INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

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CDEEP IIT BOMBAY

COMMUTATIVE ALGEBRA:

PROF. DILIP P. PATEL
DEPARTMENT OF MATHEMATICS,
IISc Bangalore

Lecture No. – 06
Support of a Module

We have now defined associated prime ideals of a module over commutative ring, and we have proved that when the ring is Noetherian and the module is finitely generated, then the set of associated prime ideals is a finite set.

Now one little observation about the associated prime ideals this is very useful for computation, so this theorem says if A is Noetherian, and V is finite A module, if S is a multiplicatively closed set in A, multiplicatively set is A also we assume in general 1 belong to S, so that means H is actually submonoid of the multiplicative monoid of A, then if I look at the module $S^{-1}V$ this is a module over the ring $S^{-1}A$, and if I want to know what are the associated prime ideals of this module as a module over $S^{-1}A$ this is precisely all $S^{-1}P$ where P is running through the associated prime ideals of A and prime ideals of V who don't intersect with S, P intersection S is empty.
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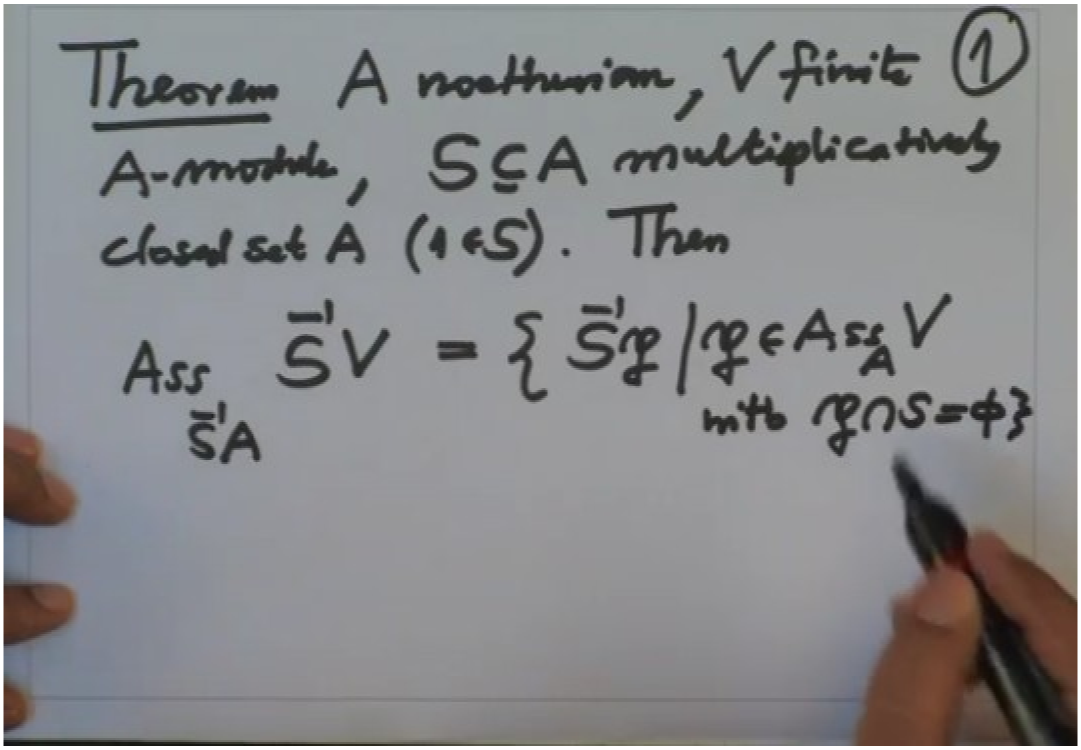

So I'm going to leave with the details of the proof as an exercise this is very easy because all that we need to check that, so for example I want to check that this inclusion, this inclusion I want to check, this inclusion, suppose I want to check this, that means what? I have to take associated prime ideals, so to check this so let P belonging to associated prime ideals of V and P don't intersect with S then I want to prove that then to prove $S^{-1}P$, first of all $S^{-1}P$ is now a proper ideal of $S^{-1}A$, because P doesn't intersect with S and this prime ideal $S^{-1}P$ this actually is associated prime ideals of $S^{-1}V$ as $S^{-1}A$ module,

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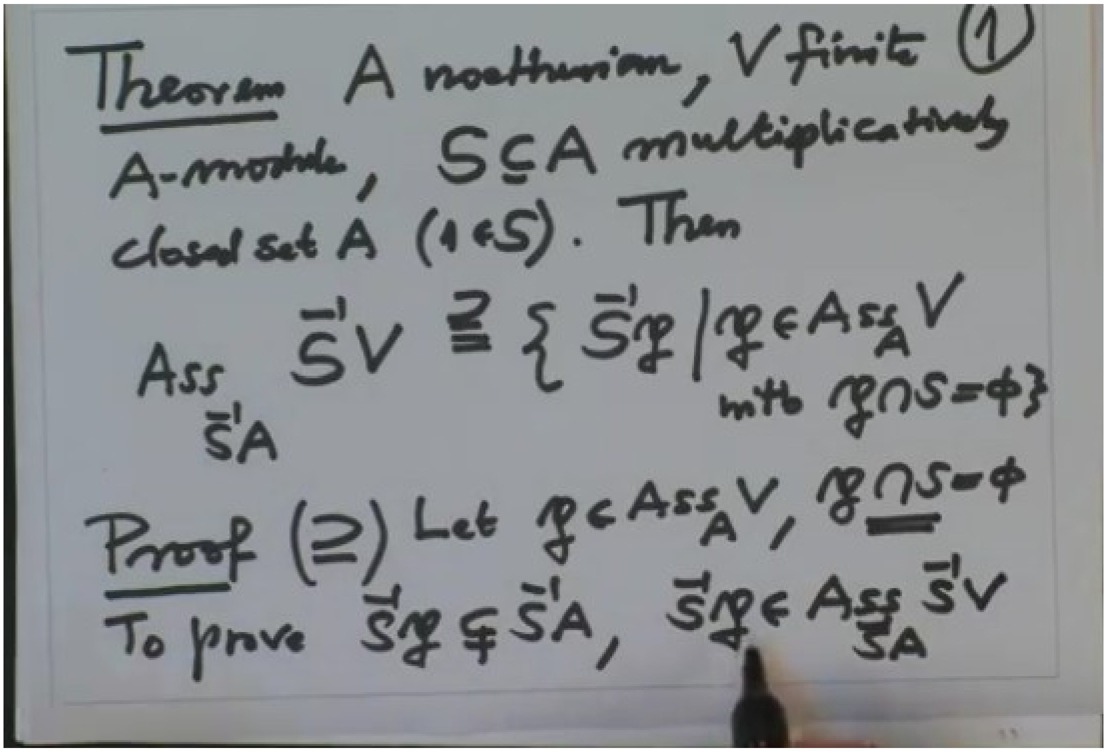

to check this I have to produce an element in $S^{-1}V$ so that it is a non-annihilator is precisely $S^{-1}P$.

Originally that P was associated therefore P is annihilator of x for some $x \in V$, x nonzero, and now the candidate we should look for is annihilator of $S^{-1}A$, as $S^{-1}A$ module of the element $\frac{x}{1}$, I should prove that this is precisely equal to $S^{-1}P$, if you prove that then we are done, but this is very easy because you have to prove that any element here annihilates this and conversely any annihilator of $\frac{x}{1}$ is contained here, so to see this first of all, so take any element here which is of the form A/S and multiply this by $\frac{x}{1}$, this is $\frac{Ax}{S}$ but this is $\frac{0}{1}$ which is 0 in the ring $S^{-1}A$, so where this $\frac{a}{s}$ is an element in $S^{-1}P$ that means a is in P and s is in S, therefore it proves this inclusion.

Conversely we know that P is a prime ideal, so therefore if $\frac{a}{s}$ is an element in $S^{-1}A$ with $\frac{a}{s}$ times $\frac{x}{1}$, this means $\frac{0}{1}$ then this means, so that means there exists a $t \in S$ with when I cross multiply, so t times ax is 0, but t is in S therefore t cannot be in P, because P and S don't intersect and this is, 0 is always in P so therefore and t is not in P therefore we conclude that, so therefore we conclude that ax is 0, therefore a belong to annihilator of x which was P, and therefore this belong to,
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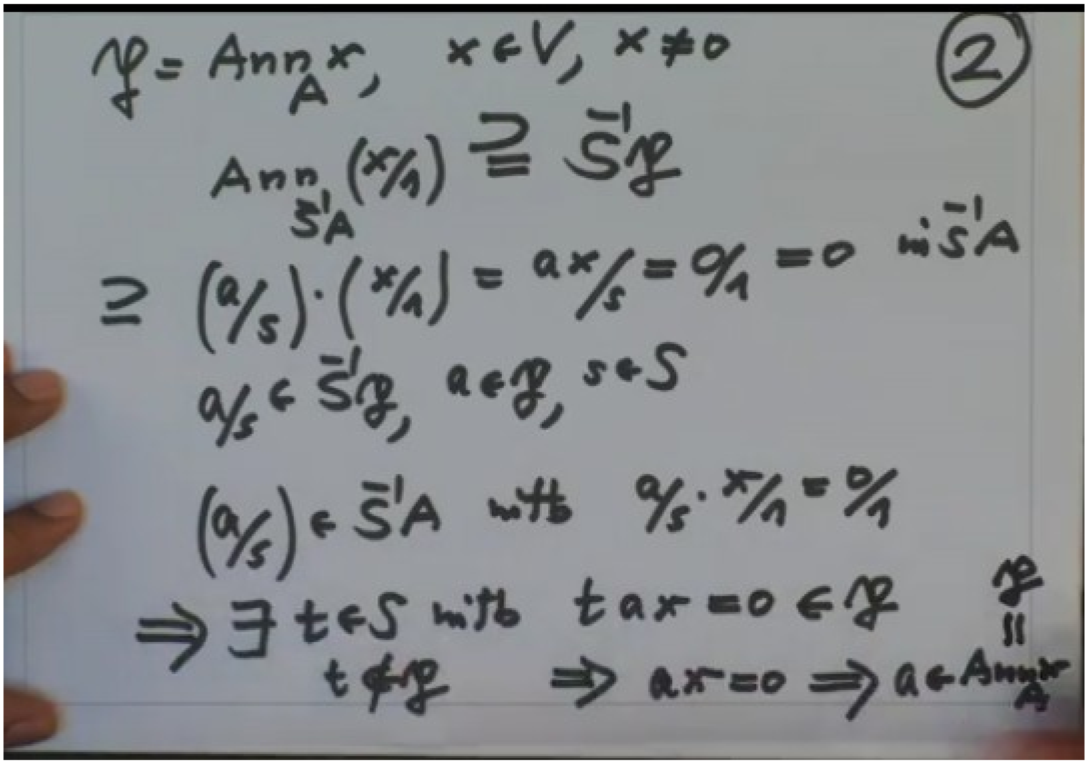

therefore this element $\frac{a}{s}$ belong to $S^{-1}P$ so that proves this inclusion. So therefore we have proved the one inclusion in the theorem and I will leave the other inclusion as an exercise, so the other inclusion this inclusion we have left it as an exercise, so it explains when you pass on to the localization what happened to the set up associated prime ideals, so among them those who intersect with the given multiplicative set they disappear, they become unit ideal and the remaining ones are still associated to this.

