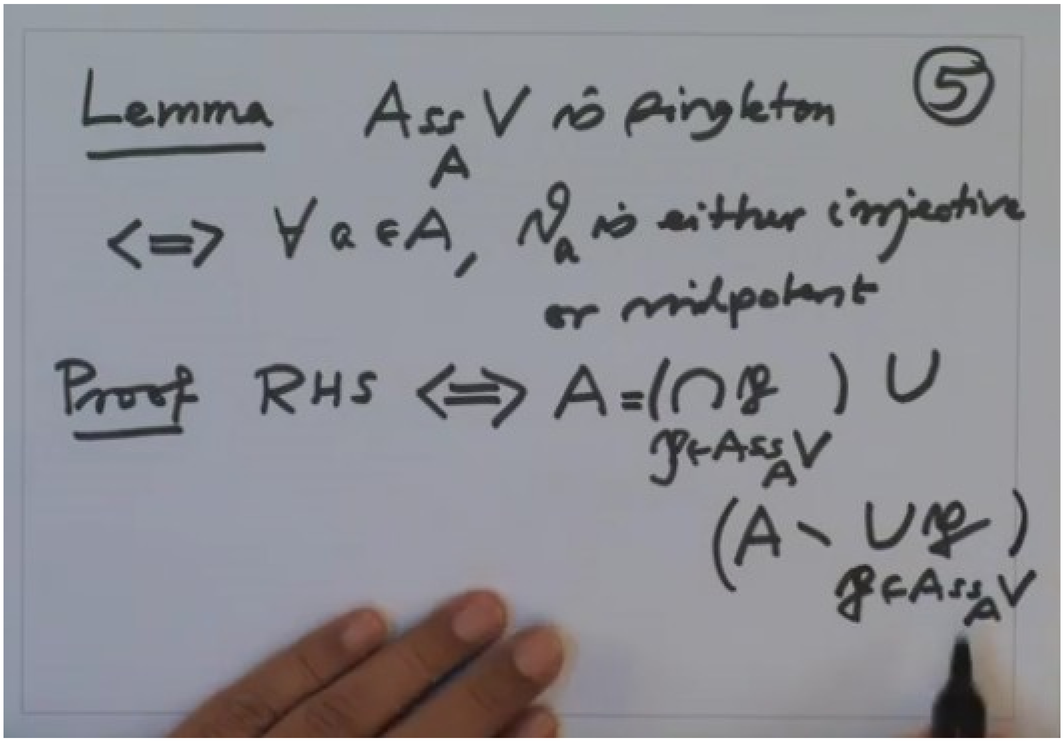


When will it not be injective? When some P, when it will be in the associated prime that is a zero divisors, so injective then it cannot be zero divisor that means no element, no nonzero element of V can relate that, that means it is a compliment of the union of the associate primes, so that is precisely the translation of this right hand side.

So this is if and only if, now this A equal to this, but that is equivalent to saying that intersection is P, P varies in the associated primes of V, this intersection is a complement of this union, this is the whole ring so this must be equal to the union of P, P is in associated primes of V, so intersection equal to union that is this, that is just a set theory, but that is precisely, when can it happen? This can happen only when, so that is if and only if the associated prime is only one is singleton, because if there were two element at least 2 or more then this side is smaller and that side will be bigger, so therefore it is singleton.

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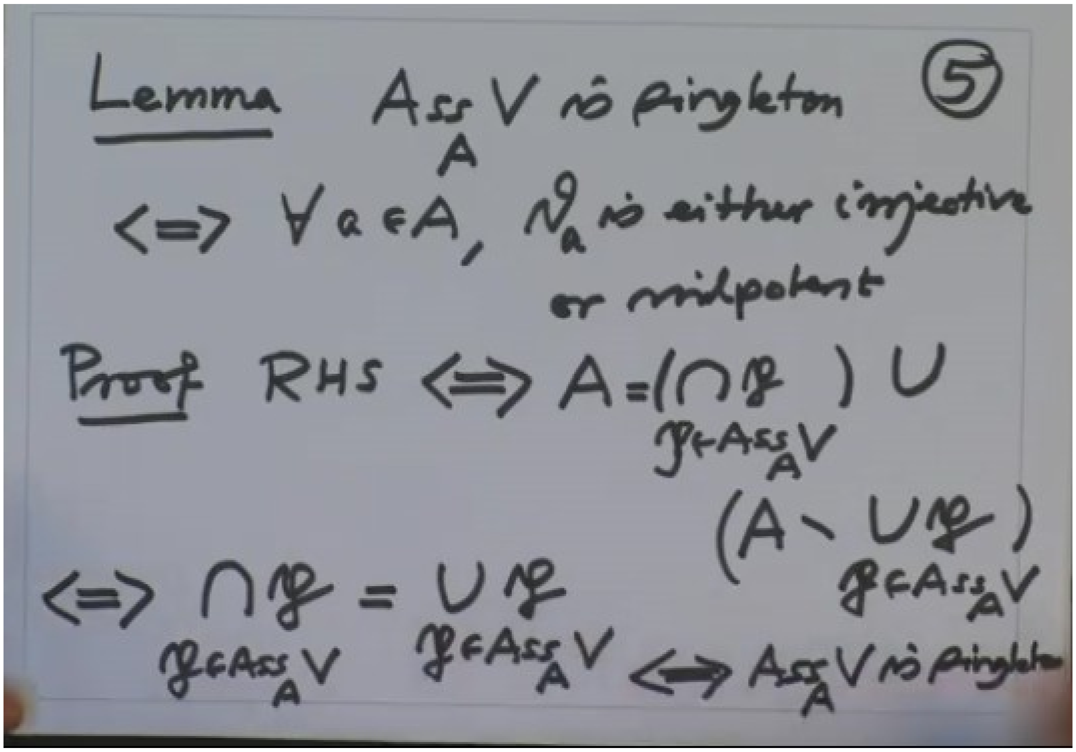


So corollary or moreover let me write it as a corollary, in this case if LHS or RHS or equivalent RHS is satisfied then in fact you can displace the associated primes, so associated primes of V that is singleton and that element, what is that prime ideal? That precisely the prime ideal all those elements A in A such that the homothecy is nilpotent, that is that prime ideal, so this is the only element in the associated primes of ring, alright, so this will allow us to define something, so let me define now, so definition, so A is a Noetherian ring, and E is finite A module, and U is a sub module of E, sub module then U is primary in E if the module V which is the residue class module $\frac{E}{U}$ satisfies equivalent conditions of the above lemma,

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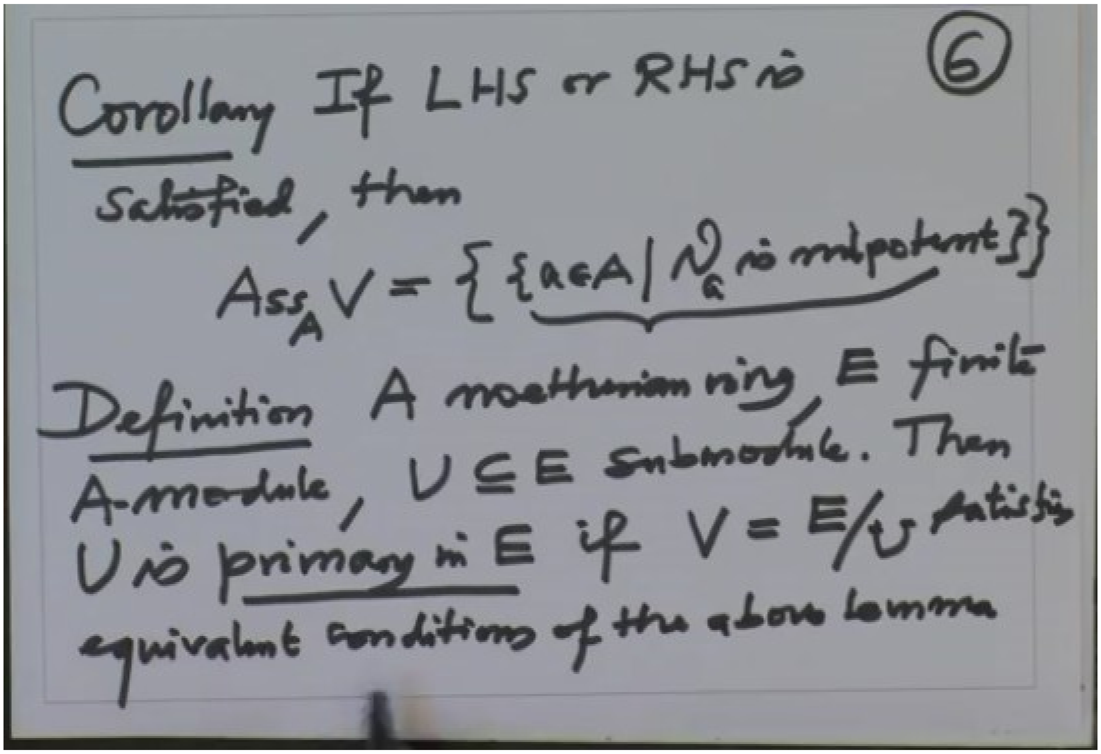


that is the associated primes of the module $\frac{E}{U}$ is only one, it's singleton P and in this case we will also say that U is P primary, that is more precisely.