Now we come to support of a module, support of a module, so support of a module and set of associated prime ideals of a module are very closely related, they are not exactly same but they are very closely related, and what is the closed relation? That we will find out now, so first of all let us define definition, so V is an A module then support of V this is by definition all those prime ideals P such that when a localize V at P this module is nonzero, this is a subset of the spectrum. And note that we have this is a topological space with Zariski topology, so where we have given,

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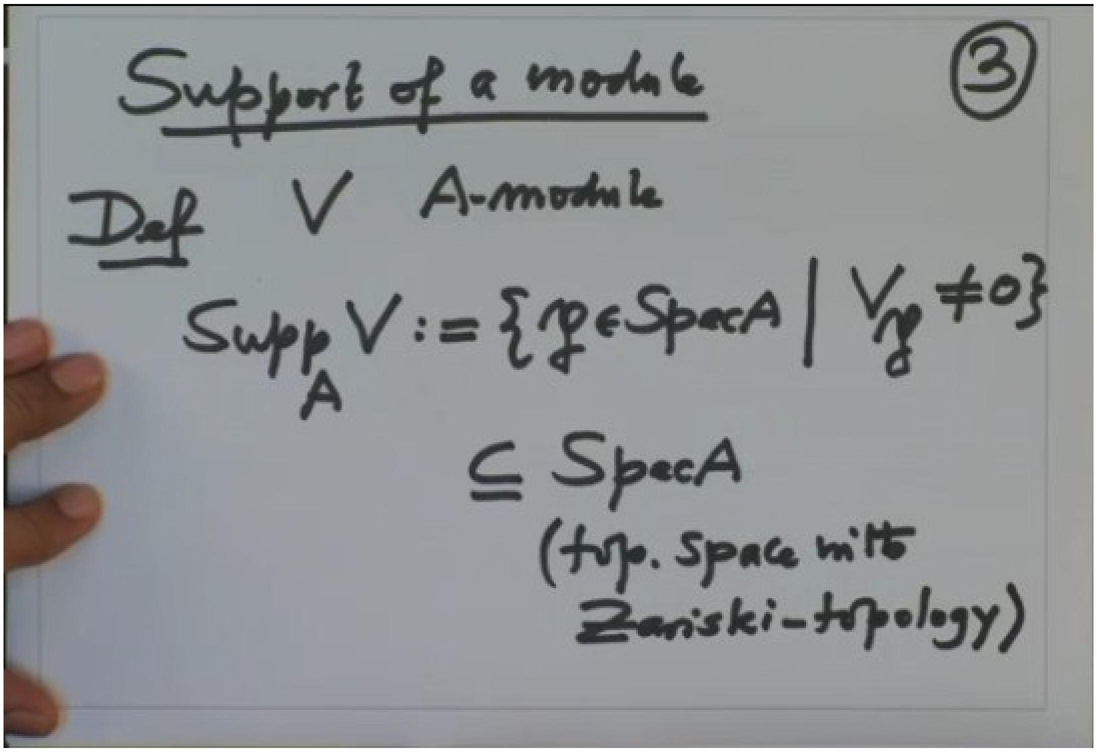

declared whatever the closed sets in this they are precisely the V of some ideals, so anyway so now this is a closed, this is subset here in a topological space so it is natural to us, what kind of a subset is it? Is it open, is it closed and so on, we will prove that this is a closed set.

And how do we find the, how do we find it? How do you find by using V that is a task, so note that if I take V = A then what is the support of A as A module? This is precisely all those prime ideals, all prime ideals because when I take any prime ideal, any prime ideal in A and if I localize these ring is nonzero because this is in fact, this is a local ring A_P is local ring with unique maximal ideal PA_P, this is an PA_P the maximal ideal therefore it is not the unit ideal, therefore this is properly contained there, therefore this ring cannot be zero.
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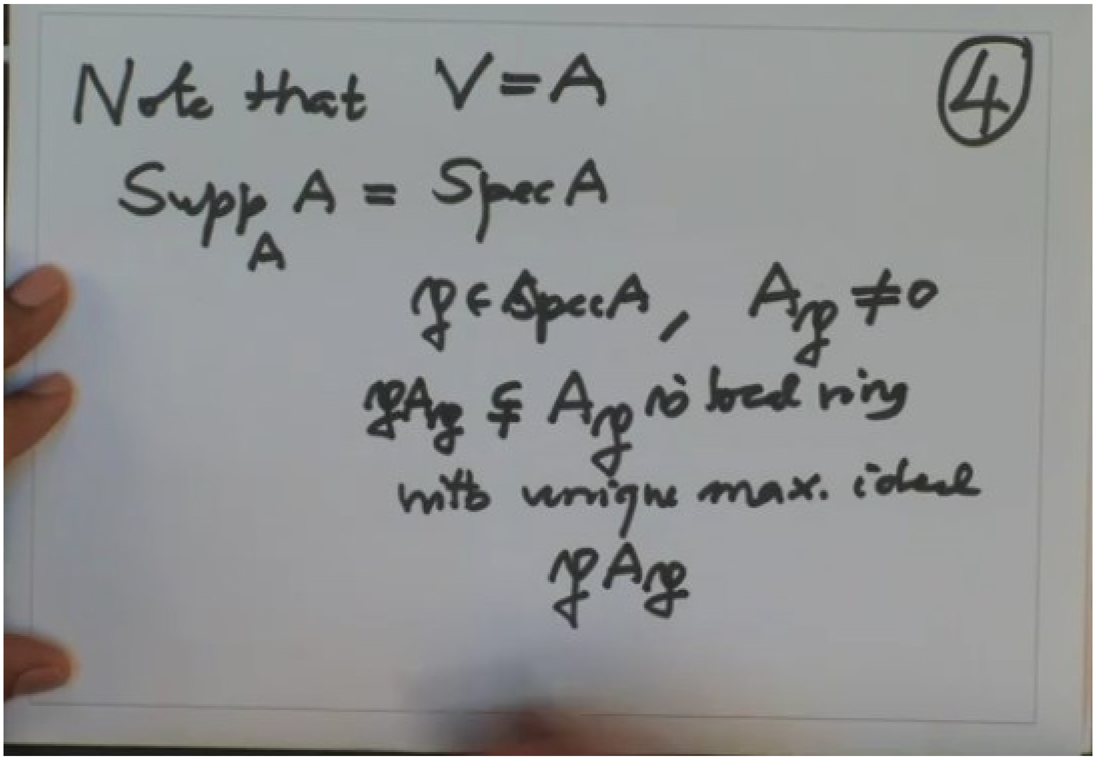

Note that here when I write this notation A localize at P this means we are taking the compliment of P, because P is a prime ideal this compliment of P is a multiplicatively closed set and one belong there, so S^{-1} of that is precisely this, that is our notation so therefore when the module is equal to the ring then the support of A as A module is the whole spectrum and therefore it is closed,

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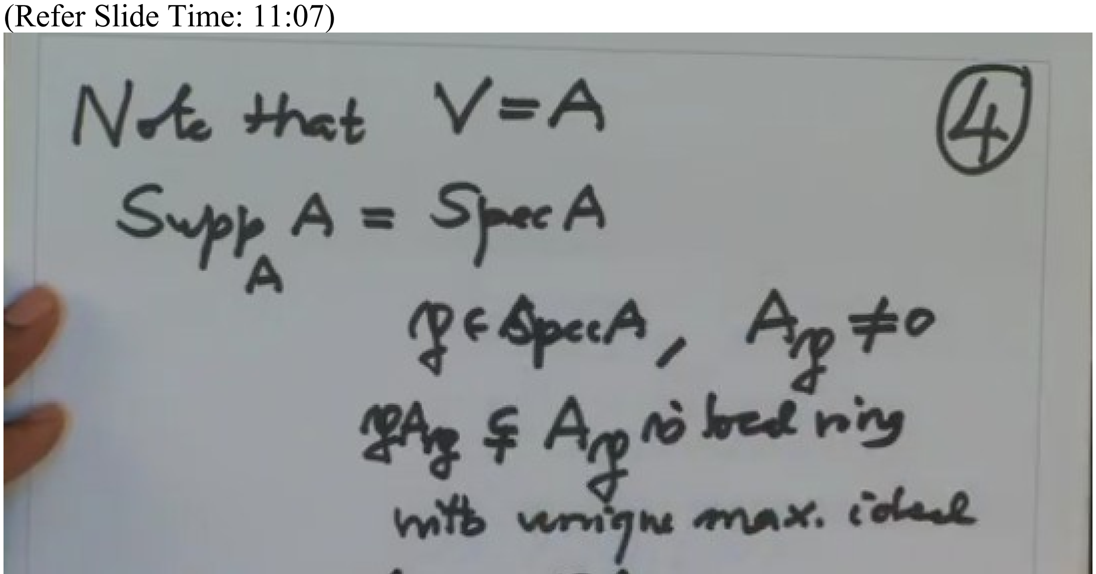

so in general that may not be the whole thing but it will be closed set, that is what we want to prove it, so in fact so the theorem we want to prove is the following.

Theorem, so let V be a finite A module, we are not assuming A is Noetherian, then support of V is precisely equal to V of annihilator of V, now this is little bit inconvenient here, but this V is by definition, this is recall the notation,

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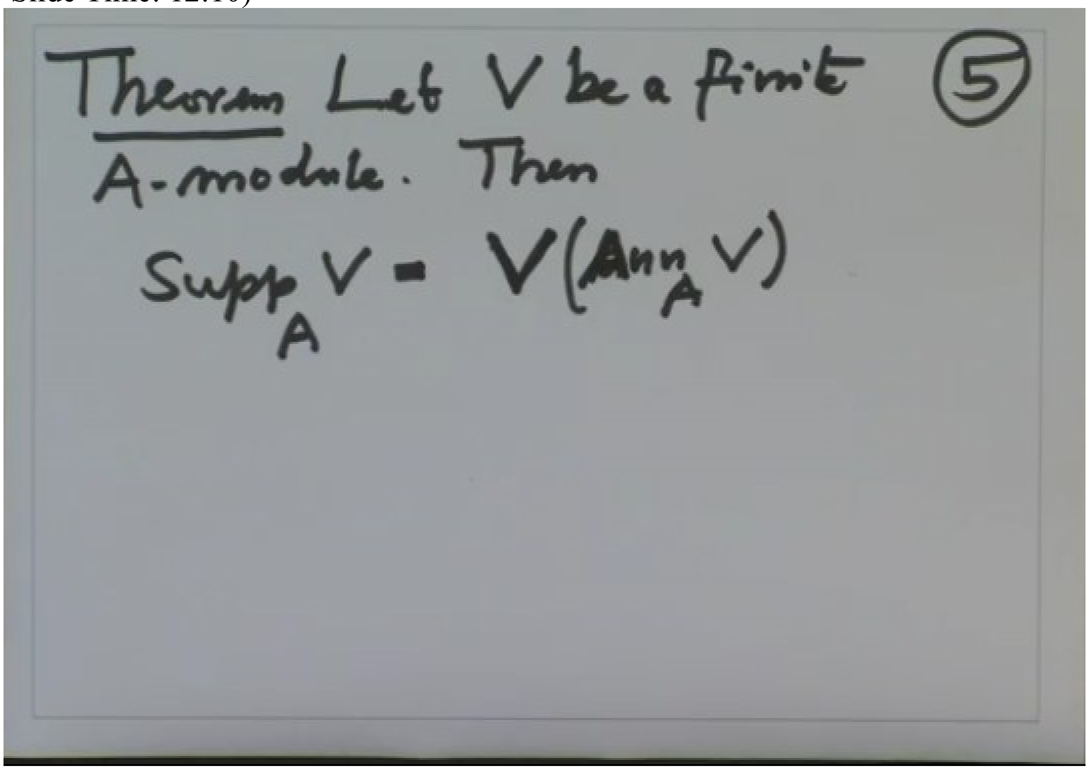

I'll recall so before we prove this we should be clear about the notation, so recall that for an ideal A in the ring A we put V(a) this is by definition, all those prime ideals in the ring which contains that given ideal A, this is a subset of the spectrum, and we know there is Zariski topology, the closed sets are precisely this collection V of an ideal where a varies in ideal A,

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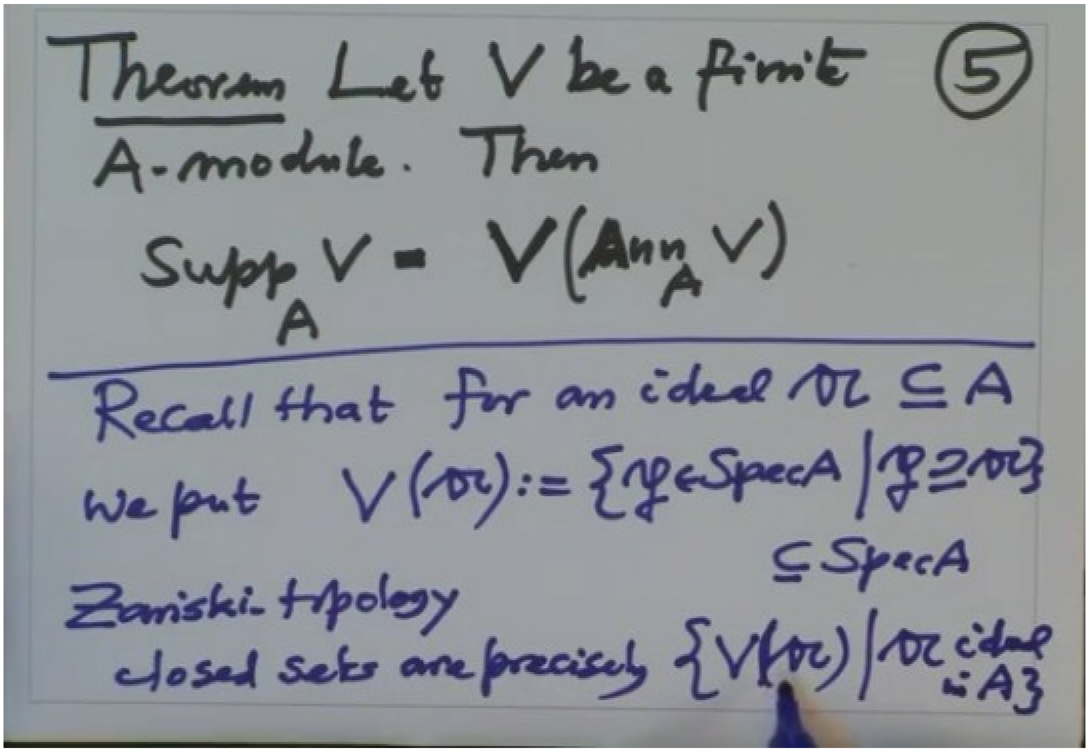

this collection forms a closed sets in a topological space they satisfy the properties of the closed sets in the topological space that we have seen in earlier lecture, and this topology on the spectrum is called the Zariski topology on spec A, this is what the notations.

And we want to prove that the support of the module is precisely V of the annihilator, alright, so what is it that we want to prove? Let us explicitly write it down, so proof, alright, so we want to prove that so to prove support of M, support of V this is precisely all those prime ideal in A such that P should contain annihilator of V, this is what we need to prove, because this is precisely V of annihilator of V, this is what we have shared, right.

So now I want to prove this, so I will prove this inclusion first, so we are proving this inclusion that means if I have an element, prime ideal in the support then I should prove that it contains the annihilator, so let P belong to the support of V so that is what, that means V localize at P is nonzero, and what do we want to prove?
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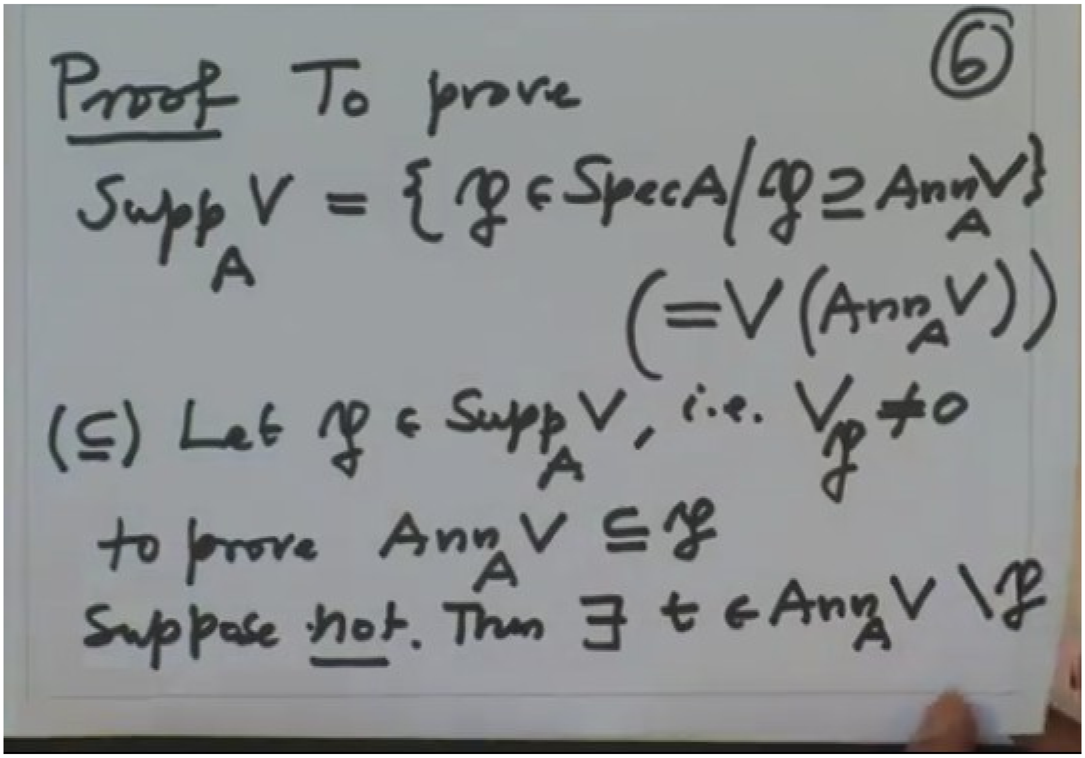

I want to prove that, so to prove annihilator of V is contained in P, so suppose not, suppose not then what? That means I can find if this is not contained here then I can find an element in the annihilator which is not in P, so then there exists T in the annihilator of V which is not in P.