In this case we say that U is P primary sub module of E, alright, let us see some examples that is, so note that if I take A Noetherian ring, A Noetherian then every prime ideal, every prime ideal P is a primary sub module of, is it P primary sub module of A, that means what?
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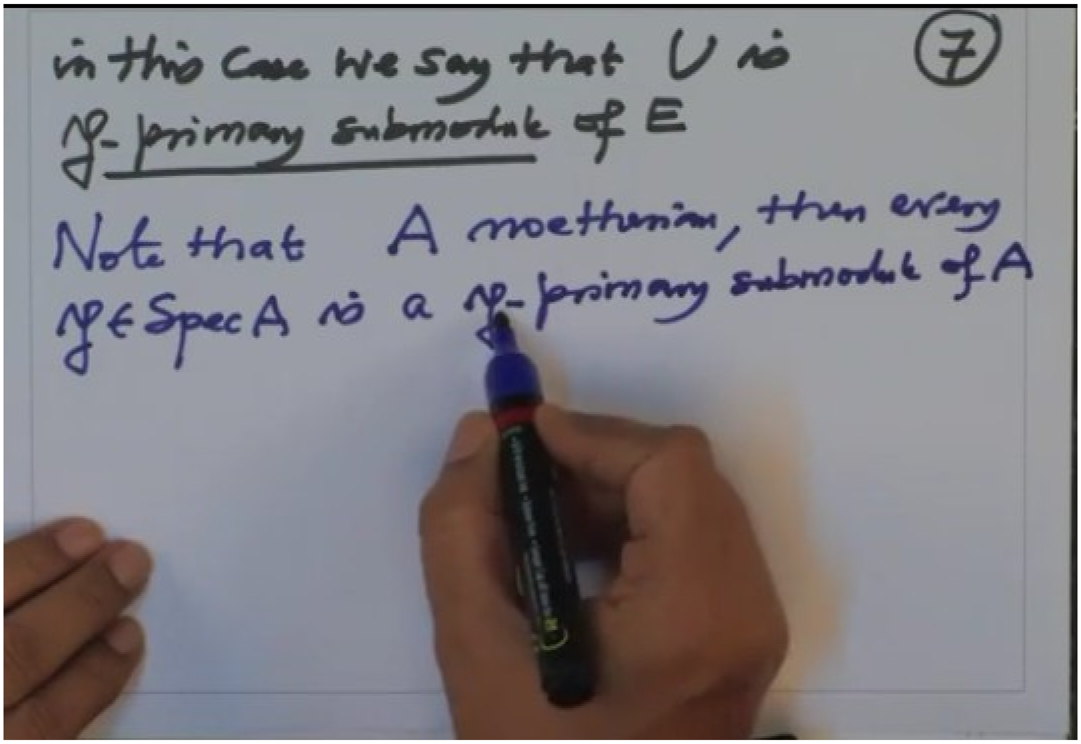


That means we have to go mod that P and check whether it has only one associated prime ideals, so that means we, so that is associated prime ideals as a module over A, $\frac{A}{P}$ just a singleton P, that is clear because the only minimal prime ideal here is P and other prime ideals they contain a nonzero element, nonzero divisors, this is an integral domain so they contain nonzero divisors so they cannot be associated primes, alright this is one.

More generally so further this is one, so this gives enough examples of a primary sub modules namely all prime ideals in a ring, there primary sub modules of this, but they are not all, so if Q is P primary ideal, now instead of saying sub module we will say ideal, ideal in A then obviously Q will be contained in P, and this Q will contain a power of P for some n, this is very easy and it follows from the following observation that if Q is P primary, Q is P primary if and only if the associated primes of $\frac{A}{Q}$ as A module this is only P

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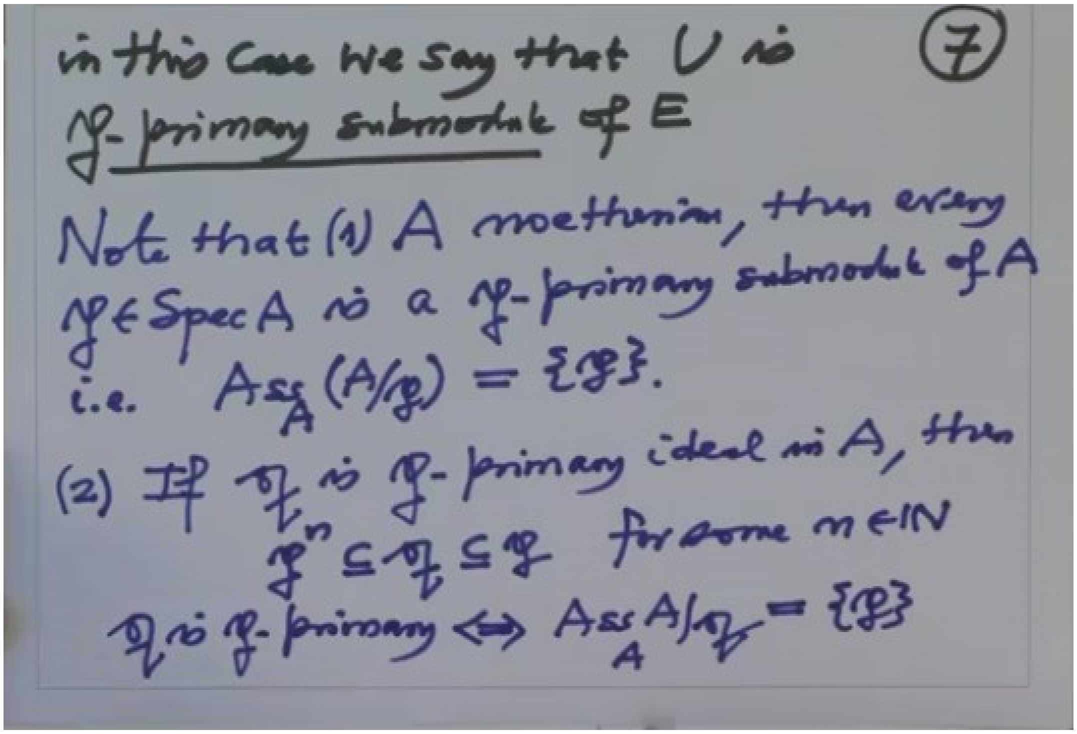


and therefore we know therefore the nilradical is only one prime ideal, so the nilradical is P, so therefore from here it follows easily that if I take the, look at the ring $\frac{A}{Q}$ it has only one minimal prime, so that nilradical of this ring is intersection of prime ideals which is and only one prime ideal is required, because this is the minimal one, so this is this and now if P the ring is Noetherian so therefore if P is generated by $a_1$ to $a_r$ then because this is the nilradical therefore the power of this, so $a_i^{n_i}$ will become 0 in the ring $\frac{A}{Q}$, but that will mean that $a_i^{n_i}$ belonging to the ideal Q for some $n_i$, this is true for every i from 1 to r and therefore from here it follows that if I take capital N to be the maximum of $n_1$ to $n_r$, it follows clearly that power of P, n-th power of P these all will be contained in Q, because when you take any element in P, write it as linear combination of $a_i$ to $a_r$ and raise it to the power n, maybe you will have to, okay,

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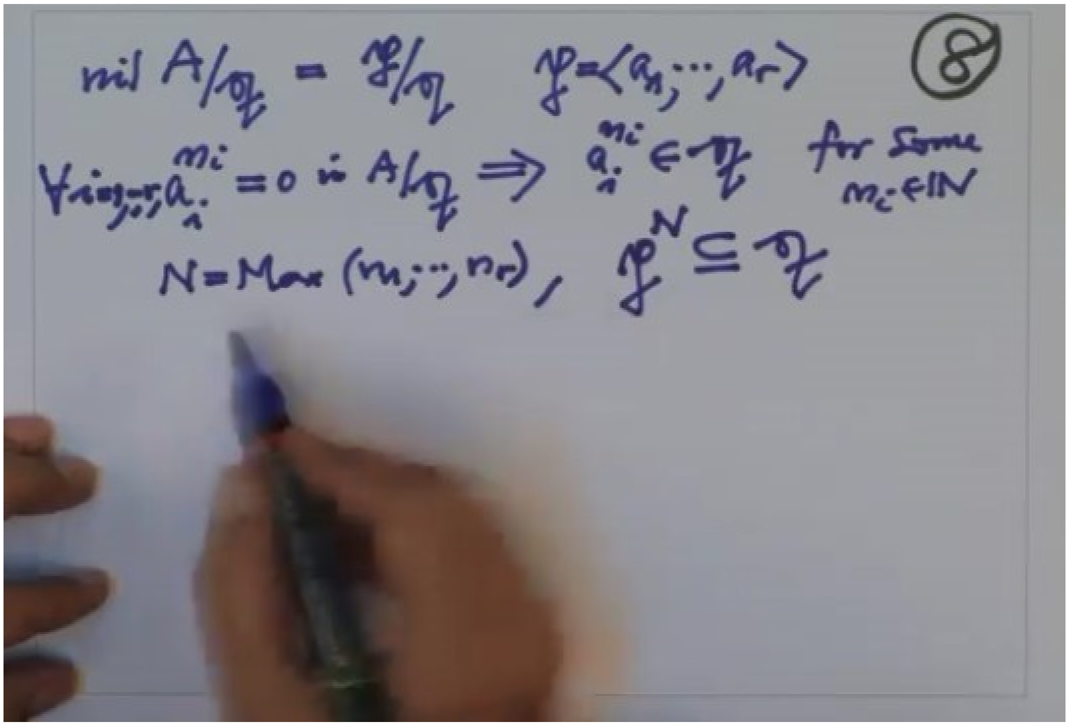


maximum of words so when you expand the polynomial expansion and then it will get all that, those elements will be in Q so that proves that, fact that primary ideals have this property.

But converse of this statement in general not true, converse is not true in general that means if ideal A, ideal Q is contained between P and the power of P then whether Q is P primary that is not always true, but it is certainly true when but true, but it is true if the prime ideal P is the maximal one, this is the set of all maximal ideals in A, so this is easy to check so I'll leave it for you to check this, alright.

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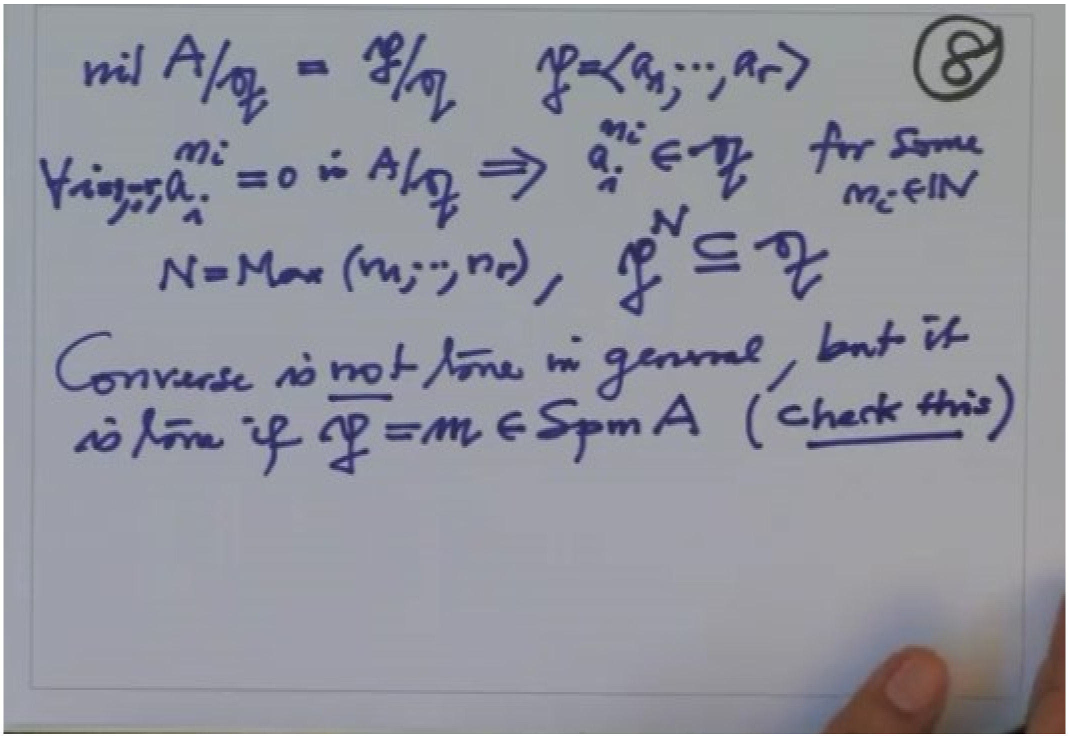