And remember what do we want to prove? We want to now get a contradiction, to what? We will get a contradiction to this, okay so now let us take any element of $\frac{V}{P}$, V localize at P, any element $\frac{x}{s}$, this means x is in V and s is in $A \setminus P$, take arbitrary element, then $\frac{x}{s}$ is also same as I can multiply up and down by that, t I have chosen so tx divided by ts, these are same that is the localization and but t is in the annihilator so therefore the numerator is 0, but then this is a 0 element in V localize at P, so $\frac{x}{s}$ so I have proved that if I take arbitrary element in the localization then it is 0, so that means we have proved that some module V localize at P is 0, but then that contradicts the fact that it is nonzero, contradiction, so therefore we have proved this inclusion.

Now conversely so this inclusion, what do we want to prove? We want to prove that, so let P is a prime ideal with P contains annihilator of V, I've given the prime ideal in the right hand side, and to prove P is in the support of V, that means V localize at P should be nonzero, this is what we want to prove, alright.

So now what is the other assumption we have? We have assumption that V is a finite module, so that means V is generated by finitely many elements,
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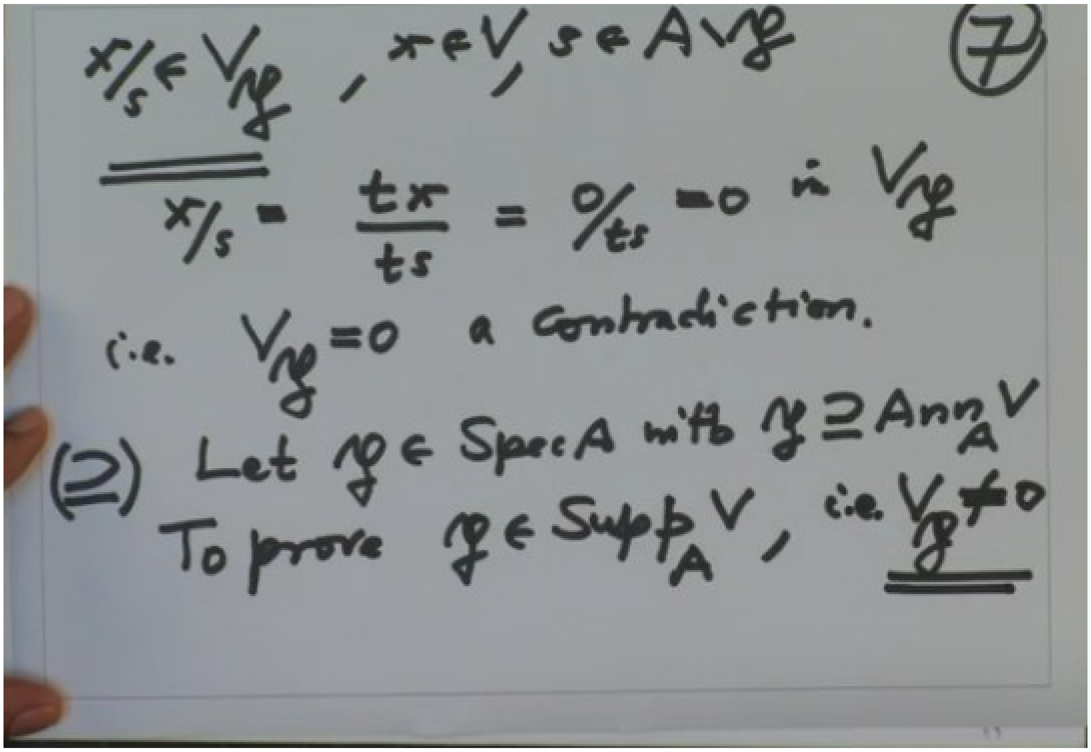

so let us say V is generated by Ax_1 etcetera, etcetera, Ax_r for some elements, r elements x_1 to x_r in V this means every element of V is A linear combination of this x_1 to x_r, that means V is generated by this x_1 to x_r, that is we have given,

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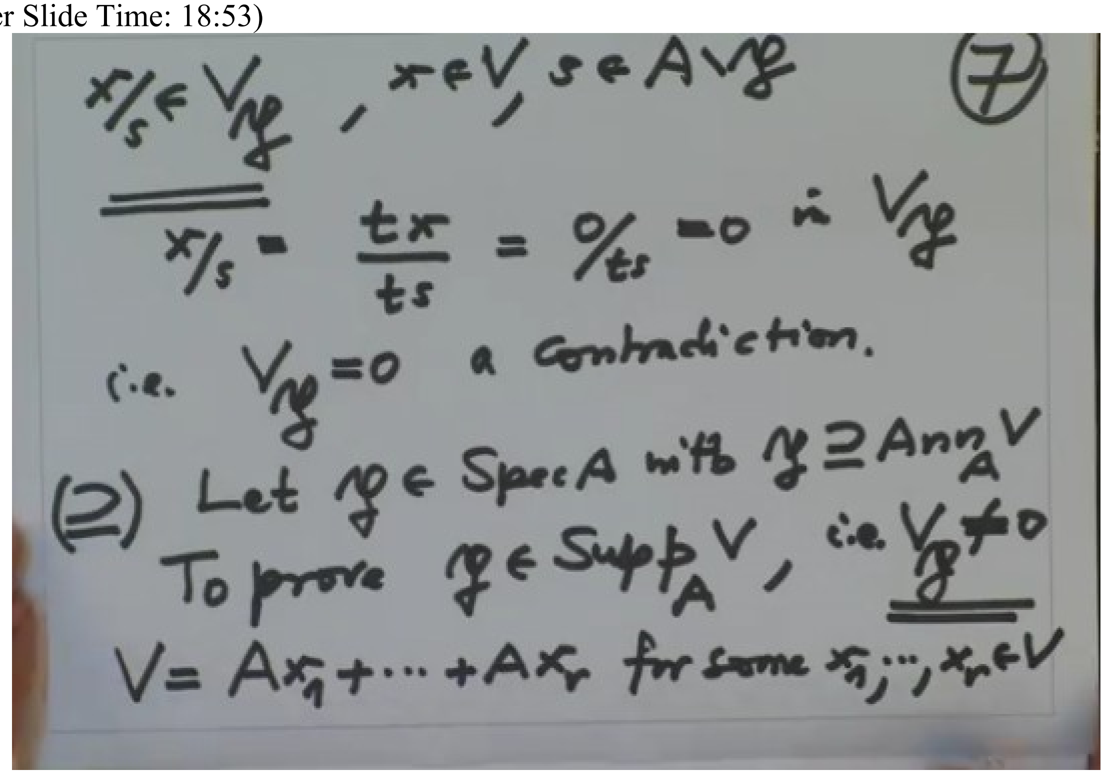

and we want to prove now that P is in the support, P contains the annihilator, so what is annihilator of V? Annihilator of V is annihilator of this sum $Ax_1 + \ldots + Ax_r$, because V is equal to this, and what is the annihilator of the sum? That is intersection, so intersection I is from 1 to R annihilators of the elements ideals, some module generated by Ax_i, this is the, but this is the intersection of this ideals, so let us call, if you call this ideals to be A_i's, these are the ideals in the ring and this is intersection of that ideals and if you have intersection of the ideals that we contained the product of the ideals, so this contains, this is intersection is bigger than the product A_1 to A_r, so this is what we know about the annihilator.

And now we had given P, so since P is a prime ideal and P contains annihilator of V which is, which contains the product and because it's a prime ideal therefore it will contain one of them, so therefore P contains A_i which is annihilator of Ax_i for some i from with less equal to 1, less equal to r, there exists an i so that this P contains this ideal A_i,
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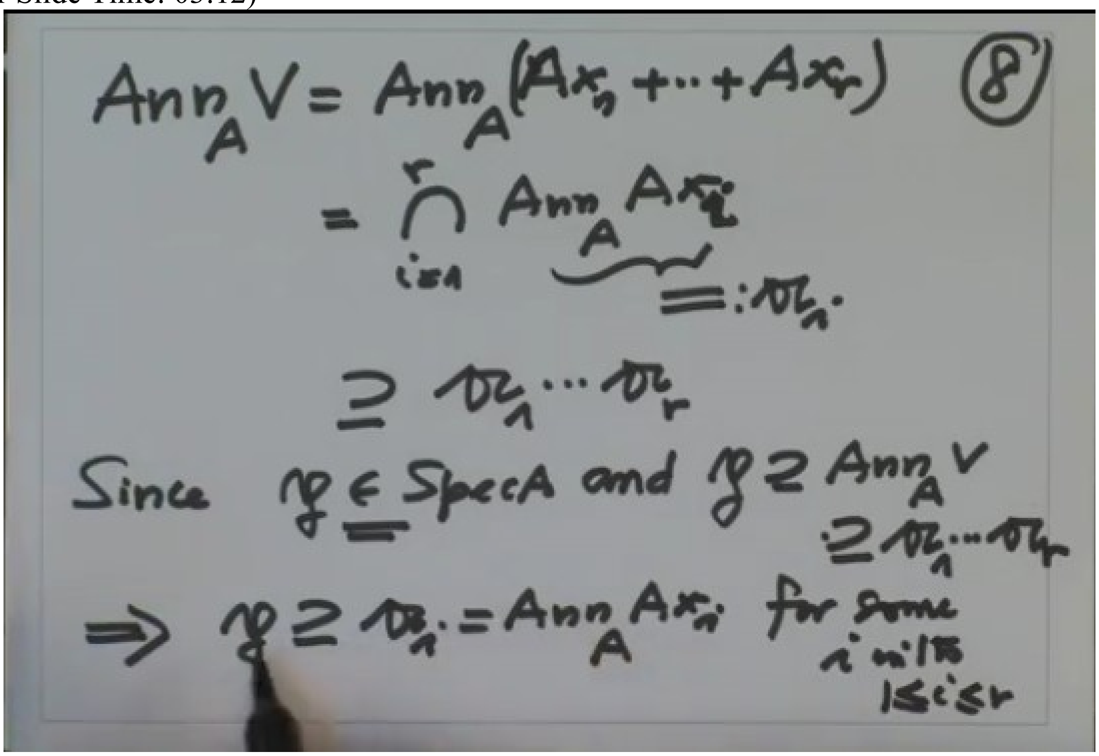

this is where we've used the fact that P is a prime ideal, but then that means what? That means if I look at from the ring A if I go to the ring, the residue class ring A by annihilator of Ax_i, this is the ring the residue class ring $\frac{A}{A_i}$, see if I pass on from here to here P is a prime ideal here and P contains this AI so therefore P will continue to be the prime ideal that I will denote by $\overline{P}$, $\overline{P}$ is $\frac{P}{A_i}$ so this is the prime ideal in the residue class ring, so this $\overline{P}$ is a prime ideal in the ring $\frac{A}{A_i}$, alright.