Now the next observation is a very important observation to prove what is called primary decomposition, and this is, so that means it's sort of given any subset, finite subset of the spectrum, can I find some module who is associated primes are precisely this given ones so that is theorem.

So theorem, so as usual A is Noetherian ring, A Noetherian and I have given now module, V is finite A module, finite may not be needed so I'll just write A module here, and we have given a subset capital phi this is a subset, this is not empty set, this is a subset of the associated primes of V, this is a subset then subset, then there exists a sub module W of V such that associated primes of W is precisely the associated primes of V minus this subset and obviously then the associated primes of A of $\frac{V}{W}$ will be the given phi, clear?

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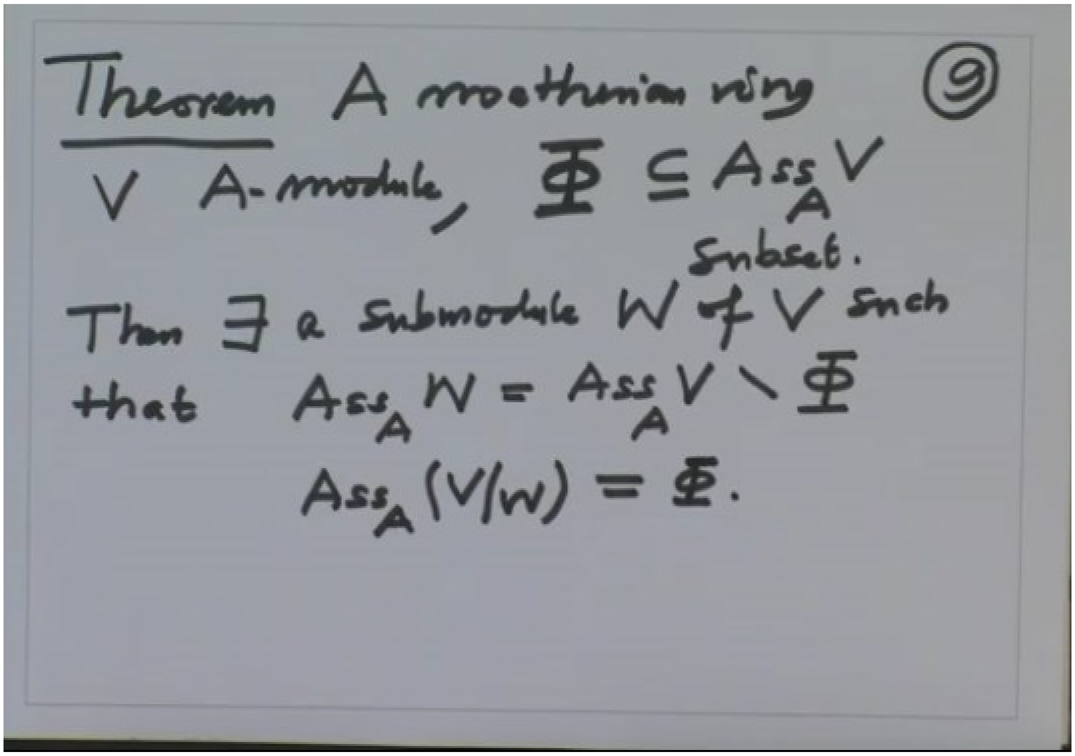


So we want to prove this, this is not so difficult, so proof so we have given a module and given a finite subset of the associated primes and we want to construct W and therefore we will construct $\frac{V}{W}$ and then we will check.