And what is now A mod annihilator of Ax_i, so why do I write so much? Look at A, modulo A_i and then the localize at this, just think of this as A module and localize at P that is same

thing as A mod A_i this ring localizing at the $\overline{P}$, that means what? That means localization commutes with the residue,
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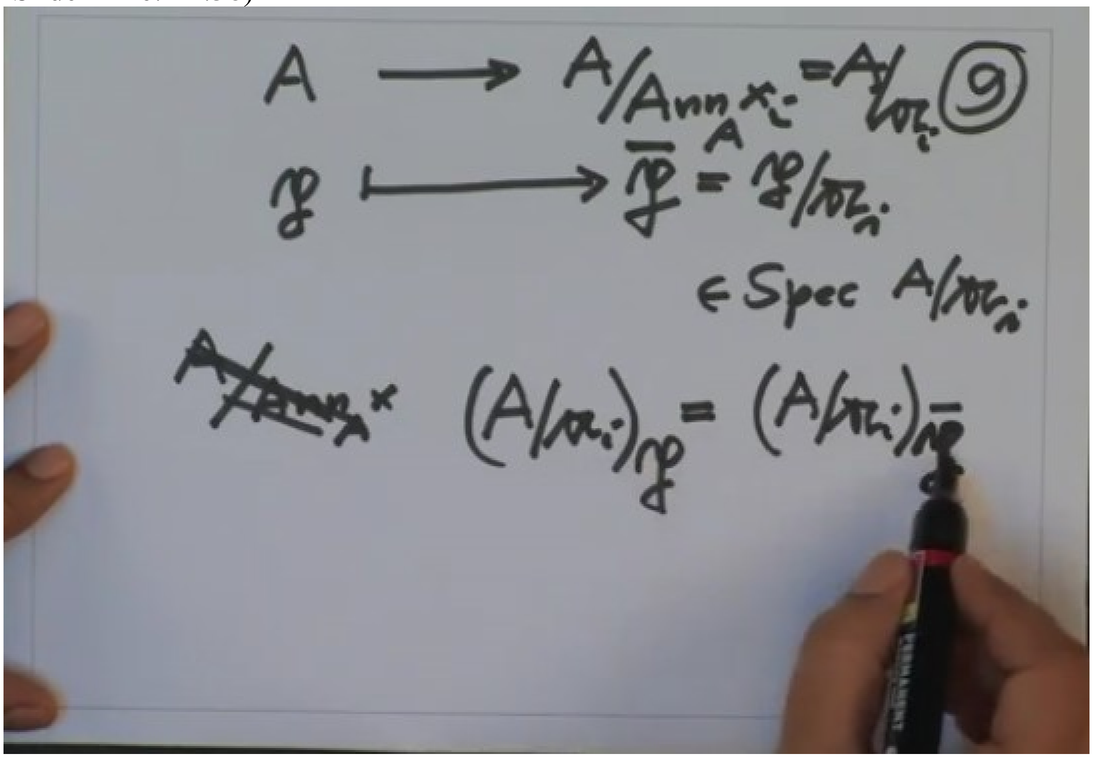

so whether I localize first or I localize, I go mod and then localize, these two operations are same, but then what is this? This is definitely nonzero, this is definitely nonzero because this is nonzero because this is a prime ideal in the ring, this ring, and it's a non, so it's nonzero because it's a proper ideal, P is a proper ideal I have noted that if you take a ring then every element in the spectrum is a support, so this P bar is in the spectrum of this, so therefore as a ring this one every element of the spectrum of that ring will be in the support, so therefore what we proved is A modulo the annihilator of Ax_i this we know, this is, this localize at P that is this set, this is this, and this inside module this is isomorphic to the sub module of V generated by x_i, we have seen that this two are isomorphic, this we have seen in the beginning so therefore they localization isomorphic, and therefore we know because this is nonzero, this is nonzero, see this is nonzero, so therefore this is nonzero, this is nonzero,
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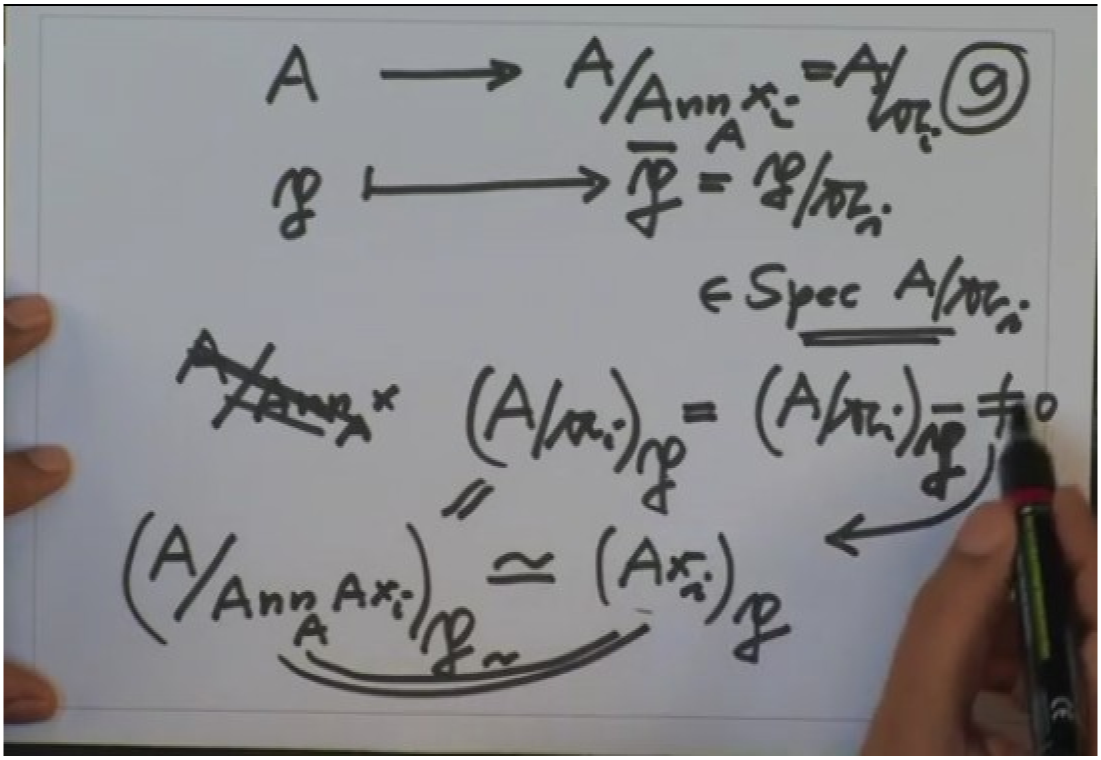

but this is a sub of V localize at P, so this is nonzero, so therefore what we proved is V localize at P as a nonzero sub module in particular the module should be nonzero.

V localize at P is nonzero and that is what we wanted to prove,

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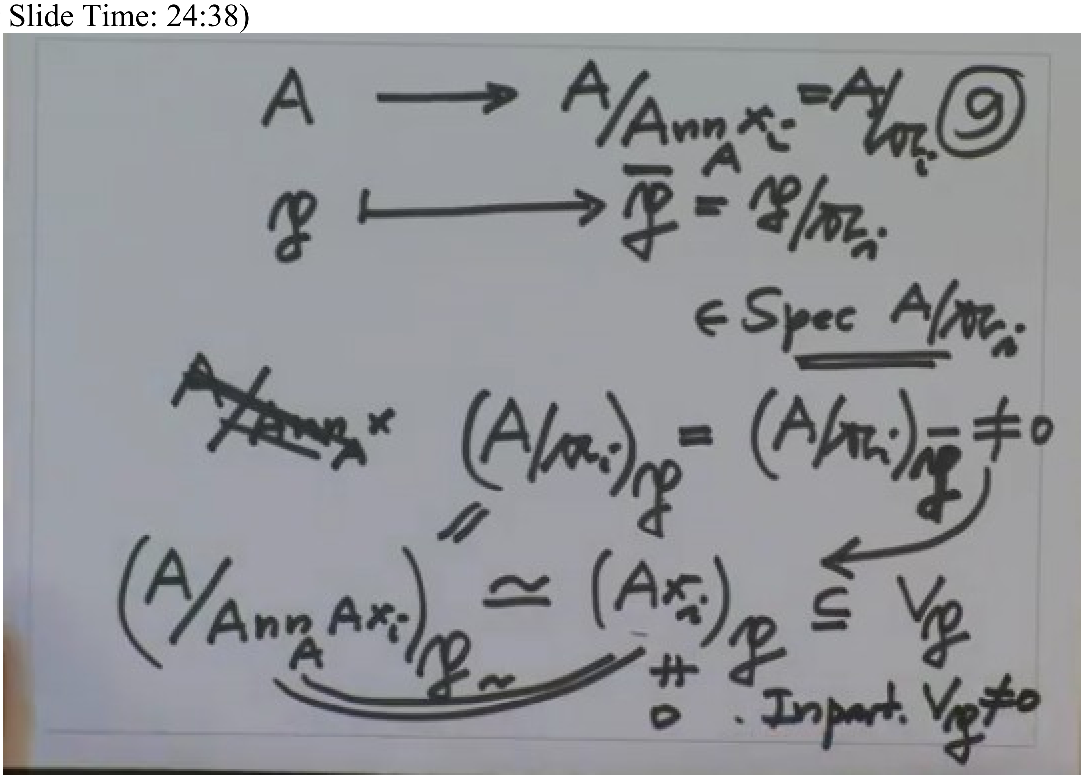

so we have proved that the support is precisely the V of the annihilator also the module V, so in particular it is a closed set in Zariski topology, so alright.