So now what do we do? You look at the family, so look at the family so consider the family, family of sub modules S, this is by definition all the sub modules, those sub modules P, P is a sub module of V, sub module such that the associated prime ideals of P of the A module P is contained in associated prime ideals of V minus that the given set phi, look at this family.
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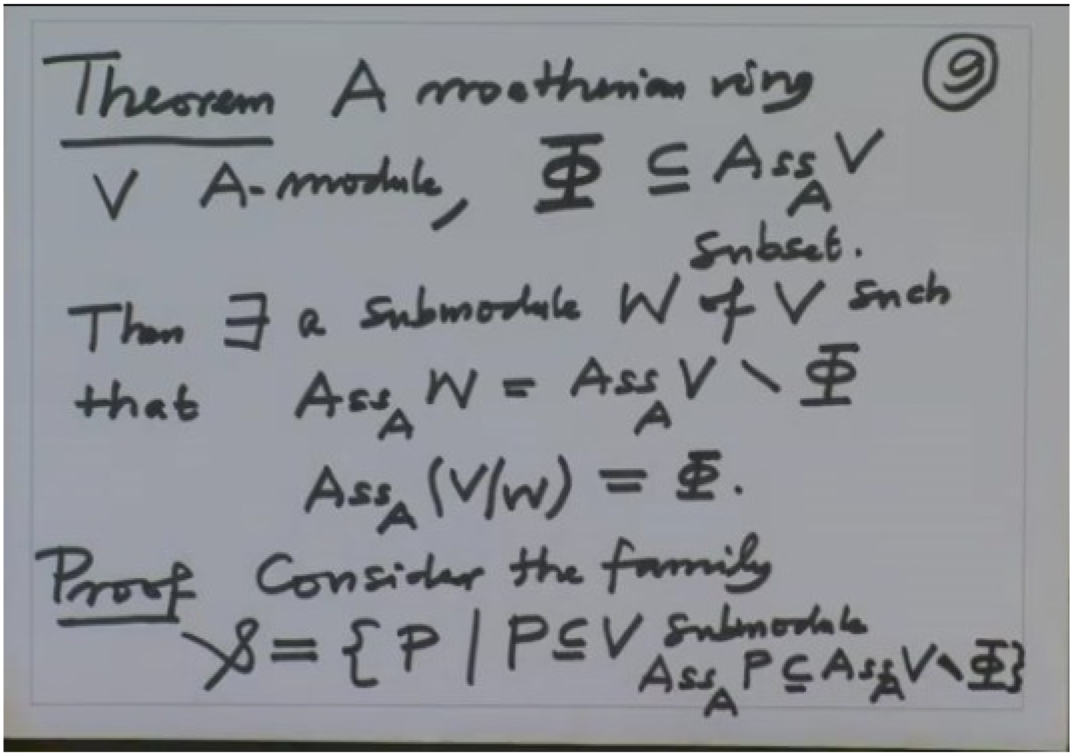


And I want to now look at the max, I want to choose a maximal element in this family with respect to the inclusion, so that means I need to apply Zorn's lemma to this family, so to apply Zorn’s lemma to the ordered set S with respect to the inclusion, I should check that I can apply that means I should check that this ordered set is inductively ordered that means I should check that given a chain in this, it has an upper bound so let C contained in S be a chain, chain means totally ordered subset of this ordered set that means any two elements of C are comparable, then look at $P_0$ , $P_0$ is by definition union of all the elements which are occurring in this C, so P, where P and C, see elements of C or elements of S which are sub modules with some property, so now look at this P naught, then obviously this P naught is a sub module of V that is clear because each P is a sub module, and this is a chain that means given any two elements here I can always find an element, both the elements contained in the same P and therefore I can add this, so it is a sub module is very easy, so for this we need C is a chain, alright.

Now I want to check that if P, so let P belonging to the associated prime ideals of P0, then I want to check that, so this means what? So that is P is annihilator of some element in P, P is annihilator of x, so x is in $P_0$ , and obviously x has to be nonzero, so take such a element P therefore this x will belong to P for some P, so that will imply, because this P0 is union so x will belong to some P with P belonging to C, and therefore this p, small p will belonging to the associated prime ideals of $V \setminus \phi$ , because any, so this means that P, this P actually belong to the associated prime ideals of P, but associated prime ideals of P is contained in this set, therefore P is contained there, alright, so we have checked that if P belonging to the associated prime ideals of P naught then P should belong to this,
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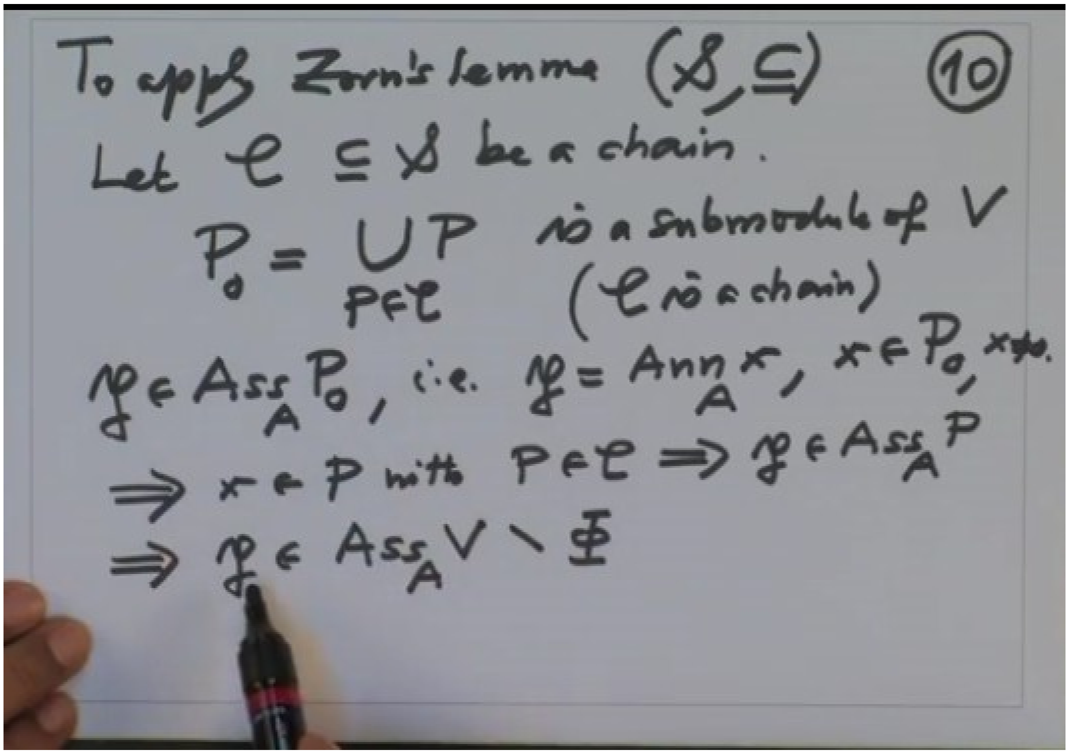


and also note that I forgot to check that this is an nonempty set that is clear because the set 0, the sub module 0, 0 sub module belong to S that is clear, because P is 0, the associated prime ideals of 0 is obviously contained here, and therefore 0 is an element in these, therefore this is a nonempty set and chain has an upper bound.