So now the next one, next theorem is also equally very important that now gives a connection between the associated prime ideals, so the next theorem is, next theorem is, now I'll assume let A be Noetherian, and V finite A module, and we have a chain of sub modules, chain of sub modules of V with the successive residues $\frac{V_i}{V_{i-1}}$, these are isomorphic to $\frac{A}{P_i}$ where P_i is are the prime ideals, i is from 1 to n, okay.

Then remember we have proved earlier that the associated prime ideals of V, this is contained in P_1 to P_n , this we have proved earlier, this is we have proved and now I'm saying that this is contained in the support of V,
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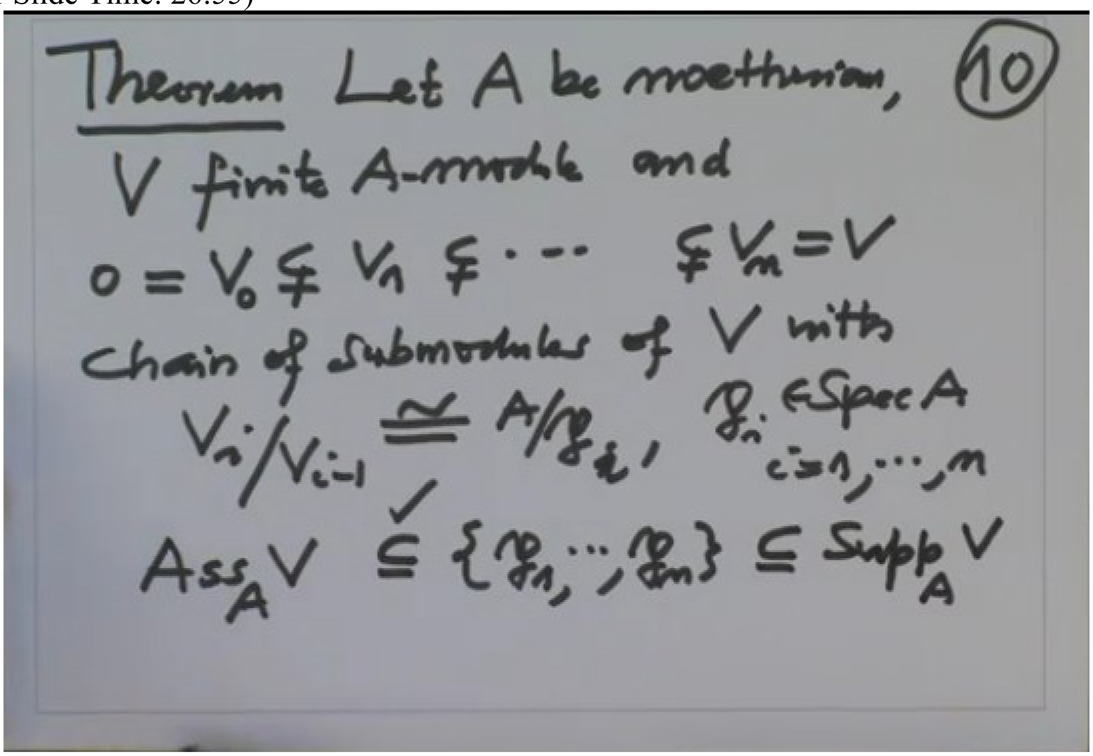

moreover associated prime ideals of V this set and support of V have the same minimal
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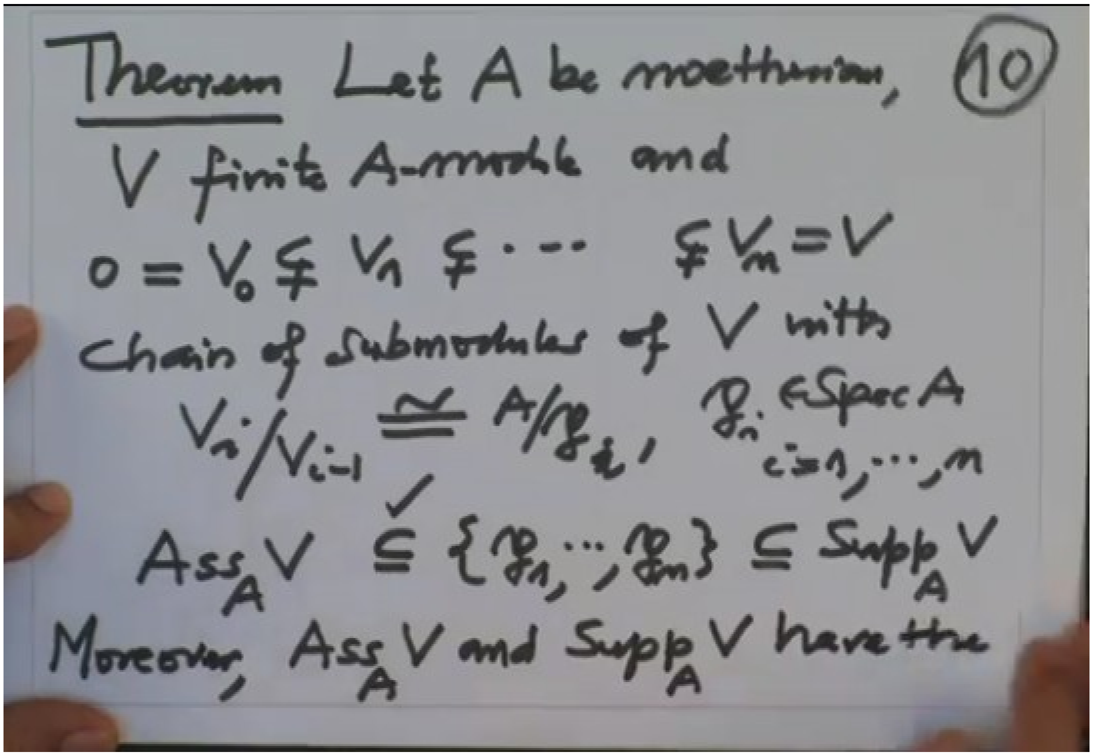

elements with respect to the natural inclusion.

Alright, so proof, we have already proof that the associated prime ideals of V is contained in P_1 to P_n, this we have already proved, proved earlier. Alright, so and what do you know? How these P_i 's are? A mod P_i 's are precisely the ith sub quotient, this is isomorphic to $\frac{V_i}{V_{i-1}}$ that is what is given to us, so therefore when I localize now this is the residue class ring, so when I localize A mod P_i at P_i this is definitely nonzero, same reason because this is the ring, this is a nonzero ring because P_i 's are the proper ideal, this is nonzero ring, and this is a localization of that ring therefore this is nonzero, and this is onwards so this isomorphic to this $\frac{V_i}{V_{i-1}}$ localize at P_i, so therefore this localize at P_i is nonzero, therefore so that definitely implies the numerator V_i localize at P_i, this is definitely nonzero, because numerator is 0 this module will be 0, but then if so therefore this V_i localize at P_i this is nonzero, and this is a sub module of V localize at P_i, because localization is,

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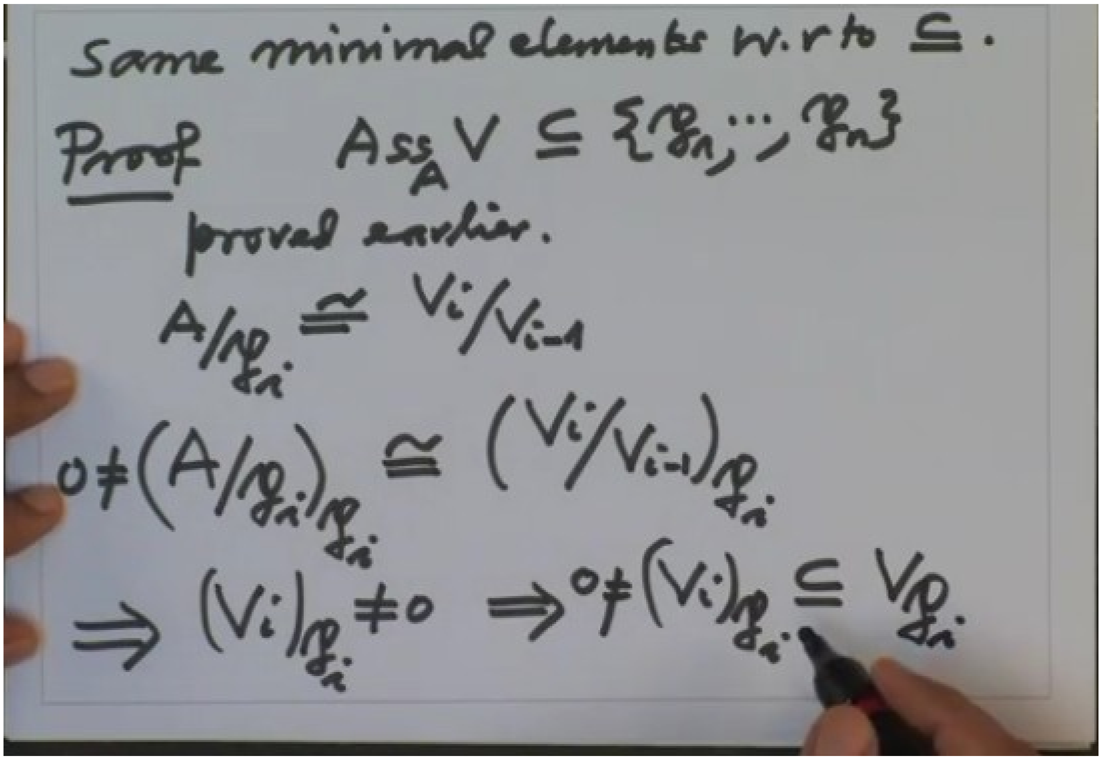
1. 

localization keep injections so that means if V_i is a sub modulo of V then further localization will also, it will be a sub module of the localization, so therefore I produce a sub module which is nonzero in particular V_{P_i} is nonzero and that shows that this prime ideals P_i 's are in the support, so and P_i 's belong to the support of V, so we are proved the other inclusion, so proved, so we have proved this P_1 to P_n this is a subset of the support.

Now we only have to prove, so only need to prove minimal elements are same, only to prove that minimal elements in the support are contained in the associated primes, so that means we need to prove, if P is in the support of V is minimal element with respect to of course the natural inclusion, then we need to prove P belonging to associated primes of V,
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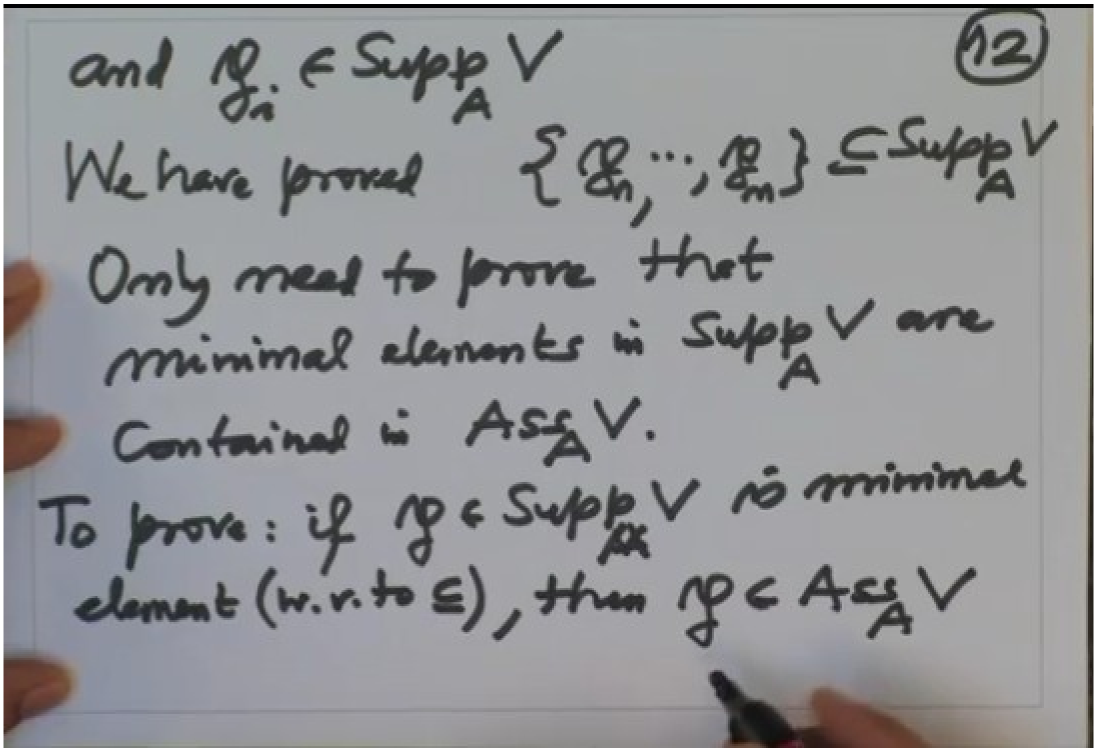

that means we want to find an element x so that P is annihilator of x, for some $x \in V$ for some, so there exists $x \in V$ so that P is annihilator of x that is what we want to prove.

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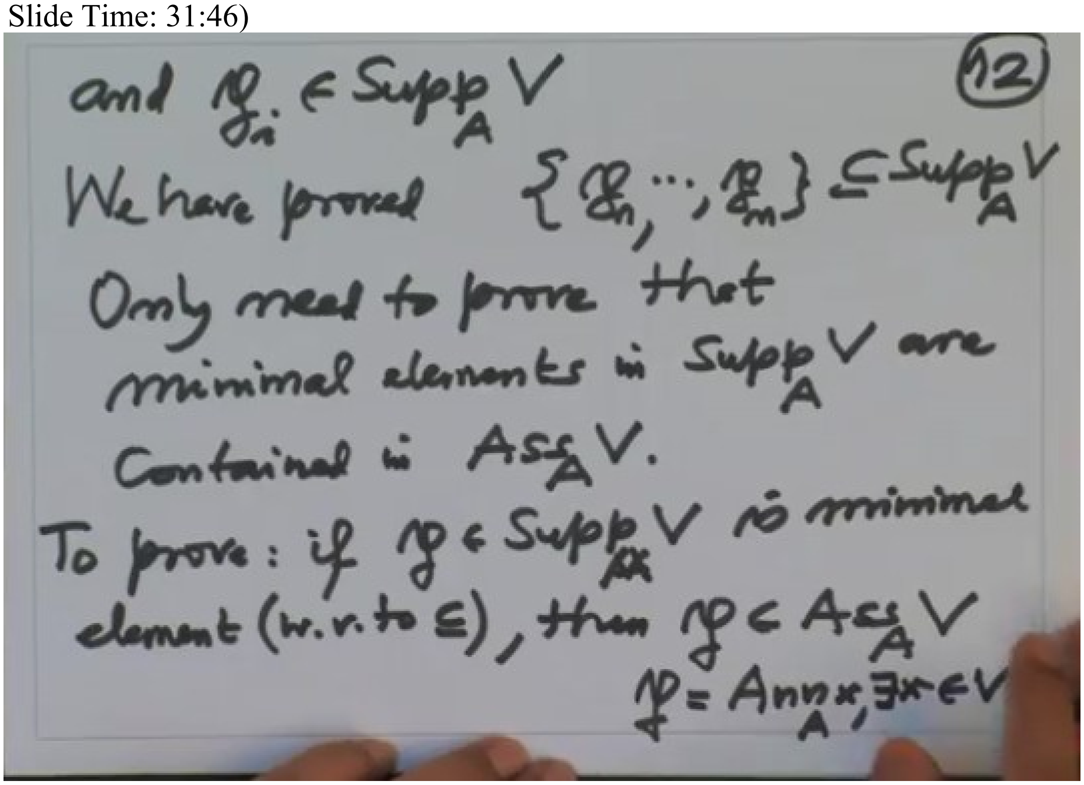

We want to prove like what we, some properties what we, I'll use one of the properties, so alright, so we started with the P in the support and P is a minimal element, now when I localize so V is here then localize V at P, so there is a natural map here, and we have the support of, this is a support, so if P is in the support of V then it will, this is nonzero and therefore, so therefore support of this module V_P as a A_P module definitely the image of P in the localization that should be there and there won't be anymore, because all the prime ideals which are smaller they cannot be in the support because we have chosen a minimal element, we have given a minimal element than the support therefore the only possibility disguise in the support and that is in the support, so therefore we have this, but alright.

So if I now, so support is this and on the other hand also what is the, what about the associated primes of V localize at P, at A localize at P? This I want to check that this is nonempty, once I check it is nonempty let us also check it is nonempty that is because if I look at,
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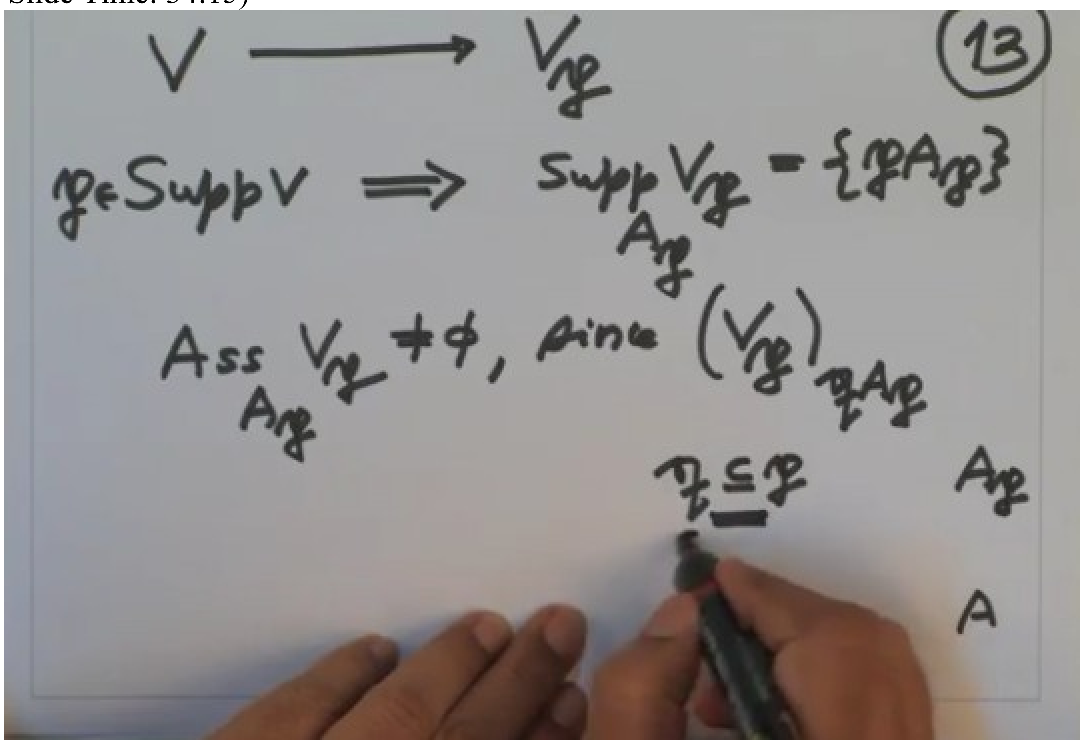