And now we want to check that this P naught if they required sub module, but that is clear because we have checked that this P, every P is contained here and therefore, so therefore what we need to check is the following, alright, so we have checked that, what do we have checked that is associated prime ideals of this union P naught is contained in associated prime ideals of $V \setminus \phi$, this is what we have checked.

So now we pick up a maximal element there, so let, we checked that this ordered set has a maximal element, so let capital W be in S, be a maximal element with respect to the inclusion, and then I want to check that, so we want to claim, claim that associated primes of A $\frac{V}{W}$ is actually contained in phi, this is what we want to check, actually we want to prove here equality but I'm checking this first, then what will happen? Once I check this, then associated primes of V which is contained in the associated primes of W union associated primes of $\frac{V}{W}$, this will be contained in, this is contained in associated primes of $V \setminus \phi$, that is how you have chosen that maximal elements, and this is contained in phi so this is union phi and therefore we will, so this will prove the theorem.

Alright, so we have to prove that this inclusion, this is what we need to prove, so let P be an associated prime of the residue class module, but that means so that is there exists an injective

module homomorphism from $\frac{A}{P}$ to $\frac{V}{W}$, because this P is annihilator of some elements so that will give you this inclusion,
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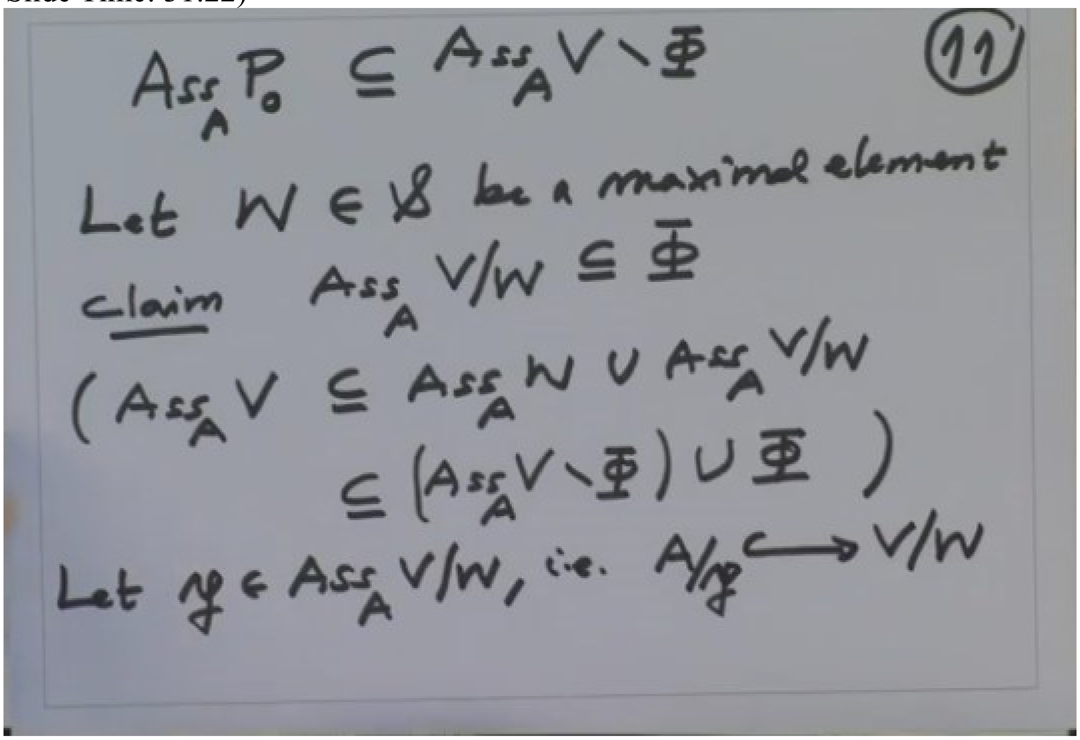