look at this modules since V localize at P and if I further localize at prime ideal Q, A localize at P, so Q is a prime ideal which is contained in P because in the ring A localize at P, the only prime ideals will come from A which are contained in P, the other prime ideals which are not contained in P they will become unit ideals, so if I take any Q in contained in P, Q prime ideal in A and further localize these V_P at this, this I get, this is isomorphic to V localized at Q, this is an easy property of the localization, so this is this,
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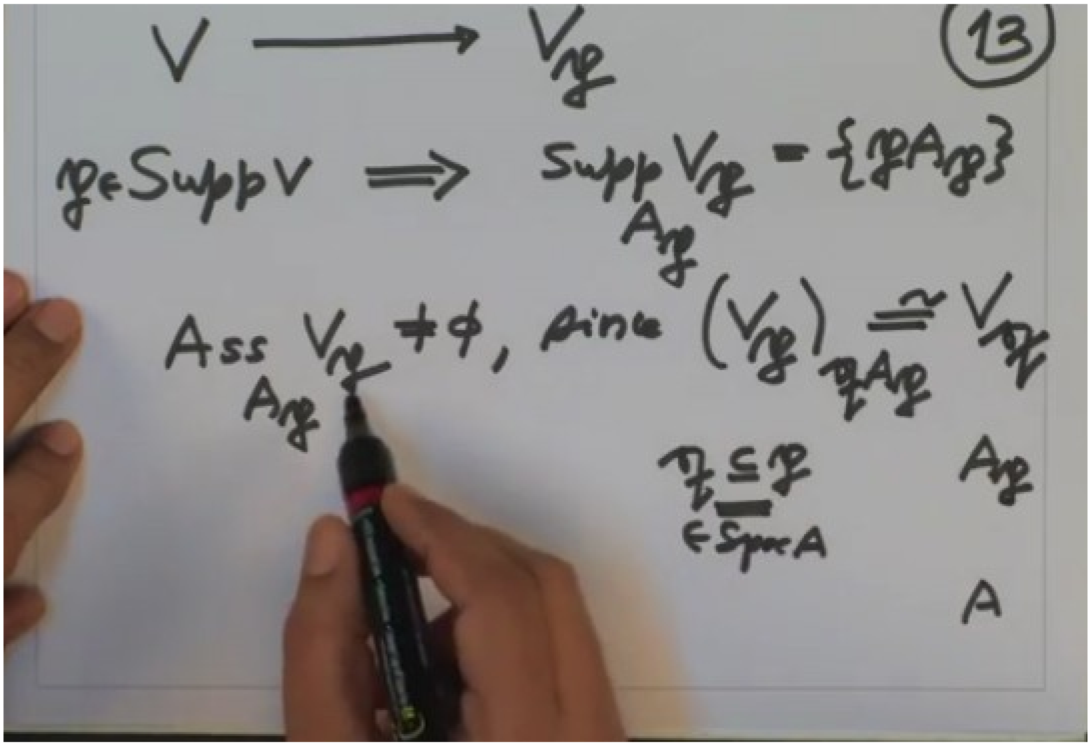

therefore definitely the associated primes of this is nonempty because this is nonzero module, this is nonempty and therefore the associated prime ideals, so that implies associated prime ideals of V_P as A_P module, this is precisely PA_P, there is no other way because we have proved that associated prime ideals is the subset of the support, so the support has only one element and this associated primes is nonempty set and it is contained in this, so there is no choice, so associated prime ideals of this must be this.

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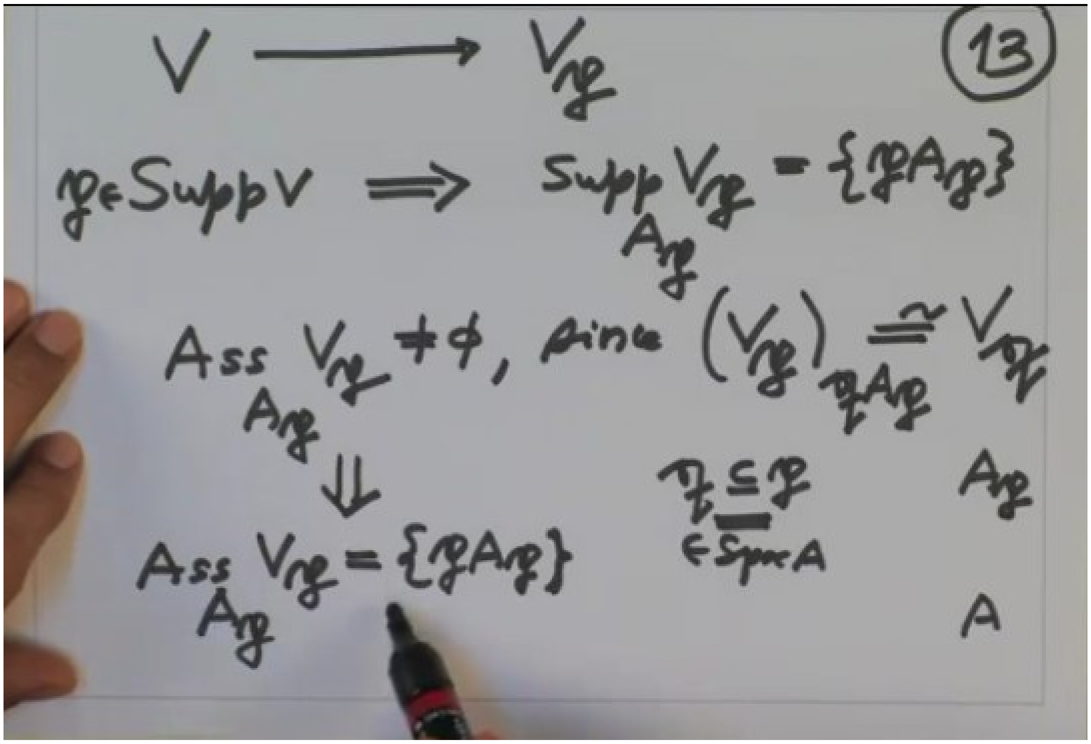

And now use the localization, after localization I have proved V_P, PA_P associated to V_P so that will prove that P belongs to associated prime ideals of V, this is by the theorem we proved in the localization case,

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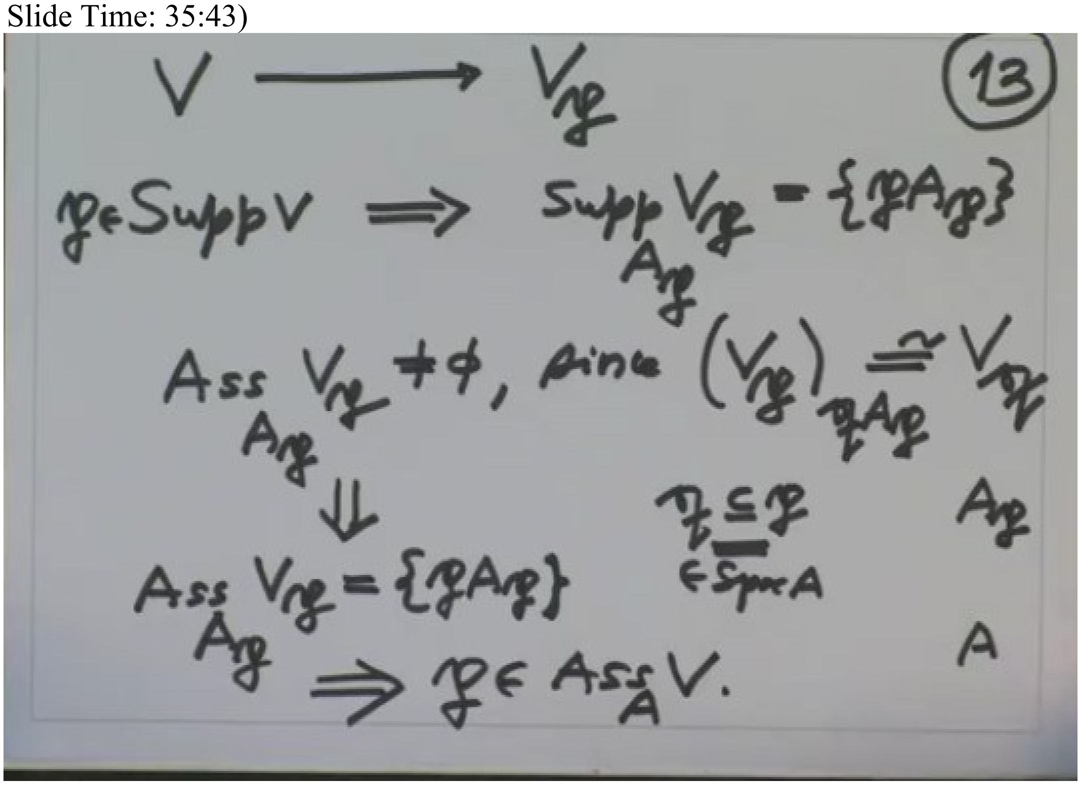

so that proves this theorem, and we will continue this discussion more in the next lecture and there we will also discuss about what is called primary decomposition of a module over Noetherian ring. Thank you very much.

Prof. Sridhar Iyer

NPTEL Principal Investigator
&
Head CDEEP, IIT Bombay

Tushar R. Deshpande
Sr. Project Technical Assistant

Amin B. Shaikh
Sr. Project Technical Assistant

Vijay A. Kedare
Project Technical Assistant

Ravi. D Paswan
Project Attendant

Teaching Assistants

Dr. Anuradha Garge

Dr. Palash Dey

Sagar Sawant

Vinit Nair

Pranjal Warade

Bharati Sakpal
Project Manager

Bharati Sarang
Project Research Associate

Riya Surange
Project Research Assistant

Nisha Thakur
Sr. Project Technical Assistant

Project Assistant
Vinayak Raut