and let $n'$, so let us look at the, so this is a sub module of this, sub modules of these are precisely the residue class modules, so therefore look at $n'$ so this is $\phi$, then I'm looking at $V'$ sub module of V with the property that $\frac{A}{P}$ is actually isomorphic to $\frac{V'}{W}$ so this is a sub module which contains W also this and because now look at this quotient module, this isomorphic to this so therefore the associated primes of $\frac{V'}{W}$ this is only singleton P because that is same as associated primes of this,
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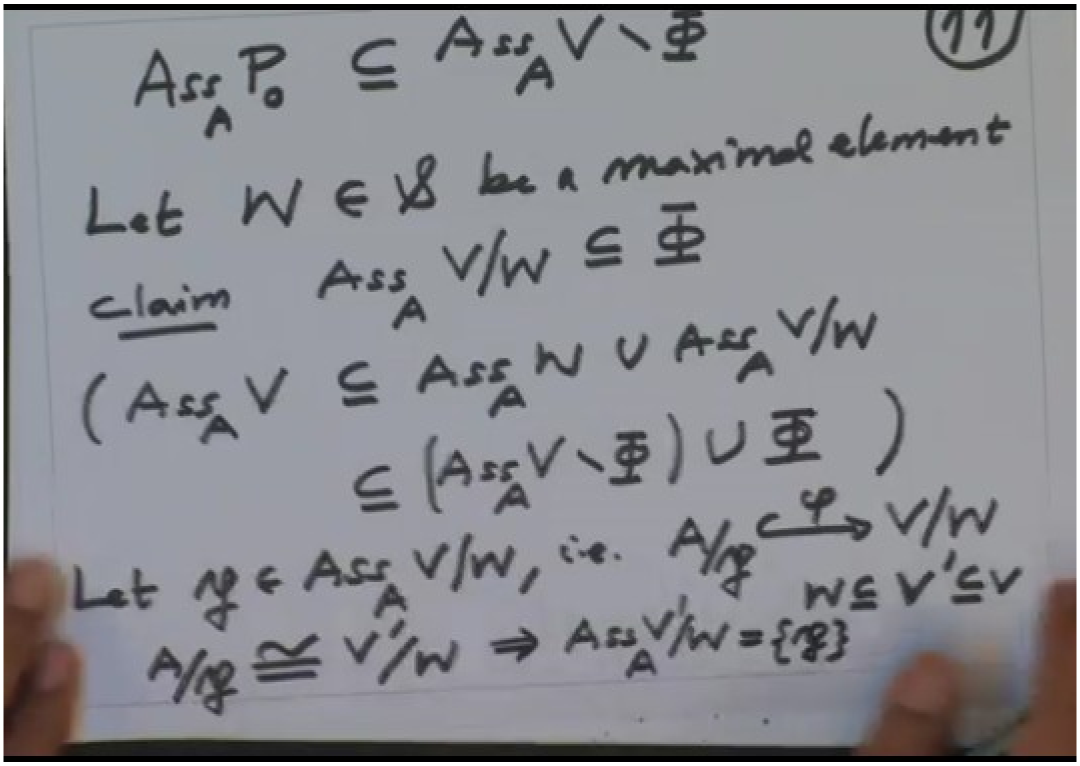


and singleton P therefore associated primes of V prime which is contained in associated primes of $W \cup P$ this is again we have applied the observations what we saw it earlier and therefore so this will prove that associated primes of V prime is contained in associated primes of V minus that given set phi, this will happen when, so see when W is not maximal in S, so therefore it follows that P belonging to associated prime ideals of V prime will be contained in associated prime ideals of V, and therefore P will belong to phi, so that proves the theorem and this is very,

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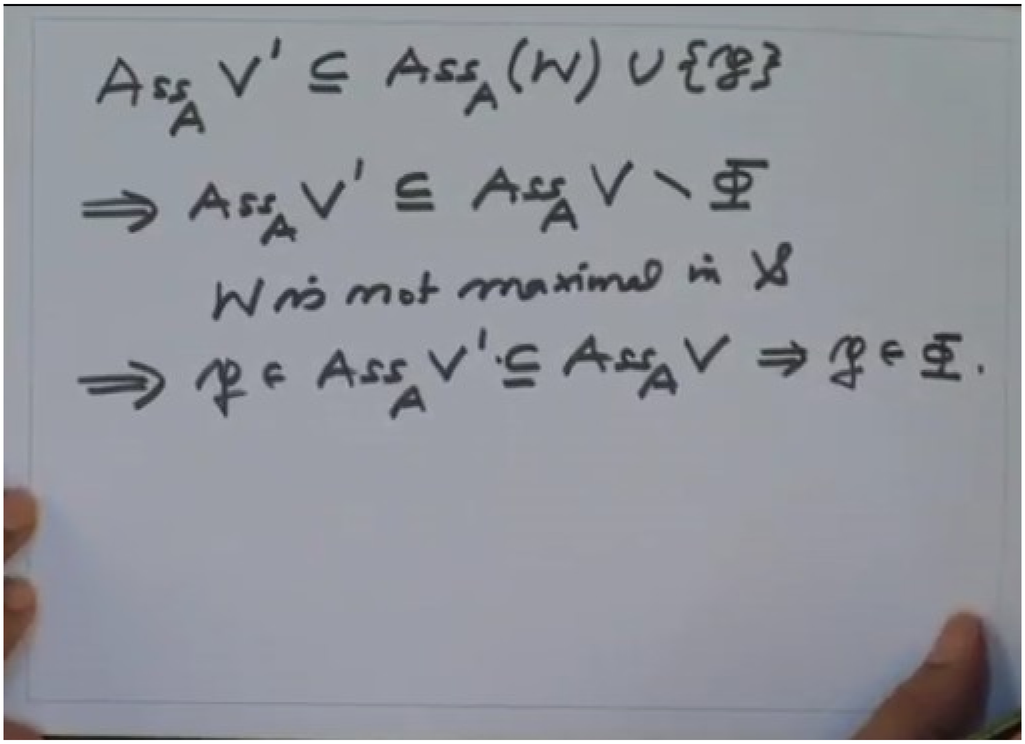


very important theorem and we will continue this to prove that every sub module of a module has a primary decomposition, so I will define after the break I will define what is the primary decomposition and then we will prove that every sub module has a primary decomposition.

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